Expansion formulas and addition theorems for Gegenbauer functions*

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We give a systematic summary of the properties of the Gegenbauer functions $C_\lambda^\alpha(x)$ and $D_\lambda^\alpha(x)$ for general complex degree and order, with emphasis on the functions of the second kind, $D_\lambda^\alpha(x)$, and on results useful in scattering theory. The results presented include Sommerfeld–Watson type expansion formulas and two reciprocal addition formulas for the functions of the second kind.

The analysis of Bethe–Salpeter wavefunctions and relativistic scattering amplitudes in terms of representations of the homogeneous Lorentz group has been considered by many authors.[1] The most common approaches have involved either the use of the representation theory of the Lorentz group and expansions in the corresponding representation functions to simplify dynamical calculations, or the use of the ideas of Fourier analysis on the Lorentz group to obtain representations of scattering amplitudes in different kinematic regions. A more general approach based on the analytic properties of relativistic scattering amplitudes, and the expansion of the Cauchy denominators in fixed-energy dispersion relations using the representation functions of the Lorentz group, was developed by the present authors.[2]

The functions which appear in any of the foregoing approaches to relativistic scattering problems for spinles particles are Gegenbauer (or hyperspherical) functions. Although the properties of the Gegenbauer polynomials $C_n^\alpha(x)$ are well known and readily available,[3–6] much less information is available in standard references about the Gegenbauer functions of the second kind, $D_\lambda^\alpha(x)$. The results obtained in Ref. 2, especially those concerned with the connections between Lorentz and Regge expansions, depended on a number of detailed properties of the $D_\lambda^\alpha(x)$ which we found it necessary to work out for ourselves. In particular, we derived several remarkable addition formulas for the functions of the second kind and general Regge-like expansion formulas for the Cauchy denominator.

In the present paper, we have attempted to collect systematically most of the results on Gegenbauer functions which we found to be useful in our earlier work. Some of the results on addition and expansion formulas are new, as noted; others are known, but not readily available. Finally, for completeness, we have included some standard results, or generalizations of standard results.

The present paper is divided into a number of short subsections which deal with particular properties of the Gegenbauer functions $C_\lambda^\alpha(x)$ and $D_\lambda^\alpha(x)$ for general values of λ, α, and x. Sections 1–5 deal with the definitions and elementary properties of the functions (integral representations, representations as hypergeometric functions, index symmetries, recurrence relations, reflection symmetries, and analytic properties). Section 6 deals with asymptotic properties of the functions for large degree and order, and contains some new relations. Section 7 deals with the expansion of the Cauchy denominator, and Secs. 8–10 with the addition formulas. The main expansion and addition formulas of Secs. 7–10 are restated for Legendre functions in Sec. 11. Representative derivations of asymptotic limits of the Gegenbauer functions for large order and degree (Sec. 6) are given in Appendix A. A detailed proof of one of the addition formulas of Sec. 9 is given in Appendix B. The rest of the results given in Secs. 6–10 can be established using similar methods, but detailed proofs are not given.

1. DEFINITIONS AND INTEGRAL REPRESENTATIONS

The Gegenbauer functions $C_\lambda^\alpha(z)$ are defined as the solution of the differential equation [HTF 3·15·2 (2)][4]

$$\left\{(z^2-1)\frac{d^2}{dz^2}+(2\lambda+1)z\frac{d}{dz}-\lambda(\lambda+2\alpha)\right\}C_\lambda^\alpha(z)=0, \tag{1.1}$$

which are regular at the singular point $z=1$. The functions of general degree λ and order α with $\mathrm{Re}(\lambda+2\alpha)>0$ are given by the integral representation

$$C_\lambda^\alpha(z)=(2\pi i)^{-1}\int_C dt\, t^{-\lambda-1}(1-2zt+t^2)^{-\alpha} \tag{1.2}$$

$$=(2\pi i)^{-1}\exp(2\pi i\alpha)\int_C dt\, t^{-\lambda-1}(t-z_+)^{-\alpha}(t-z_-)^{-\alpha}, \tag{1.3}$$

$\mathrm{Re}(\lambda+2\alpha)>0.$

Here $z_\pm=z\pm(z^2-1)^{1/2}$. The arguments of z, $(z-1)$, and $(z+1)$ are all restricted to the range $(-\pi,\pi)$. In particular, $(z^2-1)^{1/2}$ is cut from $z=-1$ to $z=+1$. The contour C in Eqs. (1.2) and (1.3) must enclose the origin in a counterclockwise sense and avoid the cuts of the integrand (see Fig. 1). The factor $(1-2zt+t^2)^{-\alpha}$

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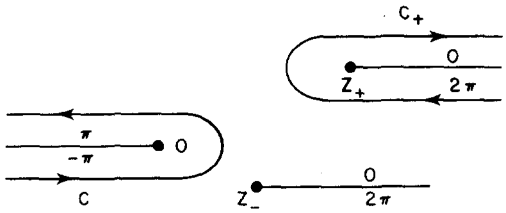

FIG. 1. The contour C in the t plane is used in the integral representation of $C_\lambda^\alpha(z)$, (1.3). The contour C_+ is used in the integral representation of $D_\lambda^\alpha(z)$, (1.5). The phases of t, $t-z_+$, $t-z_-$ above and below their respective cuts, are indicated in the figure.

is to be interpreted everywhere as $\exp(2\pi i\alpha)(t-z_+)^{-\alpha}\times(t-z_-)^{-\alpha}$ with $0<\arg(t-z_\pm)<2\pi$.

For $\mathrm{Re}\lambda<0$ and $\mathrm{Re}(\lambda+2\alpha)>0$, one can obtain an alternative integral representation for $C_\lambda^\alpha(z)$ by changing the contour integral in (1.2) to an integral along the negative real axis, and the replacing t by e^β,

$$C_\lambda^\alpha(z)=-2^{-\alpha}\frac{\sin\pi\lambda}{\pi}\int_{-\infty}^{\infty}d\beta\exp[-(\lambda+\alpha)\beta]\times(\cosh\beta+z)^{-\alpha}. \tag{1.4}$$

From this it is clear that $C_\lambda^\alpha(z)$ is analytic in the z plane cut from $z=-1$ to $z=-\infty$. The cut structure may also be deduced from (1.3) by noting that the points $z_\pm$ pinch the integration contour in Fig. 1 for $z\to-1$, $-\infty$.

A second solution to Gegenbauer's equation can be obtained by choosing a different contour in the integral representation (1.3). We choose a contour such that $D_\lambda^\alpha(z)\to0$ for $|z|\to\infty$, $\mathrm{Re}(\lambda+2\alpha)>0$, and define $D_\lambda^\alpha(z)$ by[7]

$$D_\lambda^\alpha(z)=\exp(2\pi i\alpha)(2\pi i)^{-1}\int_{C_+}dt\,t^{-\lambda-1}(t-z_+)^{-\alpha}(t-z_-)^{-\alpha}. \tag{1.5}$$

The contour C_+ is defined in Fig. 1. The singularities $z_\pm$ pinch the contour C_+ for $z\to+1$, $-\infty$; the function $D_\lambda^\alpha(z)$ is consequently cut from $z=+1$ to $z=-\infty$. For $\mathrm{Re}\alpha<1$ and $\mathrm{Re}(\lambda+2\alpha)>0$, we can obtain an alternative representation for $D_\lambda^\alpha(z)$ which is analogous to (1.4),

$$D_\lambda^\alpha(z)=2^{-\alpha}\exp(i\pi\alpha)\frac{\sin\pi\alpha}{\pi}\times\int_{\cosh^{-1}z}^{\infty}d\beta\exp[-(\lambda+\alpha)\beta](\cosh\beta-z)^{-\alpha}. \tag{1.6}$$

2. REPRESENTATIONS IN TERMS OF HYPERGEOMETRIC FUNCTIONS

The integral representation for $D_\lambda^\alpha(z)$, (1.5), is easily transformed into the standard integral representation for the hypergeometric function. One finds that

$$D_\lambda^\alpha(z)=\exp(i\pi\alpha)\frac{[z+(z^2-1)^{1/2}]^{-\lambda-2\alpha}\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+\alpha+1)}\times{}_2F_1\left(\alpha,\lambda+2\alpha;\lambda+\alpha+1;\frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right) \tag{2.1}$$

$$=\exp(i\pi\alpha)\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+\alpha+1)}\exp[-(\lambda+2\alpha)\beta]\times{}_2F_1(\alpha,\lambda+2\alpha;\lambda+\alpha+1;e^{-2\beta}), \tag{2.2}$$

$$z=\cosh\beta,\quad e^\beta=z+(z^2-1)^{1/2},\quad |e^\beta|>1,$$

$$=\exp(i\pi\alpha)\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+\alpha+1)}(2z)^{-\lambda-2\alpha}\times{}_2F_1(\tfrac12\lambda+\alpha,\tfrac12\lambda+\alpha+\tfrac12;\lambda+\alpha+1;z^{-2}),\quad |z|>1, \tag{2.3}$$

$$=\frac{1}{\sqrt{\pi}}\exp[2\pi i(\alpha-\tfrac14)]2^{-\alpha+1/2}\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}(z^2-1)^{-\alpha/2+1/4}\times Q_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z). \tag{2.4}$$

In the last expression $Q_\sigma^\mu(z)$ is the usual associated Legendre function of the second kind [HTF 3.2 (5)]. The asymptotic behavior of $D_\lambda^\alpha(z)$ as $|z|\to\infty$ is clear from (2.3).

Hypergeometric expansions of $C_\lambda^\alpha(z)$ are available in standard references, or can be derived from (1.3)[3,4] We quote only the expressions

$$C_\lambda^\alpha(z)=\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\lambda+1)\Gamma(2\alpha)}\,{}_2F_1(-\lambda,\lambda+2\alpha;\alpha+\tfrac12;\tfrac12(1-z)),\quad |\tfrac12(1-z)|\le1, \tag{2.5}$$

$$=-\frac{1}{\pi}\sin\pi\lambda\frac{\Gamma(\lambda+2\alpha)\Gamma(-\lambda-\alpha)}{\Gamma(\alpha)}[z+(z^2-1)^{1/2}]^{-\lambda-2\alpha}\times{}_2F_1\left(\lambda+2\alpha,\alpha;\lambda+\alpha+1;\frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right)$$
$$+\frac{\Gamma(\lambda+\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}[z+(z^2-1)^{1/2}]^{\lambda}\times{}_2F_1\left(-\lambda,\alpha;-\lambda-\alpha+1;\frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right),$$
$$z+(z^2-1)^{1/2}=e^\beta,\quad |e^\beta|\ge1, \tag{2.6}$$

$$=-\frac{1}{\pi}\sin\pi\lambda\frac{\Gamma(\lambda+2\alpha)\Gamma(-\lambda-\alpha)}{\Gamma(\alpha)}(2z)^{-\lambda-2\alpha}\times{}_2F_1(\tfrac12\lambda+\alpha,\tfrac12\lambda+\alpha+\tfrac12;\lambda+\alpha+1;z^{-2})$$
$$+\frac{\Gamma(\lambda+\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}(2z)^{\lambda}\times{}_2F_1(-\tfrac12\lambda,-\tfrac12\lambda+\tfrac12;-\lambda-\alpha+1;z^{-2}), \tag{2.7}$$
$$|z|>1,$$

$$=\sqrt{\pi}\,2^{-\alpha+1/2}\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}(z^2-1)^{-\alpha/2+1/4}P_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z), \tag{2.8}$$

where P_σ^μ is an associated Legendre function [HTF 3.2(3)]. The expansion of $C_\lambda^\alpha(z)$ for $|z|$ large may be obtained from (2.7).

The zeros and poles of $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ as functions of λ and α are readily deduced from the integral representations or the hypergeometric series (2.1) and

(2.5):

$$D_\lambda^\alpha \text{ has poles for } \lambda+2\alpha=0,-1,-2,\cdots,$$
$$\text{zeros for } \alpha=0,-1,-2,\cdots. \tag{2.9}$$
$$C_\lambda^\alpha \text{ has poles for } \lambda+2\alpha=0,-1,-2,\cdots,$$
$$\text{zeros for } \alpha=0,-1,-2,\cdots,$$
$$\text{and zeros for } \lambda=-1,-2,\cdots. \tag{2.10}$$

For $\lambda=0,1,2,\cdots$, $C_\lambda^\alpha(z)$ has no poles as a function of α.

3. SYMMETRY WITH RESPECT TO λ AND α

We can also obtain a number of useful symmetry properties of C_λ^α and D_λ^α from their hypergeometric expansions. From (2.5) and the properties of the gamma functions, it follows that

$$C_{-\lambda-2\alpha}^\alpha(z)=-\frac{\sin\pi(\lambda+2\alpha)}{\sin\pi\lambda}C_\lambda^\alpha(z). \tag{3.1}$$

From (2.3) and (2.7),

$$C_\lambda^\alpha(z)=\exp(-i\pi\alpha)\frac{\sin\pi\lambda}{\sin\pi(\lambda+\alpha)}[D_\lambda^\alpha(z)-D_{-\lambda-2\alpha}^\alpha(z)]. \tag{3.2}$$

It should be noted that the right-hand side of (3.2) does not vanish for $\lambda=$integer. The product $\sin\pi\lambda\, D_\lambda^\alpha(z)$ vanishes, but from (2.3) one can see that $\sin\pi\lambda\, D_{-\lambda-2\alpha}^\alpha(z)$ is nonzero.

If we rewrite (2.3) for $\alpha\to-\alpha+1$, $\lambda\to\lambda+2\alpha-1$, and apply the Kummer transformation [HTF 2.9(2)]

$${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z), \tag{3.3}$$

we obtain

$$D_{\lambda+2\alpha-1}^{-\alpha+1}(z)=-\exp(-2\pi i\alpha)2^{2\alpha-1}(z^2-1)^{\alpha-1/2}\times\frac{\Gamma(\lambda+1)\Gamma(\alpha)}{\Gamma(\lambda+2\alpha)\Gamma(-\alpha+1)}D_\lambda^\alpha(z), \tag{3.4}$$

while from (3.2) and (3.4) it follows that

$$C_{\lambda+2\alpha-1}^{-\alpha+1}(z)=2^{2\alpha-1}(z^2-1)^{\alpha-1/2}\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(-\alpha+1)\Gamma(\lambda+2\alpha)}\times\{C_\lambda^\alpha(z)-2\exp(-i\pi\alpha)\cos\pi\alpha\, D_\lambda^\alpha(z)\}. \tag{3.5}$$

These results can be re-expressed as well-known properties of the Legendre functions by using (2.4) and (2.8).

4. RECURRENCE RELATIONS

The recurrence relations for the functions $C_\lambda^\alpha(z)$ are available from standard references (HTF 3.15.2):

$$(\lambda+1)C_{\lambda+1}^\alpha(z)-2(\lambda+\alpha)zC_\lambda^\alpha(z)+(\lambda+2\alpha-1)C_{\lambda-1}^\alpha(z)=0, \tag{4.1}$$

$$(\lambda+\alpha+1)C_{\lambda+1}^\alpha(z)=\alpha[C_{\lambda+1}^{\alpha+1}(z)-C_{\lambda-1}^{\alpha+1}(z)], \tag{4.2}$$

$$2\alpha(1-z^2)C_{\lambda-1}^{\alpha+1}(z)=(\lambda+2\alpha-1)C_{\lambda-1}^\alpha(z)-\lambda zC_\lambda^\alpha(z), \tag{4.3}$$

$$\frac{d}{dz}C_\lambda^\alpha(z)=2\alpha C_{\lambda-1}^{\alpha+1}(z), \tag{4.4}$$

$$\frac{d}{dz}[C_{\lambda+1}^\alpha(z)-C_{\lambda-1}^\alpha(z)]=2(\lambda+\alpha)C_\lambda^\alpha(z). \tag{4.5}$$

The functions $D_\lambda^\alpha(z)$ defined by (1.5) or (2.1)–(2.4) satisfy the same recurrence relations[7] provided $\lambda\neq0,-1,-2,\cdots$.

5. REFLECTION SYMMETRY AND CUT STRUCTURE

One can establish the following properties of $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ from the integral representations (1.3) and (1.5) or from the hypergeometric expansions in (2.3) and (2.5).

Reflection symmetries, $z\to\exp(\pm i\pi)z$:

$$C_\lambda^\alpha[\exp(\pm i\pi)z]=\exp(\pm i\pi\lambda)C_\lambda^\alpha(z)\mp2i\exp(-i\pi\alpha)\times\exp(\mp i\pi\alpha)\sin\pi\lambda\, D_\lambda^\alpha(z), \tag{5.1}$$

$$D_\lambda^\alpha[\exp(\pm i\pi)z]=\exp[\mp i\pi(\lambda+2\alpha)]D_\lambda^\alpha(z). \tag{5.2}$$

Discontinuities and relations on the cuts:

$$C_\lambda^\alpha(-x+i0)-C_\lambda^\alpha(-x-i0)=2i\sin\pi\lambda\{C_\lambda^\alpha(x)-2\exp(-i\pi\alpha)\cos\pi\alpha\, D_\lambda^\alpha(x)\},\quad x>1, \tag{5.3}$$

$$D_\lambda^\alpha(-x+i0)-D_\lambda^\alpha(-x-i0)=-2i\sin\pi(\lambda+2\alpha)D_\lambda^\alpha(x),\quad x>1, \tag{5.4}$$

$$C_\lambda^\alpha(x)=D_\lambda^\alpha(x+i0)+\exp(-2\pi i\alpha)D_\lambda^\alpha(x-i0),\quad |x|<1. \tag{5.5}$$

In these expressions, z is an arbitrary complex number on the first sheet of the z plane, while x is real. For $\alpha=1/2$, the relations reduce to those familiar in the case of the Legendre functions.

We can use the foregoing relations to obtain an integral representation for the functions $D_n^\alpha(z)$ with integer degree. It is easily seen from (5.2) that the function

$$\bar{D}_n^\alpha(z)=(z^2-1)^{\alpha-1/2}D_n^\alpha(z),\quad n=\text{integer}, \tag{5.6}$$

is cut only from $z=-1$ to $z=+1$. The discontinuity across this cut can be calculated using (5.5),

$$\bar{D}_n^\alpha(x+i0)-\bar{D}_n^\alpha(x-i0)=(1-x^2)^{\alpha-1/2}\exp[i\pi(\alpha-\tfrac{1}{2})]C_n^\alpha(x). \tag{5.7}$$

The function $\bar{D}_n^\alpha(z)$ can clearly be expressed as a contour integral using Cauchy's theorem, with a contour which consists of a clockwise circuit around the cut from $z=-1$ to $z=+1$, and a counterclockwise loop at ∞. The contribution from the latter vanishes for $n\geqslant0$ [see (2.3)]. For $\text{Re}\,\alpha>-\frac{1}{2}$, we can express the remaining integral in terms of the discontinuity function (5.7), and obtain the desired representation,

$$D_n^\alpha(z)=\exp(i\pi\alpha)(z^2-1)^{-\alpha+1/2}\frac{1}{2\pi}\int_{-1}^{1}dt\,\frac{(1-t^2)^{\alpha-1/2}C_n^\alpha(t)}{z-t},$$
$$n=0,1,2,\ldots,\ \text{Re}\,\alpha>-\tfrac{1}{2}. \tag{5.8}$$

In the special case $\alpha=\frac{1}{2}$, this result reduces to a familiar expression for $Q_n(z)$ [cf. (2.4), (2.8)]. See also Ref. 3, (4.61.4), for the corresponding result for the more general case of Jacobi functions.

6. ASYMPTOTIC LIMITS FOR LARGE λ AND α

We shall frequently need the behavior of $C_\lambda^\alpha(z)$ and $D_\lambda^\alpha(z)$ for large values of one or both of the parameters. This behavior is easily obtained from the integral representations by saddle point methods. It can also be obtained in some cases by noting that the leading terms in the hypergeometric function ${}_2F_1(a,b;c;z)$ provide an asymptotic expansion of the function in inverse powers of c for $\mathrm{Re}\,c \to \infty$ and $|z|>1$ provided z does not lie on the interval $1 \leq z < \infty$ [HTF 2.3.2]. In the present section, we will simply collect the results which we will need later, and indicate the method of derivation of each. The saddle point calculations for two nonstandard limits [(6.1) and (6.10)] are sketched in Appendix A. The remaining saddle point calculations involve similar techniques, and the details will not be given. All of the asymptotic estimates presented are uniform for z and any free parameters (e.g., α in the case $\lambda \to \infty$) in any fixed finite domains in the complex plane which exclude the points or regions indicated. We have not determined in most cases the most general conditions under which our results hold, but only that the range of validity is adequate for our purposes.

The asymptotic behavior of $C_\lambda^\alpha(z)$ for $|\lambda| \to \infty$ along any ray in the right half λ plane is considered in Appendix A. Along rays such that $\mathrm{Im}\lambda \to \pm\infty$,

$$C_\lambda^\alpha(z) \sim \lambda^{\alpha-1} 2^{-\alpha} [\Gamma(\alpha)]^{-1} (z^2-1)^{-\alpha/2} \times \{[z+(z^2-1)^{1/2}]^{\lambda+\alpha} + \exp(\pm i\pi\alpha)[z+(z^2-1)^{1/2}]^{-\lambda-\alpha}\},$$
$$|\lambda| \to \infty,\ \mathrm{Re}\lambda \geq 0,\ \mathrm{Im}\lambda \to \pm\infty,\ |\arg(z\pm 1)| < \pi. \quad (6.1)$$

This result holds, as indicated, for all z not on the interval $-\infty < z \leq 1$. The derivation which leads to (6.1) must be altered somewhat for $\mathrm{Re}\lambda \to \infty$ with $\mathrm{Im}\lambda$ fixed [see Appendix A or HTF 2.3.2 (17)], and one finds a result with the same form, but a quite different interpretation,

$$C_\lambda^\alpha(z) \sim \lambda^{\alpha-1} 2^{-\alpha} [\Gamma(\alpha)]^{-1} (z^2-1)^{-\alpha/2} \times \{[z+(z^2-1)^{1/2}]^{\lambda+\alpha} + \exp(\pm i\pi\alpha)[z+(z^2-1)^{1/2}]^{-\lambda-\alpha}\},$$
$$|\lambda| \to \infty,\ \mathrm{Re}\lambda \geq 0,\ \mathrm{Im}z \gtrless 0,\ |\arg(z\pm 1)| < \pi. \quad (6.2)$$

This result holds for all z not on the intervals $-\infty < z \leq -1$ and $1 \leq z < \infty$. It should be modified for z on the interval $1 \leq z < \infty$ by omission of the second term (see Appendix A). We note in this connection that

$$|z+(z^2-1)^{1/2}| \geq 1$$

for all z in the complex z plane cut from -1 to $+1$. The equality is attained only for z on the cut. The second term in (6.2) is therefore exponentially small relative to the first term and can be dropped in any case for $\mathrm{Re}\lambda \to \infty$ provided z is a finite distance away from the interval $[-1,1]$.

A more detailed analysis shows that (6.1) and (6.2) are equivalent for all z a finite distance away from $[-1,1]$ and all $|\lambda| \to \infty$, $\mathrm{Re}\lambda \geq 0$, provided only the dominant terms are retained. The interval $[-1,1]$ requires special treatment. The asymptotic behavior of $C_\lambda^\alpha(z)$ for $\mathrm{Re}\lambda \to -\infty$ can be determined by using the symmetry relation (3.1) in conjunction with (6.1) or (6.2).

The asymptotic behavior of $D_\lambda^\alpha(z)$ for $|\lambda| \to \infty$ is quite simple [Appendix A or (5d) and HTF 3.2 (37), 2.3.2 (16)],

$$D_\lambda^\alpha(z) \sim \exp(i\pi\alpha)\lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}(z^2-1)^{-\alpha/2}[z+(z^2-1)^{1/2}]^{-\lambda-\alpha},$$
$$|\lambda| \to \infty,\ |\arg\lambda| < \pi,\ |\arg(z\pm 1)| < \pi. \quad (6.3)$$

The asymptotic behavior of $C_{\lambda-n}^{\alpha+n}(z)$ for $\mathrm{Re}\,n \to \infty$ can be obtained from the integral representation (1.3) using a saddle point estimate or, alternatively, by applying the Kummer transformation (3.3) to the hypergeometric function (2.5) and retaining only the leading term (HTF 2.3.2),

$$C_{\lambda-n}^{\alpha+n}(z) = 2^{\alpha+n-1/2} \frac{\Gamma(\lambda+2\alpha+n)}{\Gamma(\lambda-n+1)\Gamma(2\alpha+2n)} (z+1)^{-\alpha-n+1/2} \times {}_2F_1(\lambda+\alpha+\tfrac{1}{2}, -\lambda-\alpha+\tfrac{1}{2}; \alpha+n+\tfrac{1}{2}; \tfrac{1}{2}(1-z))$$
$$\sim \frac{\sin\pi(n-\lambda)}{(n\pi)^{1/2}} 2^{-\alpha-n+1/2}(z+1)^{-\alpha-n+1/2},$$
$$\mathrm{Re}\,n \to \infty,\ |\arg(z+1)| < \pi. \quad (6.4)$$

The corresponding result for $D_{\lambda-n}^{\alpha+n}(z)$ can be obtained either by saddle point methods, or from (5.1) and (6.4). Thus, we find from (5.1) that

$$D_{\lambda-n}^{\alpha+n}(z) = \tfrac{1}{2}\exp[i\pi(\alpha+n)] \frac{1}{\sin\pi(\lambda-n)} \times \{\exp[\pm i\pi(\alpha+n+\tfrac{1}{2})]C_{\lambda-n}^{\alpha+n}[\exp(\pm i\pi)z] - \exp[\pm i\pi(\lambda+\alpha+\tfrac{1}{2})]C_{\lambda-n}^{\alpha+n}(z)\} \quad (6.5)$$

Use of the asymptotic limit (6.4) then gives the limit

$$D_{\lambda-n}^{\alpha+n}(z) \sim 2^{-\alpha-n-1/2}(\pi n)^{-1/2}\exp[i\pi(\alpha+n)] \times \{(z-1)^{-\alpha-n+1/2} - \exp[\pm i\pi(\alpha+\lambda-\tfrac{1}{2})](z+1)^{-\alpha-n+1/2}\}, \quad (6.6)$$
$$\mathrm{Re}\,n \to \infty,\ \mathrm{Im}z \lesseqgtr 0,\ |\arg(z\pm 1)| < \pi.$$

The second term in this expression should be omitted for z on the interval $1 < z < \infty$.

The asymptotic behavior of $C_{\lambda+n}^{\alpha-n}(z)$ for $\mathrm{Re}\,n \to \infty$ can be obtained by using successively (3.5), (5.1), and (6.4). We find from (3.5) and (5.1) that

$$C_{\lambda+2\alpha-1}^{-\alpha+1}(z) = 2^{2\alpha-1}(z^2-1)^{\alpha-1/2} \frac{\Gamma(\lambda+1)\Gamma(\alpha)}{\Gamma(\lambda+2\alpha)\Gamma(-\alpha+1)\sin\pi\lambda} \times \exp[\pm i\pi(\alpha+\tfrac{1}{2})]\{\cos\pi[\lambda+\alpha]C_\lambda^\alpha(z) - \cos\pi\alpha\, C_\lambda^\alpha[\exp(\pm i\pi)z]\},\quad \mathrm{Im}z \gtreqless 0. \quad (6.7)$$

If we now replace α in (6.7) by $n-\alpha+1$ and λ by $\lambda+2\alpha-n-1$, and use the asymptotic limit in (6.4), we find that

$$C^{\alpha-n}_{\lambda+n}(z) \sim 2^{n-\alpha+1/2}(\pi n)^{-1/2}\,\frac{\sin\pi(\alpha-n)}{\sin\pi(\lambda+2\alpha-n)}$$
$$\times\{\cos\pi(n-\alpha)(z+1)^{n-\alpha+1/2}+\exp[\pm i\pi(n-\alpha+\tfrac{1}{2})]$$
$$\times\cos\pi(\lambda+\alpha)(z-1)^{n-\alpha+1/2}\},$$
$$\mathrm{Re}\,n\to\infty,\ \mathrm{Im}\,z\leqslant 0,\ |\arg(z\pm 1)|<\pi. \qquad (6.8)$$

We will also need asymptotic estimates for the functions $D^{\alpha}_{-\lambda+l}(z)$ and $C^{\lambda+\alpha-l}_{l}(z)$ for $|\lambda|\to\infty$ with $\mathrm{Re}\lambda\geqslant 0$ and the ratio $|l/\lambda|\ll 1$ and fixed. We obtain the estimate of $D^{\alpha}_{-\lambda+l}(z)$ by using (3.2) and whichever of (6.1) or (6.2) is appropriate. The results in the two cases have the same form,

$$D^{\alpha}_{-\lambda+l}(z)\sim-\exp[i\pi\alpha](\lambda-l)^{\alpha-1}2^{-\alpha}(z^2-1)^{-\alpha/2}[\Gamma(\alpha)\sin\pi$$
$$\times(\lambda-l+2\alpha)]^{-1}$$
$$\times\{\sin\pi(\lambda-l+\alpha)[z+(z^2-1)^{1/2}]^{\lambda-l-\alpha}$$
$$-\sin\pi\alpha\exp[\pm i\pi(\lambda-l+\alpha)]$$
$$\times[z+(z^2-1)^{1/2}]^{-\lambda+l+\alpha}\}, \qquad (6.9)$$

where the $\pm$ in the phase factor in the second term corresponds either to $|\lambda|\to\infty$ with $\mathrm{Im}\lambda\to\mp\infty$, $|\arg(z-1)|<\pi$, or to $\mathrm{Re}\lambda\to\infty$, $\mathrm{Im}\lambda$ fixed, $\mathrm{Im}z\leqslant 0$, depending on whether (6.1) or (6.2) is used. (The results are, of course, equivalent when nonleading terms are dropped.) For $\mathrm{Re}\lambda\to\infty$ and $|z+(z^2-1)^{1/2}|>1$, the second term in (6.9) should be dropped.

Finally, the asymptotic estimate for $C^{\lambda-l}_{l}(z)$ is derived in Appendix A for the conditions we will need,

$$C^{\lambda-l}_{l}(z)\sim\frac{\Gamma(\lambda)}{\Gamma(l+1)\Gamma(\lambda-l)}\,(2z)^{l}\left[\frac{2zt_0}{x}\right]^{-l}\left[\frac{1-zt_0}{1-\frac{1}{2}x}\right]^{-\lambda(1-x)},$$
$$t_0=x/\{z+[z^2-x(2-x)]^{1/2}\}\approx x/2z, \qquad (6.10)$$
$$|\lambda|\to\infty,\ \mathrm{Re}\lambda>0,\ x=l/\lambda,\ |x|<1\text{ and fixed},\ |z|\gg 1.$$

This result apparently cannot be derived directly from the hypergeometric representations of $C^{\lambda-l}_{l}(z)$.

7. EXPANSION OF THE CAUCHY DENOMINATOR

The Gegenbauer polynomials $C^{\alpha}_{n}(t)$ for $n=0,1,2,\cdots$ form a complete set of eigenfunctions for the differential equation (1.1).[8] These polynomials are orthogonal on the interval $-1\leqslant t\leqslant 1$ with the weight $(1-t^2)^{\alpha-1/2}$, $\mathrm{Re}\alpha>-\frac{1}{2}$ [HTF 3.15.1 (16)],

$$\int_{-1}^{1}dt(1-t^2)^{\alpha-1/2}C^{\alpha}_{m}(t)C^{\alpha}_{n}(t)$$
$$=2^{-2\alpha+1}\pi\,\frac{\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(n+1)[\Gamma(\alpha)]^2}\,\delta_{nm}. \qquad (7.1)$$

We may use the completeness and the orthogonality relations in conjunction with (5.8) to derive a standard generalization[9] of Heine's expansion of the Cauchy denominator $(z-t)^{-1}$ in terms of Legendre functions [HTF 3.10 (10)]

$$(z-t)^{-1}=\exp(-i\pi\alpha)2^{2\alpha}[\Gamma(\alpha)]^2(z^2-1)^{\alpha-1/2}$$
$$\times\sum_{n=0}(n+\alpha)\,\frac{\Gamma(n+1)}{\Gamma(n+2\alpha)}\,C^{\alpha}_{n}(t)D^{\alpha}_{n}(z). \qquad (7.2)$$

This expansion converges absolutely for

$$|[t+(t^2-1)^{1/2}]/[z+(z^2-1)^{1/2}]|<1, \qquad (7.3)$$

a result which can be established by using the asymptotic limits (6.2) and (6.3) in (7.2). The restriction (7.3) requires that the point t lie within that ellipse in the complex t-plane with foci at $t=\pm 1$ which passes through the point $t=z$. The convergence is uniform for t on and within any ellipse with foci at ± 1 inside that determined by (6.9). This is, of course, the same region of convergence as one obtains for Heine's expansion [the special case of (7.2) for $\alpha=\frac{1}{2}$, see (2.4) and (2.8)], and the more general expansion of $(z-t)^{-1}$ in a series of Jacobi functions.[9]

The series expansion of $(z-t)^{-1}$ given in (7.2) does not converge in the region needed in our earlier work.[2] We will therefore obtain an alternative expansion with a larger domain of convergence by using the Sommerfeld–Watson transformation[10] on (7.2). We first replace t by $-t$ in (7.2), and use (5.1) to write $C^{\alpha}_{n}(-t)$ as $(-1)^nC^{\alpha}_{n}(t)$. The sum in (7.2) can then be replaced by a contour integral,[10]

$$\frac{1}{z+t}=\exp(-i\pi\alpha)2^{2\alpha}[\Gamma(\alpha)]^2(z^2-1)^{\alpha-1/2}$$
$$\times\sum_{n=0}(-1)^n(n+\alpha)\,\frac{\Gamma(n+1)}{\Gamma(n+2\alpha)}\,C^{\alpha}_{n}(t)D^{\alpha}_{n}(z) \qquad (7.4)$$
$$=i\exp(-i\pi\alpha)2^{2\alpha-1}[\Gamma(\alpha)]^2(z^2-1)^{\alpha-1/2}$$
$$\times\int_{C}\frac{d\nu}{\sin\pi\nu}\,(\nu+\alpha)\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+2\alpha)}\,C^{\alpha}_{\nu}(t)D^{\alpha}_{\nu}(z), \qquad (7.5)$$

where the contour of integration runs around the positive real axis in the negative sense (Fig. 2). It is easily established from (6.1)–(6.3) that the integrand vanishes sufficiently rapidly for $|\nu|\to\infty$, $\mathrm{Re}\nu\geqslant 0$, that the contour of integration can be opened up to run parallel to the imaginary axis provided

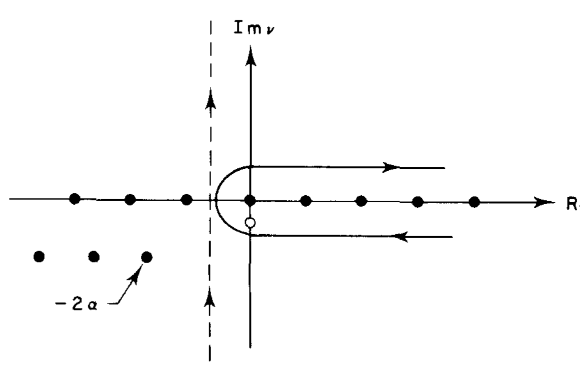

FIG. 2. The locations of the poles of the integrand in (7.5) for $\mathrm{Re}\alpha>0$, and the contours of integration used in (7.5) (solid line) and (7.7) (dashed line). The poles are located at $\nu=0$, ± 1, ± 2, $\cdots$, and at $\nu=-2\alpha-n$, $n=0, 1, 2, \cdots$.

$$|\arg[t+(t^2-1)^{1/2}]|+|\arg[z+(z^2-1)^{1/2}]|<\pi. \quad (7.6)$$

Thus for all t and z which satisfy (7.6)

$$\frac{1}{z+t}=i\exp(-i\pi\alpha)2^{2\alpha-1}[\Gamma(\alpha)]^2(z^2-1)^{\alpha-1/2}$$

$$\times\int_{-i\infty-\epsilon}^{i\infty-\epsilon}\frac{d\nu}{\sin\pi\nu}(\nu+\alpha)\frac{\Gamma(\nu+1)}{\Gamma(\nu+2\alpha)}C_\nu^\alpha(t)D_\nu^\alpha(z),$$

$$0<\epsilon<1,\ \mathrm{Re}\alpha>0. \quad (7.7)$$

In writing the result in this form, we have used the fact that $C_\nu^\alpha(t)$ and $D_\nu^\alpha(z)$ have no poles in the right-half ν-plane for $\mathrm{Re}\alpha>0$. If $\mathrm{Re}\alpha<0$, some of the poles of the integrand at $\nu=-2\alpha-n$, $n=0,1,\cdots$, will lie in the right-half plane, and the sum of the residues of these poles must be added to (7.7). Note that the region of validity of (7.7) given in (7.6) is much larger than the elliptical region (7.3).

8. GENERALIZATIONS OF STANDARD ADDITION FORMULAS

The addition formulas for the Gegenbauer functions relate the functions of argument

$$\xi=x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2} \quad (8.1)$$

or

$$\cosh\beta=\cosh\beta_1\cosh\beta_2-\cos\phi\sinh\beta_1\sinh\beta_2$$

to sums of products of Gegenbauer functions with arguments x_1, x_2, and z, or $\cosh\beta_1$, $\cosh\beta_2$, and $\cos\phi$. In this section and the following two sections, we will present some generalizations of the standard addition formula for the Gegenbauer polynomials [HTF 3.15.1 (19)] which hold for Gegenbauer functions of the first and second kind of arbitrary noninteger degree λ. The ranges of the arguments x_1, x_2, and z are also nonstandard. Our method of derivation in each case is suggestive, but not rigorous as presented. The proofs that the results presented actually represent the functions in question are based in our approach on the use of Carlson's theorem.[11] However, the proofs are rather lengthy, and we will illustrate the method in only one case, the proof of Eq. (9.3) given in Appendix B. Alternative proofs of some of the results for the special case of the ordinary Legendre functions $P_\lambda(\xi)$ and $Q_\lambda(\xi)$ are available in standard references.[12] Henrici[13] has given completely different proofs of the more general results in Eqs. (8.3), (8.6), and (9.5), a fact of which we were unaware at the time we derived these results for our own use. The "inverse" addition formulas in Eqs. (10.6) and (10.8) or (10.11) and (10.12) appear to be completely new.

The classical addition formula for the Gegenbauer polynomials of argument

$$\cos\Theta=\cos\theta_1\cos\theta_2-\cos\phi\sin\theta_1\sin\theta_2$$

is given by HTF 3.15.1 (19),

$$C_l^\alpha(\cos\Theta)=\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{n=0}^{l}(-1)^n4^n\frac{\Gamma(l-n+1)[\Gamma(\alpha+n)]^2}{\Gamma(l+2\alpha+n)}$$

$$\times(2n+2\alpha-1)(\sin\theta_1\sin\theta_2)^nC_{l-n}^{\alpha+n}(\cos\theta_1)$$

$$\times C_{l-n}^{\alpha+n}(\cos\theta_2)C_n^{\alpha-1/2}(\cos\phi),$$

$$l=0,1,\cdots. \quad (8.2)$$

This result may be derived by group-theoretical methods for $\alpha=m/2$, m an integer, by using the relation of the Gegenbauer polynomials to the (unitary) representation coefficients for the rotation group SO($m+2$),[14] and can be extended to general complex values of α by analytic continuation. It can also be proved by purely analytic methods.[15] The extension of (8.2) to complex angles, hence, to arbitrary values of x_1, x_2, and z is immediate provided n is an integer.

The standard addition theorem for the Gegenbauer polynomials has an obvious extension to the functions of noninteger degree λ, obtained formally by extending the summation in (8.2) to infinity,

$$C_\lambda^\alpha(x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2})$$

$$=\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{n=0}^{\infty}(-1)^n\frac{4^n\Gamma(\lambda-n+1)[\Gamma(\alpha+n)]^2}{\Gamma(\lambda+2\alpha+n)}$$

$$\times(2n+2\alpha-1)(x_1^2-1)^{n/2}(x_2^2-1)^{n/2}$$

$$\times C_{\lambda-n}^{\alpha+n}(x_1)C_{\lambda-n}^{\alpha+n}(x_2)C_n^{\alpha-1/2}(z). \quad (8.3)$$

For $\lambda=l=$integer, this series terminates with the term $n=l$ [see (2.5) and (2.10)] and hence, reduces to the proper result for the Gegenbauer polynomials, $l=0,1,\cdots$. The validity of (8.3) for general λ can be established using this observation and Carlson's theorem.[11] We note also that (8.3) reduces for $\alpha=\frac{1}{2}$ to a known result for the Legendre functions $P_\lambda(\xi)$ of arbitrary degree.[16] The general result has been given (for associated Legendre functions) by Henrici[17] and Vilenkin.[18]

The region of convergence of (8.3) may be established through the use of (6.2) and (6.4). It is determined by the condition

$$|z+(z^2-1)^{1/2}|<|(x_1+1)(x_2+1)/(x_1-1)(x_2-1)|, \quad (8.4)$$

where $(z\pm1)$, $(x_1\pm1)$, and $(x_2\pm1)$ all have their principal phases. This condition requires that z be inside the ellipse in the complex z plane with foci at ±1 which passes through the point

$$z=(x_1x_2+1)/[(x_1^2-1)(x_2^2-1)]^{1/2}.$$

The latter gives the location of the branch point of $C_\lambda^\alpha(\xi)$, $\xi=x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}$, considered as a function of z, that is, the location of the singularity of $C_\lambda^\alpha(\xi)$ closest to the interval $[-1,1]$ (see Szegö, Ref. 3, Theorem 9.1.1.). If we write $x_i=\cosh\beta_i$, where β_i may be complex with $|\mathrm{Im}\beta_i|<\pi$ and $z=\cosh\phi$, $|\mathrm{Im}\phi|<\pi$, the region of convergence corresponds to

$$\left|\tanh\frac{\beta_1}{2}\tanh\frac{\beta_2}{2}e^{-\phi}\right|<1. \quad (8.5)$$

We can easily derive an addition formula for the function D_λ^α from (8.3). We take x_1, x_2, and z real, $x_1>x_2$

>1, and combine the functions $C_\lambda^\alpha(\xi)$ and $C_\lambda^\alpha(e^{i\pi}\xi)$ for $\xi=x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}$ using (5.1). The second function is related to the first by the replacement $x_1\to e^{i\pi}x_1$. After a little manipulation, we obtain the desired result,

$$D_\lambda^\alpha(x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2})$$
$$=\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{n=0}(-1)^n\frac{4^n\Gamma(\lambda-n+1)[\Gamma(\alpha+n)]^2}{\Gamma(\lambda+2\alpha+n)}$$
$$\times(2n+2\alpha-1)(x_1^2-1)^{n/2}(x_2^2-1)^{n/2}$$
$$\times D_{\lambda-n}^{\alpha+n}(x_1)C_{\lambda-n}^{\alpha+n}(x_2)C_n^{\alpha-1/2}(z). \quad (8.6)$$

Note that $C_{\lambda-n}^{\alpha+n}(x_2)$ vanishes when $\lambda-n=-1,-2,\cdots$ by (2.5), so that the apparent singularities due to $\Gamma(\lambda-n+1)$ in (8.6) are not present. The addition theorem may be extended to that region of complex x_1, x_2, and z which is continuously connected with the real region and in which the series converges. This region is easily established by using (6.1), (6.4), and (6.6), and is determined for $\mathrm{Re}x_1>0$ by the condition

$$|z+(z_2^2-1)^{1/2}|<|(x_1\mp1)(x_2+1)/(x_1\pm1)(x_2-1)|^{1/2}. \quad (8.7)$$

This condition restricts z to the interior of the smaller of the two ellipses with foci at ±1 which pass through the branch points of $D_\lambda^\alpha(\xi)$ at

$$z=(x_1x_2\pm1)/[(x_1^2-1)(x_2^2-1)]^{1/2}.$$

If we write $x_i=\cosh\beta_i$, $z=\cosh\phi$, $|\mathrm{Im}\beta_i|<\pi$, $|\mathrm{Im}\phi|<\pi$, the series converges for

$$\left|\coth\frac{\beta_1}{2}\tanh\frac{\beta_2}{2}e^{-\phi}\right|<1,\quad |\mathrm{Im}\beta_1|<\frac{\pi}{2},\quad |\mathrm{Im}\beta_2|<\pi. \quad (8.8)$$

9. ADDITION FORMULAS FOR $|z|$ LARGE

The second set of addition formulas which we will discuss are those which were necessary in our earlier work on Lorentz expansions of scattering amplitudes to express Lorentz amplitudes in terms of partial wave amplitudes.[2] As these addition formulas do not appear in the standard references 3–6, we will consider their derivation in some detail. An alternative derivation of (9.5) has been given by Henrici,[13] with the results expressed in terms of associated Legendre functions.

We begin with the classical addition formula for the Gegenbauer polynomials, (8.2). Since l is an integer, both sides of (8.2) are polynomials, and the addition formula is valid for arbitrary complex values of x_1, x_2, and z, in particular, for values of z for which the condition (8.4) is violated (e.g., $z\to\infty$ with x_1, x_2 fixed). We will use the symmetry relation (5.1) to change the sign of the argument of $C_l^\alpha(\xi)$, $C_l^\alpha(\xi)=(-1)^lC_l^\alpha(-\xi)$, and rewrite (8.2) (with l replaced by λ) as

$$C_\lambda^\alpha(z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2)$$
$$=\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{n=0}^{\lambda}(-1)^{\lambda-n}\frac{4^n\Gamma(\lambda-n+1)[\Gamma(\alpha+n)]^2}{\Gamma(\lambda+2\alpha+n)}$$
$$\times(2n+2\alpha-1)(x_1^2-1)^{n/2}(x_2^2-1)^{n/2}$$
$$\times C_{\lambda-n}^{\alpha+n}(x_1)C_{\lambda-n}^{\alpha+n}(x_2)C_n^{\alpha-1/2}(z), \quad (9.1)$$
$$\lambda=0,1,\cdots.$$

We next use (3.2) to rewrite $C_n^{\alpha-1/2}(z)$ as

$$C_n^{\alpha-1/2}(z)=\exp[-i\pi(\alpha-\tfrac12)]\frac{\sin\pi n}{\sin\pi(n+\alpha-\frac12)}$$
$$\times[D_n^{\alpha-1/2}(z)-D_{-n-2\alpha+1}^{\alpha-1/2}(z)]. \quad (9.2)$$

Note that for n an integer, $\sin\pi nD_n^{\alpha-1/2}(z)$ vanishes, but the product $\sin\pi nD_{-n-2\alpha+1}^{\alpha-1/2}(z)$ does not. We therefore drop the terms which involve $D_n^{\alpha-1/2}(z)$. In the remaining series, we make the substitution $n=\lambda-l$, and formally allow the l sum to run from 0 to ∞. For λ an integer, only the terms $0\leq l\leq\lambda$ contribute. Note that we have chosen the direction of the summation, $l\to\infty$ rather than $n\to\infty$, to assure the convergence of the series for noninteger λ. Finally, since we want to extend the result to noninteger λ, where the symmetry property (3.1) holds, we add to the remaining series an equivalent series with λ replaced by $-(\lambda+2\alpha)$. [The added series vanishes for λ an integer, since $\sin\pi\lambda D_{\lambda+l+1}^{\alpha-1/2}(z)=0$ in this case.] After some rearrangement of the gamma functions, the resulting addition formula may be written in the form

$$C_\lambda^\alpha(z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2)$$
$$=2\pi i\exp(-i\pi\alpha)\frac{\sin\pi\lambda}{\sin\pi(\lambda+\alpha)}\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}$$
$$\times\Bigg\{\sum_{l=0}^{\infty}(2\lambda+2\alpha+2l+1)\frac{\Gamma(l+1)\Gamma(2\lambda+2\alpha+l+1)}{[\Gamma(\lambda+\alpha+l+1)]^2}$$
$$\times4^{-\lambda-2\alpha-l}[(x_1^2-1)(x_2^2-1)]^{-(\lambda+2\alpha+l)/2}$$
$$\times C_l^{-\lambda-\alpha-l}(x_1)C_l^{-\lambda-\alpha-l}(x_2)D_{\lambda+l+1}^{\alpha-1/2}(z)$$
$$\times\sum_{l=0}^{\infty}(2\lambda+2\alpha-2l-1)\frac{\Gamma(l+1)\Gamma(-2\lambda-2\alpha+l+1)}{[\Gamma(-\lambda-\alpha+l+1)]^2}$$
$$\times4^{\lambda-l}[(x_1^2-1)(x_2^2-1)]^{(\lambda-l)/2}$$
$$\times C_l^{\lambda+\alpha-l}(x_1)C_l^{\lambda+\alpha-l}(x_2)D_{-\lambda-2\alpha+l+1}^{\alpha-1/2}(z)\Bigg\}. \quad (9.3)$$

The right-hand side of this equation (which we will denote by S_λ^α) is equal to C_λ^α by construction for $\lambda=0,1,2,\cdots$. It is easily checked that it has the symmetries (3.1) and (5.1) of C_λ^α for $\lambda\to-\lambda-2\alpha$ and for $x_i\to\exp(\pm i\pi)x_i$ for arbitrary noninteger λ, and has the asymptotic behavior (6.1) of $C_\lambda^\alpha(\xi)$ for $z\to\infty$. It is therefore plausible that S_λ^α is in fact equal to C_λ^α for arbitrary complex λ, that is, that (9.3) is a correct addition theorem. We will prove in Appendix B that this is the case. Our method consists of showing that the function

$$G_\lambda^\alpha=\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2\alpha)}[\xi+(\xi^2-1)^{1/2}]^{-\lambda}[C_\lambda^\alpha-S_\lambda^\alpha]$$

is suitable for the application of Carlson's theorem, namely, that it is regular in the right-half λ plane and bounded by $\exp[|\lambda|(\pi-\epsilon)]$, $\epsilon>0$, for $|\lambda|\to\infty$, $\mathrm{Re}\lambda\geq0$. Since G_λ^α vanishes by construction for $\lambda=0,1,2,\cdots$, it vanishes identically by Carlson's theorem. We conclude that $C_\lambda^\alpha = S_\lambda^\alpha$ for all λ, α, x_1, x_2, and z for which the series converges.

The region of convergence of (9.3) is easily established by using the asymptotic estimates given in (6.2) and (6.8). The series converges absolutely for fixed x_1 and x_2 and z such that

$$|z+(z^2-1)^{1/2}| > |(x_1\pm 1)(x_2\pm 1)/(x_1\mp 1)(x_2\mp 1)|^{1/2},$$

$$|\arg(x_1\pm 1) < \pi, \quad |\arg(x_2\pm 1)| < \pi, \quad |\arg(\xi\pm 1)| < \pi,$$

$$\xi = z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2} - x_1x_2, \tag{9.4}$$

where all combinations of the signs in $(x_1\pm 1)$ and $(x_2\pm 1)$ are to be considered. This requires that z lie outside the larger of the ellipses with foci at ± 1 which pass through the branch points of $D_\lambda^\alpha(\xi)$ in the z plane. The convergence is uniform for z strictly outside the ellipse. For z real, $z>1$, and the product $[(x_1+1)(x_2+1)/(x_1-1)(x_2-1)]$ real and positive, the convergence condition (9.4) becomes simply $\xi > 1$, the form used in Ref. 2.

The addition formula for $D_\lambda^\alpha(\xi)$ which corresponds to (9.3) can be obtained by using the connection between $C_\lambda^\alpha(\xi)$ and $C_\lambda^\alpha(e^{\pm i\pi}\xi)$ given in (5.1). If we take x_1, x_2, and z real with $\xi > 1$, and combine the series for $C_\lambda^\alpha(\xi)$ and $C_\lambda^\alpha(e^{i\pi}\xi)$, with the second function being obtained from the first by the replacement $x_1 \to e^{i\pi}x_1$, the series which involve $D^{\alpha-1/2}_{-\lambda-2\alpha+l+1}(z)$ in (9.3) drop out. The series which involve $D^{\alpha-1/2}_{\lambda+l+1}(z)$ can be combined, and give the result that

$$D_\lambda^\alpha(z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2} - x_1x_2)$$

$$= 2\pi i\,\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\sum_{l=0}^{\infty}(2\lambda+2\alpha+2l+1)$$

$$\times\frac{\Gamma(l+1)\Gamma(2\lambda+2\alpha+l+1)}{[\Gamma(\lambda+\alpha+l+1)]^2}$$

$$\times 4^{-\lambda-2\alpha-l}[(x_1^2-1)(x_2^2-1)]^{-(\lambda+2\alpha+l)/2}$$

$$\times C_l^{-\lambda-\alpha-l}(x_1)C_l^{-\lambda-\alpha-l}(x_1)C_l^{-\lambda-\alpha-l}(x_2)D^{\alpha-1/2}_{\lambda+l+1}(z). \tag{9.5}$$

The addition formula (9.5) converges for complex x_1, x_2, and z such that conditions (9.4) are satisfied and is valid for arbitrary λ and α. In the limit $\alpha \to \frac{1}{2}+$, the function $D_\lambda^{1/2}(\xi) = (i/\pi)Q_\lambda(\xi)$ appears on the left. The product $\Gamma(2\alpha-1)D^{\alpha-1/2}_{\lambda+\alpha+1}(z)$ on the right approaches $\frac{1}{2}(\lambda+l+1)^{-1}[z+(z^2-1)^{1/2}]^{-\lambda-l-1}$, and (9.5) can be reduced to the classical addition formula for $Q_\lambda(\xi)$ given by Heine.[12] An addition formula equivalent to (9.5) has been derived by Henrici[13] using entirely different methods based on the differential equation for the biaxially symmetric potential. Henrici's result extends the classical addition formula for $Q_\lambda(\xi)$ to the case of general $Q_\nu^\mu(\xi)$, hence, by (2.4), to general $D_\lambda^\alpha(\xi)$ [see Ref. 13, Eq. (89)].

It remains for us at this point to demonstrate the validity of (9.5) for general complex λ. Since the identities which lead from (9.3) to (9.5) are valid for arbitrary λ and α, this demonstration is equivalent to the proof of (9.3). This is given in Appendix B.

10. INVERSE ADDITION THEOREMS

We turn finally to an entirely different class of addition theorems. A special case was used in Ref. 2, namely, the expansion of the Legendre function

$$Q_j\left(\frac{\cosh\beta+\cosh\beta_1\cosh\beta_2}{\sinh\beta_1\sinh\beta_2}\right)$$

in terms of the functions $D^1_{\nu-1}(\cosh\beta)$. We will derive a corresponding addition formula here for an arbitrary D_λ^α. Let

$$z = \frac{\cosh\beta+\cosh\beta_1\cosh\beta_2}{\sinh\beta_1\sinh\beta_2} \tag{10.1}$$

and

$$\zeta = t\sinh\beta_1\sinh\beta_2 - \cosh\beta_1\cosh\beta_2, \tag{10.2}$$

and rewrite the integral representation (5.8) for $D_n^\alpha(z)$, n an integer, in the form

$$(z^2-1)^{\alpha-1/2}D_n^\alpha(z) = \frac{1}{2\pi}\exp(i\pi\alpha)\sinh\beta_1\sinh\beta_2\int_{-1}^{1}dt$$

$$\times\frac{(1-t^2)^{\alpha-1/2}}{\cosh\beta-\zeta}C_n^\alpha(t). \tag{10.3}$$

We suppose for simplicity that z is real and $z>1$. Then $(\cosh\beta-\zeta)>0$. We may then use (7.2) with $\alpha\to\alpha+\frac{1}{2}$ to expand the denominator,

$$(\cosh\beta-\zeta)^{-1} = \exp[-i\pi(\alpha+\tfrac{1}{2})]2^{2\alpha+1}[\Gamma(\alpha+\tfrac{1}{2})]^2(\sinh\beta)^{2\alpha}$$

$$\times\sum_{m=0}^{\infty}(m+\alpha+\tfrac{1}{2})\frac{\Gamma(m+1)}{\Gamma(m+2\alpha+1)}$$

$$\times C_m^{\alpha+1/2}(\zeta)D_m^{\alpha+1/2}(\cosh\beta), \tag{10.4}$$

and the addition theorem (8.2) to expand $C_m^{\alpha+1/2}(\zeta)$,

$$C_m^{\alpha+1/2}(t\sinh\beta_1\sinh\beta_2 - \cosh\beta_1\cosh\beta_2)$$

$$= (-1)^m\frac{\Gamma(2\alpha)}{[\Gamma(\alpha+\frac{1}{2})]^2}\sum_{l=0}^{m}(-1)^l\frac{\Gamma(m-l+1)[\Gamma(\alpha+l+\frac{1}{2})]^2}{\Gamma(m+2\alpha+l+1)}$$

$$\times(2l+2\alpha)(4\sin\beta_1\sinh\beta_2)^l C^{\alpha+l+1/2}_{m-l}(\cosh\beta_1)$$

$$\times C^{\alpha+l+1/2}_{l}(\cosh\beta_2)C_l^\alpha(t). \tag{10.5}$$

When we insert (10.4) and (10.5) into (10.3), the integral on t can be evaluated by use of the orthogonality relations for the Gegenbauer polynomials, (7.1). The interchanges of the orders of summation and integration cause no problem. We are left with a sum in which the summation index runs from $m=n$ to $m=\infty$. Upon changing the summation index to $l=m-n$ and formally changing n to λ, we obtain the desired addition theorem,

$$(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z) = -\tfrac{1}{2}i(4\sinh\beta_1\sinh\beta_2)^{\lambda+1}(\sinh\beta)^{2\alpha}$$

$$\times\frac{\Gamma(\lambda+2\alpha)\Gamma(2\alpha)[\Gamma(\lambda+\alpha+\frac{1}{2})]^2}{[\Gamma(\alpha)]^2\Gamma(\lambda+1)}\sum_{l=0}^{\infty}(-1)^l(2\lambda+2\alpha+2l+1)$$

$$\times\frac{\Gamma(l+1)\Gamma(\lambda+l+1)}{\Gamma(\lambda+2\alpha+l+1)\Gamma(2\lambda+2\alpha+l+1)}$$

$$\times C_l^{\lambda+\alpha+1/2}(\cosh\beta_1)C_l^{\lambda+\alpha+1/2}(\cosh\beta_2)$$

$$\times D_{\lambda+l}^{\alpha+1/2}(\cosh\beta), \quad \cosh\beta = z\sinh\beta_1\sinh\beta_2 - \cosh\beta_1\cosh\beta_2. \tag{10.6}$$

The derivation of (10.6) given above holds only for $\lambda = 0, 1, 2, \cdots$. However, the validity of the expansion for arbitrary λ can be established by an argument using Carlson's theorem[11] similar to that given in connection with the addition formulas (9.3) and (9.5) (see Appendix B). The details add nothing new, and will not be given here.

It only remains to establish the region of convergence for (10.6). From the known asymptotic behavior of the C and D functions, (6.1)–(6.3), one immediately finds that the series converges for

$$\mathrm{Re}\beta > |\mathrm{Re}\beta_1| + |\mathrm{Re}\beta_2|. \tag{10.7}$$

As this range of β is too restrictive for our previous applications,[2] we again use the Sommerfeld–Watson transformation[10] to obtain a result with a broader domain of validity. We begin in the region (10.7) in which the series converges, and suppose that $\mathrm{Re}\lambda > -\frac{1}{2}$, $\mathrm{Re}\alpha > 0$ so that the summand in (10.6), considered as a function of l, has no poles in the right-half l-plane. Application of the Sommerfeld–Watson transformation to (10.6) then gives

$$(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z) = \tfrac{1}{4}(4\sinh\beta_1\sinh\beta_2)^{\lambda+1}(\sinh\beta)^{2\alpha}$$

$$\times \frac{\Gamma(\lambda+2\alpha)\Gamma(2\alpha)[\Gamma(\lambda+\alpha+\frac{1}{2})]^2}{[\Gamma(\alpha)]^2\Gamma(\lambda+1)}$$

$$\times \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{dl}{\sin\pi l}$$

$$\times \frac{(2\lambda+2\alpha+2l+1)\Gamma(l+1)\Gamma(\lambda+l+1)}{\Gamma(\lambda+2\alpha+l+1)\Gamma(2\lambda+2\alpha+l+1)}$$

$$\times C_l^{\lambda+\alpha+1/2}(\cosh\beta_1)C_l^{\lambda+\alpha+1/2}(\cosh\beta_2)$$

$$\times D_{\lambda+l}^{\alpha+1/2}(\cosh\beta), \quad 0<\epsilon<1. \tag{10.8}$$

The restrictions on the values of λ and α can be eliminated by deforming the integration contour to avoid the singularities of the integrand which move into the right-half λ-plane for general values of λ, α. The integral is well-defined provided

$$|\mathrm{Im}\beta_1| + |\mathrm{Im}\beta_2| + |\mathrm{Im}\beta| < \pi, \tag{10.9}$$

a result which follows from (6.1) and (6.3). This condition places no restrictions on the relative magnitudes of the β's if they are all real. This is the desired result. For the special case needed in Ref. 2, $\alpha = \frac{1}{2}$, (10.8) becomes an addition theorem for $Q_\lambda(z)$,

$$Q_\lambda(z) = -\tfrac{1}{2}i(4\sinh\beta_1\sinh\beta_2)^{\lambda+1}\sinh\beta[\Gamma(\lambda+1)]^2$$

$$\times \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{dl}{\sin\pi l}\frac{\Gamma(l+1)}{\Gamma(2\lambda+l+2)}$$

$$\times C_l^{\lambda+1}(\cosh\beta_1)C_l^{\lambda+1}(\cosh\beta_2)$$

$$\times D_{\lambda+l}^1(\cosh\beta), \tag{10.10}$$

where z is given by (10.1).

If we use the definitions $x_1 = \cosh\beta_1$, $x_2 = \cosh\beta_2$, $\xi = \cosh\beta$, (10.6) and (10.8) can be rewritten as

$$(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z) = -i2^{2\lambda+1}[(x_1^2-1)(x_2^2-1)]^{(\lambda+1)/2}(\xi^2-1)^\alpha$$

$$\times \frac{\Gamma(\lambda+2\alpha)\Gamma(2\alpha)[\Gamma(\lambda+\alpha+\frac{1}{2})]^2}{[\Gamma(\alpha)]^2\Gamma(\lambda+1)}$$

$$\times \sum_{l=0}^{\infty}(-1)^l(2\lambda+2\alpha+2l+1)$$

$$\times \frac{\Gamma(l+1)\Gamma(\lambda+l+1)}{\Gamma(\lambda+2\alpha+l+1)\Gamma(2\lambda+2\alpha+l+1)}$$

$$\times C_l^{\lambda+\alpha+1/2}(x_1)C_l^{\lambda+\alpha+1/2}(x_2)D_{\lambda+l}^{\alpha+1/2}(\xi),$$

$$\xi = z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2} - x_1x_2, \tag{10.11}$$

$$|\xi + (\xi^2-1)^{1/2}| > |x_1 + (x_1^2-1)^{1/2}|\,|x_2 + (x_2^2-1)^{1/2}|,$$

where all quantities have their principal phases, $|\arg(\xi\pm1)|\le\pi$, $|\arg(x_1\pm1)|\le\pi$, $|\arg(x_2\pm1)|\le\pi$. Similarly,

$$(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z) = 2^{2\lambda}[(x_1^2-1)(x_2^2-1)]^{(\lambda+1)/2}(\xi^2-1)^\alpha$$

$$\times \frac{\Gamma(\lambda+2\alpha)\Gamma(\alpha)[\Gamma(\lambda+\alpha+\frac{1}{2})]^2}{[\Gamma(\alpha)]^2\Gamma(\lambda+1)}$$

$$\times \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{dl}{\sin\pi l}(2\lambda+2\alpha+2l+1)$$

$$\times \frac{\Gamma(l+1)\Gamma(\lambda+l+1)}{\Gamma(\lambda+2\alpha+l+1)\Gamma(2\lambda+2\alpha+l+1)}$$

$$\times C_l^{\lambda+\alpha+1/2}(x_1)C_l^{\lambda+\alpha+1/2}(x_2)D_{\lambda+l}^{\alpha+1/2}(\xi),$$

$$|\arg[\xi+(\xi^2-1)^{1/2}]| + |\arg[x_1+(x_1^2-1)^{1/2}]|$$

$$+ |\arg[x_2+(x_2^2-1)^{1/2}]| < \pi. \tag{10.12}$$

The corresponding results for $C_\lambda^\alpha(z)$ may be obtained by using (3.2).

We note finally that if the integration contour in (10.12) is pushed to ∞ in the left-half l-plane, the sum of the residues of the poles in that half-plane gives an addition formula for $(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z)$ which converges for

$$|\xi+(\xi^2-1)^{1/2}| < |x_1-(x_2-1)^{1/2}|\,|x_2-(x_2^2-1)^{1/2}|. \tag{10.13}$$

11. EXPANSION FORMULAS AND ADDITION FORMULAS FOR LEGENDRE FUNCTIONS

The expansion formulas and addition theorems for the Gegenbauer functions given in the preceding sections can be converted through the use of (2.4) and (2.8) into equivalent results expressed in terms of associated Legendre functions. We will collect the most important of those results in this section.

The expansion formulas for the Cauchy denominator in terms of Gegenbauer functions given in (7.2) and (7.7) lead to the following expressions in terms of Legendre functions,

$$(z-t)^{-1}=\exp(-i\pi\mu)\times\sum_{n=0}^{\infty}(2n+2\mu+1)(t^2-1)^{-\mu/2}\times P^{-\mu}_{n+\mu}(t)(z^2-1)^{\mu/2}Q^{\mu}_{n+\mu}(z),$$

$$\left|\frac{t+(t^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right|<1, \tag{11.1}$$

$$(z+t)^{-1}=\frac{i}{2}\exp(-i\pi\mu)\int_{-i\infty-\epsilon}^{i\infty-\epsilon}\frac{d\nu}{\sin\pi\nu}\times(2\nu+2\mu+1)(t^2-1)^{-\mu/2}P^{-\mu}_{\nu+\mu}(t)\times(z^2-1)^{\mu/2}Q^{\mu}_{\nu+\mu}(z), \tag{11.2}$$

$$|\arg[t+(t^2-1)^{1/2}]|+|\arg[z+(z^2-1)^{1/2}]|<\pi.$$

The result in (11.2) gives a Regge-type expansion for the Cauchy denominator for arbitrary complex μ (complex helicity).

The addition theorems given in (8.3) and (8.6) for the Gegenbauer functions of argument

$$\xi=x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2} \tag{11.3}$$

are generalizations of the classical addition formulas for the Legendre functions $P_\nu(\xi)$ and $Q_\nu(\xi)$ [19] to arbitrary ν,μ,

$$(\xi^2-1)^{\mu/2}P^{\mu}_{\nu}(\xi)=(2\pi)^{1/2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\sum_{n=0}^{\infty}(-1)^n(n-\mu)\times\frac{\Gamma(\nu-\mu+n+1)\Gamma(n-2\mu)}{\Gamma(\nu+\mu-n+1)\Gamma(n+1)}\times(x_1^2-1)^{\mu/2}(x_2^2-1)^{\mu/2}P^{\mu-n}_{\nu}(x_1)P^{\mu-n}_{\nu}(x_2)\times(z^2-1)^{\mu/2+1/4}P^{\mu+1/2}_{n-\mu-1/2}(z), \tag{11.4}$$

$$|z+(z^2-1)^{1/2}|<\left|\frac{(x_1+1)(x_2+1)}{(x_1-1)(x_2-1)}\right|^{1/2},$$

$$(\xi^2-1)^{\mu/2}Q^{\mu}_{\nu}(\xi)=(2\pi)^{1/2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\sum_{n=0}^{\infty}(-1)^n(n+\mu)\times\frac{\Gamma(\nu-\mu+n+1)\Gamma(n-2\mu)}{\Gamma(\nu+\mu-n+1)\Gamma(n+1)}\times(x_1^2-1)^{\mu/2}(x_2^2-1)^{\mu/2}Q^{\mu-n}_{\nu}(x_1)P^{\mu-n}_{\nu}(x_2)\times(z^2-1)^{\mu/2+1/4}P^{\mu+1/2}_{n-\mu-1/2}(z), \tag{11.5}$$

$$|z+(z^2-1)^{1/2}|<\left|\frac{(x_1\mp1)(x_2+1)}{(x_1\pm1)(x_2-1)}\right|^{1/2}.$$

The addition formula (9.5) for the Gegenbauer function $D^{\alpha}_{\lambda}(\zeta)$ of argument

$$\zeta=z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2 \tag{11.6}$$

leads to the following addition formula for the Legendre function $Q^{\mu}_{\nu}(\zeta)$,

$$(\zeta^2-1)^{\mu/2}Q^{\mu}_{\nu}(\zeta)=-i(2\pi)^{1/2}\frac{\cos\pi\nu}{\sin\pi\nu}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\times\sum_{l=0}^{\infty}(-1)^l(\nu+l+1)\times\frac{\Gamma(\nu-\mu+l+1)\Gamma(-2\nu-l-1)}{\Gamma(\nu+\mu+l+2)\Gamma(l+1)}\times(x_1^2-1)^{\mu/2}(x_2^2-1)^{\mu/2}P^{\nu+l+1}_{\nu}(x_1)P^{\nu+l+1}_{\nu}(x_2)\times(z^2-1)^{\mu/2+1/4}Q^{\mu+1/2}_{\nu+l+1/2}(z), \tag{11.7}$$

$$|z+(z^2-1)^{1/2}|>\left|\frac{(x_1\pm1)(x_2\pm1)}{(x_1\mp1)(x_2\mp1)}\right|^{1/2}.$$

A result for $P^{\mu}_{\nu}(\zeta)$ analogous to (9.3) can be obtained by using (2.8), (3.2), and (11.7).

Finally, the addition formula for $(z^2-1)^{\alpha-1/2}D^{\alpha}_{\lambda}(z)$ given in (10.11) leads to a new addition formula for $Q^{\mu}_{\nu}(z)$,

$$(z^2-1)^{-\mu/2}Q^{\mu}_{\nu}(z)=-i(2\pi)^{1/2}[(x_1^2-1)(x_2^2-1)]^{\mu/2+1/4}\times(\zeta^2-1)^{-\mu/2+1/4}\sum_{l=0}^{\infty}(-1)^l(\nu+l+1)\times\frac{\Gamma(2\nu+l+2)}{\Gamma(l+1)}P^{-\nu-1/2}_{\nu+l+1/2}(x_1)\times P^{-\nu-1/2}_{\nu+l+1/2}(x_2)Q^{\mu-1/2}_{\nu+l+1/2}(\zeta), \tag{11.8}$$

$$|\zeta+(\zeta^2-1)^{1/2}|>|x_1+(x_1^2-1)^{1/2}|\ |x_2+(x_2^2-1)^{1/2}|.$$

The corresponding result for $P^{\mu}_{\nu}(z)$ can be obtained by using (2.8), (3.2), and (11.8).

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APPENDIX A

1. Asymptotic behavior of $D^{\alpha}_{\lambda}(z)$ and $C^{\alpha}_{\lambda}(z)$ for $|\lambda|\to\infty$

In the present section, we will derive the expressions (6.1)–(6.3) which give the asymptotic behavior of $C^{\alpha}_{\lambda}(z)$ and $D^{\alpha}_{\lambda}(z)$ for $|\lambda|\to\infty$ by using the method of steepest descents[20] to estimate the leading contributions to the integrals (1.3) and (1.5). The case of $D^{\alpha}_{\lambda}(z)$ is quite simple. From (1.5),

$$D^{\alpha}_{\lambda}(z)=\exp(2\pi i\alpha)(2\pi i)^{-1}\int_{C_+}dt\,t^{-\lambda-1}(t-z_+)^{-\alpha}(t-z_-)^{-\alpha}=\int_{C_+}\exp[\phi(t)]dt, \tag{A1}$$

where $z_\pm=z\pm(z^2-1)^{1/2}$, C_+ is the contour shown in Fig. 1, and $\phi(t)$ is defined by (A1). The saddle points of the integrand are determined by the condition $\phi'(t)=0$, and are located for $|\lambda|\gg|\alpha|$ at the points

$$t_\pm=z_\pm(1-\alpha/\lambda),\quad |\lambda|\gg|\alpha|. \tag{A2}$$

The motion of the saddle points as $\arg\lambda$ increases from $-\pi$ to π with $|\lambda|$ fixed is shown in Fig. 3. Only the saddle point t_+ near z_+ is relevant for the asymptotic

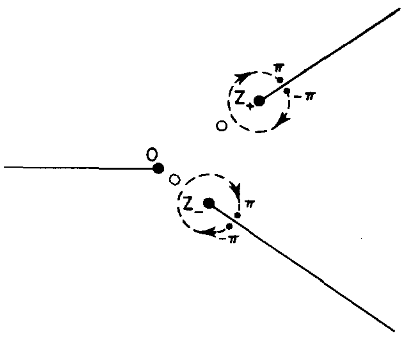

FIG. 3. Motion of the saddle points in the integrand of (A1) or (A6) for $|\lambda| \gg 1$ as the phase of λ changes from $-\pi$ to π. The branch cuts of Fig. 1 have been rotated to run parallel to the lines $\arg t = \arg z_\pm$. We take α real for simplicity.

estimation of $D_\lambda^\alpha(z)$ provided z is fixed a finite distance from 1. The path of steepest descent passes through t_+ in the direction determined by

$$\arg(t - t_+) = \pi/2 - \arg\lambda + \tfrac{1}{2}\arg\alpha + \arg z_+. \tag{A3}$$

The function $\phi(t)$ can be approximated for $t \sim t_+$ as

$$\phi(t) \approx \phi(t_+) + \tfrac{1}{2}\phi''(t_+)(t - t_+) + \cdots. \tag{A4}$$

The contour of integration C_+ may be distorted to pass through t_+ in the direction of steepest descent as shown in Fig. 4. Evaluation of the remaining Gaussian integral then gives our asymptotic estimate (6.3) for $D_\lambda^\alpha(z)$

$$D_\lambda^\alpha(z) \sim \exp(i\pi\alpha)\lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}(z^2-1)^{-\alpha/2}$$
$$\times[z + (z^2-1)^{1/2}]^{-\lambda-\alpha}[1 + O(\lambda^{-1})], \tag{A5}$$

$$|\lambda| \to \infty, \quad |\arg\lambda| < \pi, \quad |\arg(z \pm 1)| < \pi.$$

The asymptotic behavior of $C_\lambda^\alpha(z)$ for $|\lambda| \to \infty$ is somewhat trickier to determine. We begin with the integral representation (1.3),

$$C_\lambda^\alpha(z) = (2\pi i)^{-1}\int_C dt\, t^{-\lambda-1}(t - z_+)^{-\alpha}(t - z_-)^{-\alpha}$$
$$= \int_C \exp[\phi(t)]\,dt, \tag{A6}$$

where the contour C is shown in Fig. 1. The location of the saddle points is again given by (A2), but both must now be considered, and the way in which the contour C may be distorted to pass through them is not uniquely determined. We will consider two cases which lead to (6.1) and (6.2). The choice of contours for the first case is shown in Fig. 5. The contours in this case can be deformed continuously to pass through the saddle points as $\arg z$ is increased from $-\pi + \epsilon$ to $\pi - \epsilon$, with $|\arg(z \pm 1)| < \pi - \epsilon$, z not on $[-1, 1]$. However, the contours are necessarily different for $\mathrm{Im}\lambda > 0$ and $\mathrm{Im}\lambda < 0$, and cannot be deformed into each other by varying $\mathrm{Im}\lambda$ while continuing to pass through the saddle points. Estimation of the integrals by the method of steepest descents is straightforward, and one finds for this choice of contours that the asymptotic form of $C_\lambda^\alpha(z)$ is given by (6.1),

$$C_\lambda^\alpha(z) \sim \lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}(z^2-1)^{-\alpha/2}$$
$$\times\{[z + (z^2-1)^{1/2}]^{\lambda+\alpha}[1 + O(\lambda^{-1})]$$
$$+ \exp(\pm i\pi\alpha)[z + (z^2-1)^{1/2}]^{-\lambda-\alpha}[1 + O(\lambda^{-1})]\}, \tag{A7}$$

$$|\lambda| \to \infty, \quad \mathrm{Re}\lambda \geq 0, \quad \mathrm{Im}\lambda \to \pm\infty, \quad |\arg(z \pm 1)| < \pi.$$

Corresponding results for $\mathrm{Re}\lambda \leq 0$ can be obtained by using (3.1) in conjunction with (A6).

The result in (A7) is apparently discontinuous across the positive real axis in λ even though $C_\lambda^\alpha(z)$ has no such

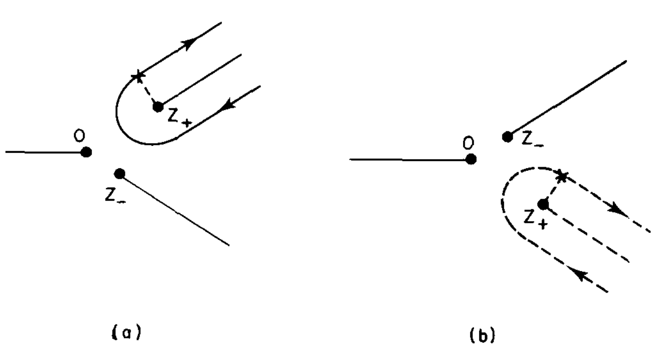

FIG. 4. Contours used in obtaining the asymptotic estimate (A5) for $D_\lambda^\alpha(z)$ for $|\lambda| \to \infty$, $|\arg\lambda| < \pi$. Only the regions near the saddle points (marked by ×) give significant contributions. The locations of the saddle points are shown for $\mathrm{Im}\lambda \gg 1$, $\mathrm{Re}\lambda = 0$, α real. (a) The contour for $\mathrm{Im}z > 0$. (b) The contour for $\mathrm{Im}z < 0$. The contour is on the second sheet of $(t - z_-)^{-\alpha}$, and is obtained by following z_+ and the contour of part (a) through the branch cut as $\mathrm{Im}z$ is decreased through zero.

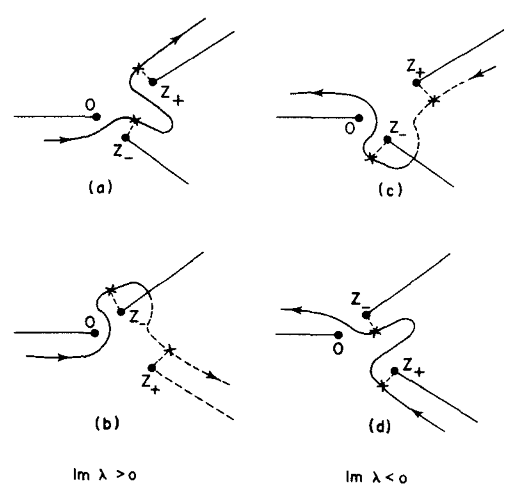

FIG. 5. Contours used in obtaining the asymptotic estimates for $C_\lambda^\alpha(z)$ for $|\lambda| \to \infty$, $\mathrm{Im}\lambda \to \pm\infty$, given in (A7). The locations of the saddle points are shown for $|\mathrm{Im}\lambda| \gg 0$, $\mathrm{Re}\lambda = 0$, α real. (a) The contour for $\mathrm{Im}\lambda > 0$, $\mathrm{Im}z > 0$. (b) The contour for $\mathrm{Im}\lambda > 0$, $\mathrm{Im}z < 0$, obtained by continuously deforming (a) to follow the saddle point as $\mathrm{Im}z$ is decreased through zero. The dashed part of the contour is on the second sheet of $(t - z_-)^{-\alpha}$. (c) The contour for $\mathrm{Im}\lambda < 0$, $\mathrm{Im}z > 0$. Note that this cannot be obtained by continuously deforming (a) to follow the saddle points as $\mathrm{Im}\lambda$ is decreased through zero (see Fig. 3 for the motion of the saddle points with $\arg\lambda$). (d) The contour for $\mathrm{Im}\lambda < 0$, $\mathrm{Im}z < 0$, obtained by deforming (c) as $\mathrm{Im}z$ is decreased through zero.

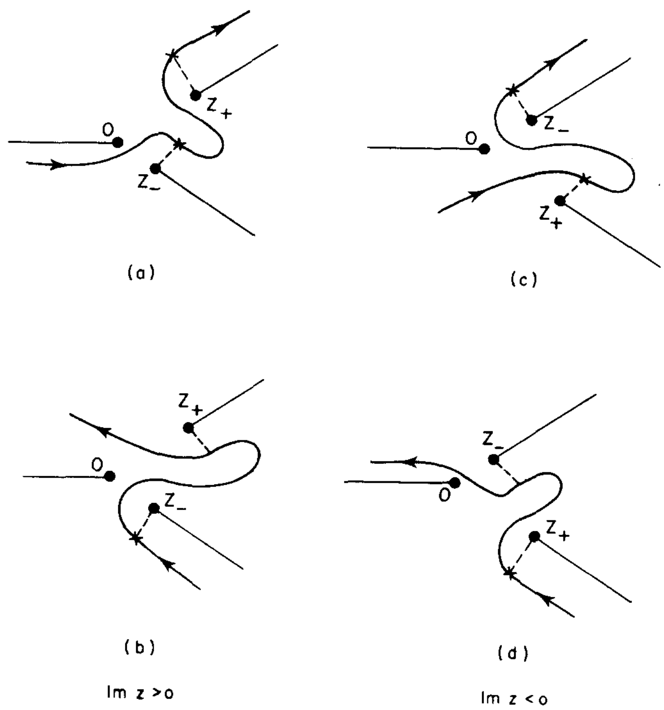

FIG. 6. Contours used in obtaining Watson's asymptotic estimates for $C_\lambda^\alpha(z)$ for $|\lambda|\to\infty$, $\mathrm{Re}\lambda \geq 0$, given in (A8). The locations of the saddle points are shown for $|\mathrm{Im}\lambda| \gg 0$, $\mathrm{Re}\lambda = 0$, α real. (a) The contour for $\mathrm{Im}\lambda > 0$, $\mathrm{Im}z > 0$. (b) The contour for $\mathrm{Im}\lambda < 0$, $\mathrm{Im}z > 0$, obtained by continuously deforming (a) to follow the saddle points as $\mathrm{Im}\lambda$ is decreased through zero (see Fig. 3 for the motion of the saddle points with $\arg\lambda$). (c) The contour for $\mathrm{Im}\lambda > 0$, $\mathrm{Im}z < 0$. Note that this cannot be obtained by deforming (a) to follow the saddle points as $\mathrm{Im}z$ is decreased through zero. (d) The contour for $\mathrm{Im}\lambda < 0$, $\mathrm{Im}z < 0$, obtained by continuous deformation of (c).

discontinuity. Note, however, that $|z + (z^2-1)^{1/2}| > 1$ for z not on the interval $[-1, 1]$. This interval is excluded for (A7). Thus the second term in (A7) is exponentially small compared to the first term for $\mathrm{Re}\lambda \to \infty$, and should be neglected relative to the corrections to the first term. The apparent discontinuity is not significant, and the second term in (A7) is relevant only for $\lambda \to \pm i\infty$.

We can obtain a different asymptotic estimate of $C_\lambda^\alpha(z)$ which does not exhibit the apparent discontinuity in λ by choosing the contours shown in Fig. 6. We must now distinguish between the cases $\mathrm{Im}z > 0$ and $\mathrm{Im}z < 0$, but for a fixed sign of $\mathrm{Im}z$ can deform the contour for $\mathrm{Im}\lambda > 0$ continuously into that for $\mathrm{Im}\lambda < 0$ while continuing to pass through the (moving) saddle points. Evaluation of the saddle point integrals is again straightforward, and one finds that the asymptotic form of $C_\lambda^\alpha(z)$ is given by (6.2),

$$C_\lambda^\alpha(z) \sim \lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}(z^2-1)^{-\alpha/2}$$
$$\times\{[z+(z^2-1)^{1/2}]^{\lambda+\alpha}[1+O(\lambda^{-1})]$$
$$+\exp(\pm i\pi\alpha)[z+(z^2-1)^{1/2}]^{-\lambda-\alpha}[1+O(\lambda^{-1})]\}, \tag{A8}$$

$|\lambda|\to\infty$, $\mathrm{Re}\lambda \geq 0$, $\mathrm{Im}z \gtrless 0$, $|\arg(z\pm 1)| < \pi$.

This result corresponds to that derived by Watson [HTF 2.3.2 (17)]. The expression in (A8) has an apparent discontinuity across the real z axis for $z > 1$ which is not present in $C_\lambda^\alpha(z)$. However, for $\mathrm{Re}\lambda \to \infty$, the discontinuity is exponentially small compared to the corrections to the leading term, and is not significant. It is in fact easily checked that the leading (and only significant) terms in (A7) and (A8) are identical for $|\lambda| \to \infty$ whatever combination of the conditions $\mathrm{Im}\lambda \gtrless 0$ is considered. We note finally that for z real, $z > 1$, and $\mathrm{Re}\lambda \to \infty$ with $\mathrm{Im}\lambda$ fixed, the saddle point at t_- does not contribute, and the second terms in both (6.1) and (6.2) should be dropped.

2. Asymptotic behavior of $C_l^{\lambda-l}(z)$ for $|\lambda|\to\infty$, l/λ fixed

The proof the addition formula (9.3) given in Appendix B requires an asymptotic estimate of $C_l^{\lambda-l}(z)$ for $|\lambda| \to \infty$, $\mathrm{Re}\lambda \geq 0$, with the ratio $|l/\lambda| \ll 1$ fixed. Thus $|l| \to \infty$ with $|\lambda|$. The relevant limit is easily obtained by the method of steepest descents for $|z| \gg 1$. From (1.3),

$$C_l^{\lambda-l}(z) = \exp[2\pi i(\lambda - l)(2\pi i)^{-1}$$
$$\times \int_C dt\, t^{-l-1}(t-z_+)^{-\lambda+l}(t-z_-)^{-\lambda+l}. \tag{A9}$$

The important saddle point is located at

$$t_0 = x/[z + [z^2 - x(2-x)]^{1/2}] \approx x/2z,$$
$$x = l/\lambda,\ \ |l| \gg 1,\ \ |\lambda| \gg 1,\ \ |z| \gg 1,\ \ |x| < 1. \tag{A10}$$

For $x > 0$ and z real, this saddle point lies to the right of $t = 0$, and the contour C should be distorted to run through t_0 parallel to the imaginary axis (see Fig. 7a). For $x < 0$, a second saddle point appears from the second sheet in t, and there are saddle points both above and below the negative real axis. Both must be taken into account. The situation for $\arg z$ near π is shown in Fig. 7b.

The results in the two cases (and intermediate cases) are easily shown to be identical if we properly identify some Γ functions which appear only in their asymptotic form. Thus, for x real, $x > 0$, a saddle point estimate of (A9) using the approximate value of t_0, $t_0 \approx x/2z$, with terms of order z^{-2} dropped in the exponent, gives

$$C_l^{\lambda-l}(z) \sim \frac{1}{\sqrt{2\pi}}\left[\frac{x(1-x)}{\lambda+1}\right]^{1/2}\frac{1}{2z}t_0^{-l-1}(1-2zt_0+t_0^2)^{-\lambda+l}$$

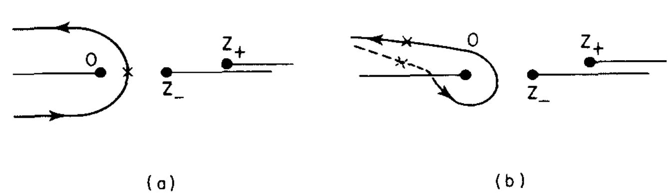

FIG. 7. The location of the saddle points and the contours used in obtaining the asymptotic estimate of $C_l^{\lambda-l}(z)$ given in (A14). We take z real. (a) The situation for $x = l/\lambda$ real and positive, $x < 1$. The second saddle point is on the second sheet in t^{-l}, and does not contribute to the asymptotic expression. (b) The case for $\arg x$ near π. The saddle points are at t_0 on the first sheet, and at $e^{-2\pi i}t_0$ on the second sheet. Both give important contributions to (A14).

$$\approx \frac{1}{\sqrt{2\pi}}(2z)^l \exp\{-[(l+\tfrac{1}{2})\ln l - l]$$

$$-[(\lambda-l-\tfrac{1}{2})\ln(\lambda-l)-\lambda+l]$$

$$+[(\lambda-\tfrac{1}{2})\ln\lambda-\lambda]\}. \qquad \text{(A11)}$$

The various terms in the exponential can be identified with Γ functions by using Stirling's formula,

$$\Gamma(z)\sim\sqrt{2\pi}\exp[(z-\tfrac{1}{2})\ln z - z], \qquad \text{(A12)}$$

and we conclude that

$$C_l^{\lambda-l}(z)\sim\frac{\Gamma(\lambda)}{\Gamma(l+1)\Gamma(\lambda-l)}(2z)^l. \qquad \text{(A13)}$$

A similar calculation in the case of two saddle points ($l\to-\infty$, $\lambda\to\infty$, $|z|\gg 1$, $x<0$, $|x|<1$) gives an identical result. Note that the zeros of $C_l^{\lambda-l}(z)$ for $\lambda-l=0,-1,-2,\cdots$ and $l=-1,-2,\cdots$ are given properly by this expression.

The result in (A13) neglects some factors which are important for $|\lambda x^2/z|\gtrsim 1$. A more careful calculation using the exact form of t_0 gives the correct limit,

$$C_l^{\lambda-l}(z)\sim\frac{\Gamma(\lambda)}{\Gamma(l+1)\Gamma(\lambda-l)}(2z)^l\left(\frac{2zt_0}{x}\right)^{-l}$$

$$\times\left(\frac{1-zt_0}{1-\frac{1}{2}x}\right)^{-\lambda(1-x)}, \qquad \text{(A14)}$$

$$|\lambda|\to\infty,\ \operatorname{Re}\lambda\geq 0,\ x=l/\lambda \text{ fixed},\ |x|<1,$$

where t_0 is given in (A10).

APPENDIX B

We will show in this Appendix that the series in (9.3) actually represents the function $C_\lambda^\alpha(\xi)$. Our proof will be based on Carlson's theorem.[11] For convenience, we will denote the series on the right-hand side of (9.3) by S_λ^α,

$$S_\lambda^\alpha(x_1,x_2,z)=2\pi i\exp(-i\pi\alpha)\frac{\sin\pi\lambda}{\sin\pi(\lambda+\alpha)}\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}$$

$$\times\Bigg\{\sum_{l=0}^{\infty}(2\lambda+2\alpha+2l+1)$$

$$\times\frac{\Gamma(l+1)\Gamma(2\lambda+2\alpha+l+1)}{[\Gamma(\lambda+\alpha+l+1)]^2}$$

$$\times 4^{-\lambda-2\alpha-l}[(x_1^2-1)(x_2^2-1)]^{-(\lambda+2\alpha+l)/2}$$

$$\times C_l^{-\lambda-\alpha-l}(x_1)C_l^{-\lambda-\alpha-l}(x_2)D_{\lambda+l+1}^{\alpha-1/2}(z)$$

$$+\sum_{l=0}^{\infty}(2\lambda+2\alpha-2l-1)\frac{\Gamma(l+1)\Gamma(-2\lambda-2\alpha+l+1)}{[\Gamma(-\lambda-\alpha+l+1)]^2}$$

$$\times 4^{\lambda-l}[(x_1^2-1)(x_2^2-1)]^{(\lambda-l)/2}$$

$$\times C_l^{\lambda+\alpha-l}(x_1)C_l^{\lambda+\alpha-l}(x_2)D_{-\lambda-2\alpha+l+1}^{\alpha-1/2}(z)\Bigg\}. \qquad \text{(B1)}$$

For λ an integer, (B1) reduces by construction to the addition formula (8.2) for $C_\lambda^\alpha(\xi)$, $\xi=z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2$. That formula has only a finite number of polynomial terms, and is valid for arbitrary x_1, x_2, and z.

Thus,

$$C_\lambda^\alpha(\xi)-S_\lambda^\alpha(x_1,x_2,z)=0,\quad \lambda=0,1,2,\cdots, \qquad \text{(B2)}$$

x_1, x_2, z arbitrary.

Carlson's theorem[11] states that a function $f(\lambda)$ which is analytic in the right-half λ-plane, bounded by $\exp[|\lambda|(\pi-\epsilon)]$ for $|\lambda|\to\infty$, $\operatorname{Re}\lambda\geq 0$, and equal to zero for $\lambda=0,1,2,\cdots$, is identically zero. Before we can apply this result to (B2), we will need to establish the asymptotic behavior of C_λ^α and S_λ^α for $|\lambda|\to\infty$. However, from (6.1) and (6.2),

$$C_\lambda^\alpha(\cosh\beta)\sim\lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}$$

$$\times(\sinh\beta)^{-\alpha}\{\exp[(\lambda+\alpha)\beta]$$

$$+\exp(\pm i\pi\alpha)\exp[-(\lambda+\alpha)\beta]\},\quad \operatorname{Re}\beta\geq 0. \qquad \text{(B3)}$$

Thus, $C_\lambda^\alpha(\cosh\beta)$ grows as $\exp(\lambda\operatorname{Re}\beta)$ for $\lambda\to\infty$, and is not suitable for the application of Carlson's theorem. We will therefore consider instead of $[C_\lambda^\alpha-S_\lambda^\alpha]$ the function

$$G_\lambda^\alpha(\cosh\beta_1,\cosh\beta_2,\cosh\phi)=\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2\alpha)}\exp(-\lambda\beta)$$

$$\times[C_\lambda^\alpha(\cosh\beta)-S_\lambda^\alpha(\cosh\beta_1,\cosh\beta_2,\cosh\phi)], \qquad \text{(B4)}$$

where

$$x_1=\cosh\beta_1,\quad x_2=\cosh\beta_2,\quad z=\cosh\phi,\quad\text{and}$$

$$\cosh\beta=\cosh\phi_1\sinh\beta_1\sinh\beta_2-\cosh\beta_1\cosh\beta_2. \qquad \text{(B5)}$$

The ratio of Γ functions is introduced for convenience. We will assume that x_1, x_2, and z are all real and in the range specified by (9.4). The hyperbolic angles β_1, β_2, and ϕ are then real. The theorem can later be extended to values of x_1, x_2, and z throughout the region of convergence of the expansion (9.3) by analytic continuation.

It can be verified using (2.5) and (6.1) or (6.2) that the function

$$\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2\alpha)}\exp(-\lambda\beta)C_\lambda^\alpha(\cosh\beta)$$

is an entire function of λ (and α) with the asymptotic form

$$\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2\alpha)}\exp(-\lambda\beta)C_\lambda^\alpha(\cosh\beta)\sim\lambda^{-\alpha}(2\sinh\beta)^{-\alpha}e^{\alpha\beta}$$

$$\times\{1+\exp(\pm i\pi\alpha)\exp[-2(\lambda+\alpha)\beta]\},\quad \operatorname{Re}\beta\geq 0, \qquad \text{(B6)}$$

and is therefore suitable for the application of Carlson's theorem for $|\operatorname{Im}\beta|<\pi/2$. We will assume for convenience that $\operatorname{Re}\alpha\geq 0$. The burden of the proof of the addition theorem (9.3) for arbitrary λ thus amounts simply to a demonstration that the function S_λ^α is regular for $\operatorname{Re}\lambda\geq 0$, and is appropriately bounded for $|\lambda|\to\infty$, $\operatorname{Re}\lambda\geq 0$. This will certainly be the case if S_λ^α has (as it should) the same asymptotic form as $C_\lambda^\alpha(\cosh\beta)$. Carlson's theorem will then establish that $C_\lambda^\alpha(\cosh\beta)$ is identical to the series S_λ^α throughout the right-half λ-plane, and by analytic continuation in λ and α, throughout the common domain of analyticity of the two functions.

It is easy to show that the function

$$\frac{\Gamma(\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2\alpha)}\exp(-\lambda\beta)S_\lambda^\alpha$$

has no singularities for finite λ in the right-half λ-plane. The functions $C_l^{-\lambda-\alpha-l}$ and $C_l^{\lambda+\alpha-l}$ in (B1) have no poles as functions of λ (or α) for l an integer. Moreover, in the second series in (B1), the poles of $\Gamma(-2\lambda-2\alpha+l+1)$ for $2\lambda+2\alpha$ an integer are cancelled either by zeros of $C_l^{\lambda+\alpha-l}$, or by the zeros of $[\Gamma(-\lambda-\alpha+l+1)]^{-2}$. The apparent poles introduced by the factor $[\sin\pi(\lambda+\alpha)]^{-1}$ are also absent. For $\lambda+\alpha$ equal to an integer, the terms in the second series in (B1) with $l<2\lambda+2\alpha$ are proportional to $\sin\pi(\lambda+\alpha)$. The remainder of the series, $l\geq 2\lambda+2\alpha$, cancels term by term with the first series in (B1), again introducing first order zeros at the poles of $[\sin\pi(\lambda+\alpha)]^{-1}$. Finally, the poles of $D_{\lambda+l+1}^{\alpha-1/2}(z)$ for $(\lambda+2\alpha+l)$ a negative integer are eliminated by the factor $[\Gamma(\lambda+2\alpha)]^{-1}$, and the poles of $D_{-\lambda-2\alpha+l+1}^{\alpha-1/2}(z)$ for $(-\lambda+l)$ a negative integer are eliminated by the factor $\sin\pi\lambda$.

The calculation necessary to establish the asymptotic form of S_λ^α for $|\lambda|\to\infty$, $\mathrm{Re}\lambda>0$, is somewhat lengthy, and we will only sketch the procedure. It is convenient as a first step to replace the sum in (B1) by a contour integral through the use of the Sommerfeld–Watson transformation.[10] The expression for S_λ^α (or C_λ^α) then reads

$$S_\lambda^\alpha(\cosh\beta_1,\cosh\beta_2,\cosh\phi)$$
$$=\tfrac{1}{2}\exp(-i\pi\alpha)\,\frac{\sin\pi\lambda}{\cos\pi(\lambda+\alpha)}\,\frac{\Gamma(2\alpha-1)}{[\Gamma(\alpha)]^2}\int_C\frac{dl}{\sin\pi l}$$

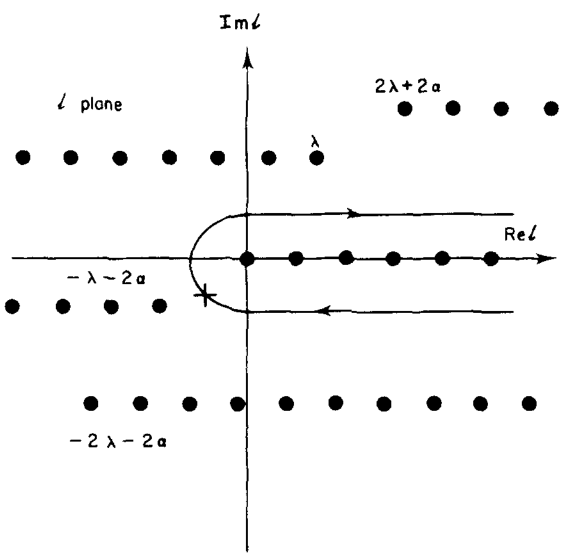

FIG. 8. The integration contour in the complex l plane for the integral representation (B7) for $C_\lambda^\alpha(\cosh\beta)$, with $\cosh\beta=(\cosh\phi\,\sinh\beta_1\,\sinh\beta_2-\cosh\beta_1\,\cosh\beta_2)$. The locations of the poles of the integrand for general values of λ and α are indicated by dots. The poles on a given line $\mathrm{Im}l$ = constant are separated by integer steps in $\mathrm{Re}l$. Some of the poles disappear for integer values of λ, $\lambda+2\alpha$, or $2\lambda+2\alpha$. The location of the saddle point which yields the main contribution to the integral for $\|\lambda|\to\infty$, ϕ, β_1 and β_2 large, is indicated by a cross.

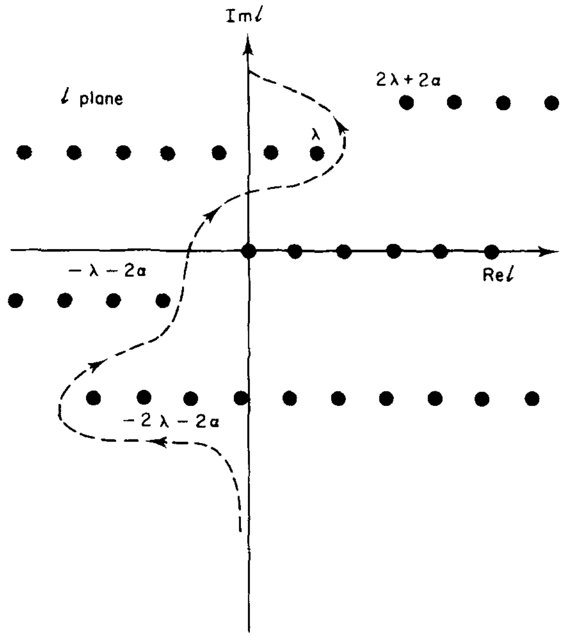

FIG. 9. The contour which yields an integral representation for $C_\lambda^\alpha(\cosh\beta)$ with extended range of validity.

$$\times\Big\{(2\lambda+2\alpha+2l+1)\,\frac{\Gamma(l+1)[\Gamma(-\lambda-\alpha-l)]^2}{\Gamma(-2\lambda-2\alpha-l)}$$
$$\times(4\sinh\beta_1\sinh\beta_2)^{-\lambda-2\alpha-l}C_l^{-\lambda-\alpha-l}(\cosh\beta_1)$$
$$\times C_l^{-\lambda-\alpha-l}(\cosh\beta_2)D_{\lambda+l+1}^{\alpha-1/2}(\cosh\phi)$$
$$+(-2\lambda-2\alpha+2l+1)\,\frac{\Gamma(l+1)[\Gamma(\lambda+\alpha-l)]^2}{\Gamma(2\lambda+2\alpha-l)}$$
$$\times(4\sinh\beta_1\sinh\beta_2)^{\lambda-l}C_l^{\lambda+\alpha-l}(\cosh\beta_1)$$
$$\times C_l^{\lambda+\alpha-l}(\cosh\beta_2)D_{-\lambda-2\alpha+l+1}^{\alpha-1/2}(\cosh\phi)\Big\},\qquad\text{(B7)}$$

where we have rearranged some of the gamma functions. The integration contour is shown in Fig. 8. The region of convergence of this representation is that given in (9.4).

It is interesting to note that for

$$\left|\mathrm{Im}\left\{\phi-\ln\left(\tanh\frac{\beta_1}{2}\tanh\frac{\beta_2}{2}\right)\right\}\right|<\pi,\qquad\text{(B8)}$$

C_λ^α can be written as the same integral multiplied by 1/2, with the integration contour taken as that in Fig. 9. The addition theorem which holds for the opposite sense of the inequality in (9.4) is obtained by pushing the contour in Fig. 9 to ∞ in the left-half l-plane.

The asymptotic form of S_λ^α for $|\lambda|\to\infty$ can now be estimated using the method of steepest descents to evaluate the integral. For $\mathrm{Re}\lambda\to\infty$, the term in (B7) which involves the function $D_{\lambda+l+1}^{\alpha-1/2}$ is exponentially small compared to the term which involves $D_{-\lambda-2\alpha+l+1}^{\alpha-1/2}$, and can be dropped [see (6.3)]. The relevant saddle point is that which occurs for $l\sim\exp(i\pi)\lambda x$, $x\lesssim 1$, $|\lambda x|\gg 1$ (see Fig. 8). The integrand decreases exponentially in l as $\mathrm{Re}l$ increases from its value at the saddle point.

We consequently require estimates of the remaining C and D functions for $|\lambda|$ and $|l|$ large. Equation (6.9) gives an estimate of $D^{\alpha-1/2}_{-\lambda-2\alpha+l+1}(\cosh\phi)$. If we assume that β_1 and β_2 are large and that $|l/\lambda|<1$, $z\gg 1$, then (6.10) gives the estimate required for $C_l^{\lambda+\alpha-l}(\cosh\beta)$. Upon combining terms, one finds that the integral which represents S_λ^α behaves for l in the neighborhood of the saddle point as

$$-\frac{i}{\sqrt{2\pi}}\frac{1}{\Gamma(\alpha)}2^{2\lambda+2\alpha-1/2}\lambda^{2\lambda+2\alpha-1}(1-e^{-2\phi})^{-\alpha+1/2}$$
$$\times(e^\phi\sinh\beta_1\sinh\beta_2)^\lambda\int_C dl\exp\{(\alpha-\tfrac{1}{2})\ln(\lambda-l)$$
$$-(2\lambda+2\alpha-l-\tfrac{1}{2})\ln(2\lambda-l)-(l+\tfrac{1}{2})\ln(-l)$$
$$+l\ln[e^{-\phi}\coth\beta_1\coth\beta_2]\}. \tag{B9}$$

The integration contour is to be taken in the direction of steepest descent through the saddle point indicated in Fig. 8. The location of the saddle point can be obtained by careful expansion of the exponent in (B9) in powers of λ. It is convenient in this calculation to assume that ϕ is large enough that

$$|\lambda|^{-1}\lesssim 2e^{-\phi}\coth\beta_1\coth\beta_2\ll 1.$$

The saddle point then occurs at $l\sim\exp(i\pi)2\lambda\exp(-\phi)\coth\beta_1\coth\beta_2$. The result of the calculation is as follows:

$$S_\lambda^\alpha\sim\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}\exp\{\lambda\ln[e^\phi\sinh\beta_1\sinh\beta_2-2\cosh\beta_1\cosh\beta_2$$
$$+O(e^{-\phi},e^{-\beta_1},e^{-\beta_2})]\},\quad \mathrm{Re}\lambda\to\infty. \tag{B10}$$

This is equal, within the accuracy of the approximations, to the asymptotic form of $C_\lambda^\alpha(\cosh\beta)$ for $\mathrm{Re}\lambda\to\infty$, $\beta\gg 1$,

$$C_\lambda^\alpha(\cosh\beta)\to[1/\Gamma(\alpha)]\lambda^{\alpha-1}e^{\lambda\beta}, \tag{B11}$$

with $\cosh\beta$ given by (B5). The consistency of the restrictions on β_1, β_2, ϕ, and the ratio $|l/\lambda|$ is easily checked.

The case $\lambda\to\pm i\infty$ is slightly more complicated, as both terms in the asymptotic expression for $D^{\alpha-1/2}_{-\lambda-2\alpha+l+1}$, (6.9) must be taken into account. The term in (B7) which involves $D^{\alpha-1/2}_{\lambda+l+1}$ also contributes. However, the asymptotic behavior obtained for S_λ^α is again consistent with that given for C_λ^α in (B3).

If we now combine C_λ^α and S_λ^α to obtain G_λ^α, (B4), and use the asymptotic limits for C_λ^α and S_λ^α, we find that G_λ^α is bounded for $|\lambda|\to\infty$, $\mathrm{Re}\lambda\geq 0$, by $\exp[|\lambda|(\pi-\epsilon)]$, $|\mathrm{Im}\beta|<\pi/2$. Since G_λ^α is also analytic in the right-half λ-plane and vanishes for $\lambda=0,1,2,\cdots$ by (B2), it is identically zero by Carlson's theorem.[11] Hence, $C_\lambda^\alpha(\cosh\beta)$ is the unique analytic continuation of S_λ^α, and the addition theorem is proved for β_1, β_2, ϕ in the ranges implied by the foregoing restrictions. The result follows for arbitrary λ, α, and x_1, x_2, z throughout the region (9.4) by analytic continuation.

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[7]This definition differs from that used by Robin, Ref. 5, Sec. 170, (93), by a factor $\pi^{-1}e^{2\pi i(\alpha-1/4)}$ which simplifies the connection between the C's and D's. Robin's definition, on the other hand, relates the C's and D's to the Legendre functions in a symmetrical way [cf. (2.4), (2.8)]. The C's and D's satisfy the same recurrence relations in λ and α for either of these definitions. However, the definition of $D_\lambda^\alpha(z)$ given in HTF 3.15.2 (32), corrections on p. 2 of errata to Vol. 1, is inappropriate even as corrected, as the function so defined does not satisfy the same recurrence relations in α as $C_\lambda^\alpha(z)$.

[8]See Ref. 3, Chap. 9, or Ref. 4, Sec. 10.19, for discussions of the general theory of expansions in orthogonal polynomials.

[9]The expansion of $(z-t)^{-1}$ in a series of Jacobi functions is given by Szegö, Ref. 3, Theorem 9.2.1. The Gegenbauer expansion (7.2) is obtained as a special case of the Jacobi expansion for $\alpha=\beta$. The derivation of (7.2) given above is based on the completeness of the Gegenbauer polynomials for the space of L^2 functions with respect to the weight $(1-t^2)^{\alpha-1/2}$. This is probably the simplest and most intuitive derivation of (7.2) from the point of view of the physicist familiar with eigenfunction expansions. However, we can obtain a simple alternative derivation free from the completeness assumption by using the recurrence relation (4.1) for $C_n^\alpha(t)$ and $D_n^\alpha(z)$ and the explicit values of these functions for $n=0$, -1 to derive the identity

$$\frac{1}{z-t}=e^{-i\pi\alpha}2^{2\alpha}[\Gamma(\alpha)]^2(z^2-1)^{\alpha-1/2}\sum_{n=0}^{N}(n+\alpha)\frac{\Gamma(n+1)}{\Gamma(n+2\alpha)}$$
$$\times C_n^\alpha(t)D_n^\alpha(z)+e^{-i\pi\alpha}2^{2\alpha-1}[\Gamma(\alpha)]^2\frac{\Gamma(N+2)}{\Gamma(N+2\alpha)}$$
$$\times\frac{(z^2-1)^{\alpha-1/2}}{z-t}\left[D_{N+1}^\alpha(z)C_N^\alpha(t)-D_N^\alpha(z)C_{N+1}^\alpha(t)\right].$$

It can then be established from (6.2) and (6.3) that the second term on the right-hand side of this equation vanishes for $N\to\infty$ and t inside the ellipse determined by (7.3).

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Bifurcation of solutions with crystalline symmetry

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We consider the BBGKY equation for the single particle probability density in a hard sphere system. We investigate whether there is bifurcation from the fluid phase to functions which have crystalline symmetry. We find that as the density of the fluid increases from zero, there is bifurcation in one, two, and three dimensions. The bifurcation is shown to be characteristic of metastability and in general it does not occur at the equilibrium coexistence of two phases. The direction of branching and the stability of solutions near bifurcation is also discussed.

1. INTRODUCTION

We consider the integral equation

$$h(x) = -1 + \frac{|\omega| \exp[-\mu(q) K(q) h(x)]}{\int_\omega \exp[-\mu(q) K(q) h(x)]\, dx}, \tag{1.1}$$

for $x \in \mathbb{R}^n$, where

$$K(q) h(x) = (q/d)^n \int_{|y| \le d/q} h(x-y)\, dy,$$

and where ω denotes the basic cell of some space lattice in $\mathbb{R}^n$. We seek solutions h of the Eq. (1.1) which satisfy the condition

$$\int_\omega h(x)\, dx = 0, \tag{1.2}$$

and which have all of the symmetries of the lattice under consideration. As is discussed in Refs. 1 and 2, the Eqs. (1.1) and (1.2) are equivalent to the first equation of the BBGKY hierarchy for particles which interact like hard spheres of diameter d/q. The real number q is the ratio of characteristic length in the lattice to the diameter of the spheres. The value $q=1$ corresponds to closest packing and increasing q corresponds to decreasing the density. Indeed for any given lattice, q and the density ρ are related by

$$\rho = c(1/dq)^n,$$

where only the constant c depends upon the lattice. The function μ of q is related to the pressure P in the system. In fact there are constants β and κ such that

$$\mu(q) = (\beta P/\rho - 1)\kappa,$$

where P is the pressure and ρ is the density. We regard μ as a known function of q. In Refs. 1 and 2, we take our information about μ as a function of q from computer experiments (see Ref. 2 for references) which plot the pressure P as a function of ρ. (See Fig. 1.)

The number $h(x)+1$ determines the probability that there is a particle at x and (1.2) is just a normalization. Clearly $h \equiv 0$ is a solution of (1.1) and (1.2) for all values of q and this represents the uniform fluid phase. The computer experiments give $\beta P/\rho$ and consequently μ as a function of ρ for this phase. We show in two and three dimensions that as ρ is increased from zero towards the freezing density (i.e., q is decreased from $+\infty$) a value ρ^* of the density is reached at which a branch of crystalline solutions of (1.1) and (1.2) bifurcates from the fluid curve. By crystalline solutions we mean functions h which satisfy (1.1) and (1.2) and have all of the symmetries of the lattice under consideration, but which are not constant. We associate the sites for particles with the maxima of h and consider only functions h which have exactly the number of maxima per unit cell corresponding to the sites in the lattice which is considered. The value ρ^* at which bifurcation occurs is always below the freezing density. The computer experiments also give results on crystalline structures which exist well below the actual freezing density. These are the so-called metastable crystals and the experimental value of the density at which these crystals approach the fluid line is in close agreement with the value ρ^* of the density at which we find bifurcation. In one dimension, there is no phase transition and $\beta P/\rho$, and consequently μ is known exactly as a function of ρ in the fluid phase. The relationship of the branching solutions to the results of the experiments is summarized in the discussion section.

Finally we relate our bifurcation analysis to the

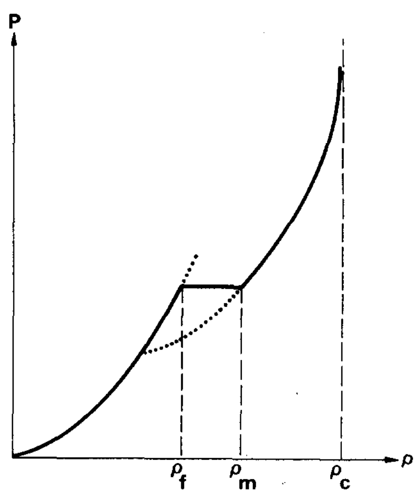

FIG. 1. Schematic of hard sphere isotherm, pressure, P, versus density ρ, for two and three dimensions. The unbroken curve illustrates results from computer simulations and ρ_f, ρ_m, and ρ_c denote the freezing, melting, and closest-packing densities respectively. The dotted curve above ρ_f and below ρ_m denote the metastable fluid and crystal, respectively. See Ref. 2.

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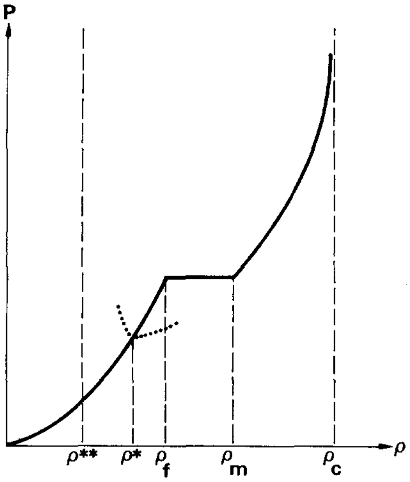

FIG. 2. Schematic of hard sphere isotherm with bifurcation density, ρ^*, and the Kirkwood instability density, ρ^{**}. The dotted curve indicates pressure associated with the bifurcating solutions that have crystalline symmetries and the correct number of maxima per unit cell. Other quantities are as defined in Fig. 1.

somewhat controversial "Kirkwood instability criterion.[3]" It is our view that this criterion gives a lower bound for the densities at which bifurcation can occur without reference to a specific lattice structure and without the restriction that the function h should have maxima at (and only at) the lattice sites. That is, Kirkwood's criterion gives a density ρ^{**} below which bifurcation cannot occur no matter what lattice is considered. We claim that for any given lattice there is a density ρ^* greater than ρ^{**} at which bifurcation does occur. Furthermore the crystalline solutions which bifurcate at ρ^* have particles at and only at the lattice sites. We believe that in two and three dimensions these branches of crystalline solutions coincide with the branches of metastable crystals below the freezing density found by the computer experiments. Thus by continuing these branches to the freezing density, they will meet the branch of stable crystalline solutions at that density. (See Fig. 2.)

2. FORMULATION IN HILBERT SPACE

In this section we reduce the problem discussed in the Introduction to an equation in an appropriate Hilbert space of periodic functions. Then, in Sec. 3, the bifurcation of nontrivial solutions of this equation is considered. In one dimension, the construction of the appropriate Hilbert space is simple and we do it directly in Sec. 3. In two and three dimensions, there are many different space lattices. For definiteness and in order to compare our results with the computer experiments, we consider the square planar and hexagonal arrays in two dimensions and the face centered cubic array in three dimensions. In Refs. 1 and 2 we also consider the hexagonal close packing array in three dimensions.

The real line is denoted by $\mathbb{R}$ and the integers by Z. Given a set $\{a_j \in \mathbb{R}^n: 1 \leq j \leq n\}$ of linearly indepdent points in $\mathbb{R}^n$, the set $\{\sum_{j=1}^n k_j a_j : k_j \in Z\}$ is called the lattice with basic vectors $\{a_j : 1 \leq j \leq n\}$. For a lattice L, we denote by $\Gamma(L)$ the full space group of L (i.e., the group generated by the translations, rotations, and reflections which leave L invariant). The basic cell $\omega(L)$ in L is the set $\{\sum_{j=1}^n \alpha_j a_j : 0 \leq \alpha_j \leq 1 \text{ for } 1 \leq j \leq n\}$ and its volume is denoted by $|\omega(L)|$.

We consider the following three lattices: L_1 is the lattice in $\mathbb{R}^2$ which has basic vectors

$$a_1^1 = d(1,0) \quad \text{and} \quad a_2^1 = d(0,1).$$

This is the square planar array of closest packing for spheres of diameter d in two dimensions. L_2 is the lattice in $\mathbb{R}^2$ which has basic vectors

$$a_1^2 = d(1,0) \quad \text{and} \quad a_2^2 = (d/2)(1, \sqrt{3}).$$

This is the hexagonal close packing array for spheres of diameter d in two dimensions. L_3 is the lattice in $\mathbb{R}^3$ which has basic vectors

$$a_1^3 = \frac{d}{\sqrt{2}}(1,1,0), \quad a_2^3 = \frac{d}{\sqrt{2}}(0,1,1), \quad \text{and} \quad a_3^3 = \frac{d}{\sqrt{2}}(1,0,1).$$

This is the face centered cubic array for the closest packing of spheres of diameter d in three dimensions.

For these lattices L_1, L_2, and L_3, let H_i denote the complex Hilbert space of functions which are invariant under the translations in $\Gamma(L_i)$. The inner product in H_i is given by

$$\langle h, g\rangle_i = \frac{1}{|\omega(L_i)|} \int_{\omega(L_i)} h(x)\overline{g(x)}\,dx \quad \text{for } f, g \in H_i.$$

For a lattice L with basic vectors $\{a_j : 1 \leq j \leq n\}$, the reciprocal lattice L^* is the lattice with basic vectors $\{A_j : 1 \leq j \leq n\}$, where the A_j are such that $A_j \cdot a_i = 2\pi\delta_{ij}$ for $1 \leq i, j \leq n$ (δ_{ij} is the Kronecker delta).

For $G \in \mathbb{R}^n$, we use ϕ_G to denote the plane wave $\phi_G(x) = \exp(iG\cdot x)$ for $x \in \mathbb{R}^n$. Clearly a plane wave ϕ_G belongs to H_i if and only if $G \in L_i^*$. Furthermore, $B_i = \{\phi_G : G \in L_i^*\}$ is an orthogonal basis for H_i.

For each $q \geq 1$, we define a linear integral operator $K(q)$ by

$$K(q)h(x) = \left(\frac{q}{d}\right)^n \int_{|y| \leq d/q} h(x-y)\,dy.$$

Then we see that $K(q)$ maps H_i into H_i and that $K(q)$ is a compact self-adjoint operator in H_i (see Ref. 1, Appendix B). In fact, $K(q)h(x)$ is a continuous function of x for each $h \in H_i$. Also $K(q)$ is diagonal with respect to the basis B_i and

$$K(q)\,\phi_G(x) = \left(\frac{q}{d}\right)^n \int_{|y| \leq d/q} \exp(iG\circ(x-y))\,dy = \lambda(q,G)\exp(iG\circ x),$$

where

$$\lambda(q,G) = \frac{2q}{d|G|}\sin\frac{d|G|}{q} \quad \text{if } n=1,$$

$$\lambda(q,G) = \frac{2\pi q}{d|G|} J_1\left(\frac{d|G|}{q}\right) \quad \text{if } n=2,$$

and

$$\lambda(q,G) = \frac{4\pi q}{d|G|} j_1\left(\frac{d|G|}{q}\right) \quad \text{if } n=3,$$

where we use the standard notation for Bessel functions. Note that for fixed q, $\lambda(q, G)$ is real and depends only on $|G|$. Hence for fixed q, the spectrum of $K(q)$ is the set of values of $\lambda(q, G)$ as G varies over the reciprocal lattice L_i^* and every eigenvalue [except $\lambda(q, 0)$] has multiplicity greater than one. To eliminate the degeneracy of these eigenvalues we pass to a real subspace of H_i which is invariant under $K(q)$. Let $\mathcal{H}_i$ denote the real subspace of H_i consisting of all functions h in H_i which are real-valued and which are invariant under the full space group $\Gamma(H_i)$ and such that $\int_{\omega(L_i)} h(x)\,dx = 0$. Then for $G \in L_i^*$, if we consider the smallest value (say) δ_i of $|G|$ such that $K(q)$ has an eigenfunction in $\mathcal{H}_i$ corresponding to the eigenvalue $\lambda(q, \delta_i)$, we find that $\lambda(q, \delta_i)$ is an eigenvalue of multiplicity one of $K(q)$ in the Hilbert space $\mathcal{H}_i$. In fact, the only eigenfunction of $K(q)$ is given by $\sum \phi_G$, where the summation is taken over all $G \in L_i^*$ such that $|G| = \delta_i$. Note that $\sum \phi_G$ is real-valued since $|G| = |-G|$. The actual value of δ_i for the various lattices is

$$\delta_1 = \frac{2\pi}{d}, \quad \delta_2 = \frac{4\pi}{\sqrt{3}d}, \quad \delta_3 = \frac{\sqrt{6}\pi}{d},$$

and the corresponding symmetrized eigenfunctions are respectively

$$\psi_1(x, y) = \cos\frac{2\pi x}{d} + \cos\frac{2\pi y}{d},$$

$$\psi_2(x, y) = \cos\frac{2\pi}{d}\left(x + \frac{y}{\sqrt{3}}\right) + \cos\frac{2\pi}{d}\left(x - \frac{y}{\sqrt{3}}\right) + \cos\frac{4\pi}{\sqrt{3}d}\, y,$$

and

$$\psi_3(x, y, z) = \sum_{k,l,m=\pm 1} \cos\frac{\sqrt{2}\pi}{d}(kx + ly + mz).$$

We have introduced a real Hilbert space $\mathcal{H}_i$ and considered how the linear integral operator $K(q)$ in (1.1) acts in $\mathcal{H}_i$. Let us now show that the problem of finding solutions of (1.1) and (1.2) with all the symmetries of L_i is equivalent to solving a nonlinear equation in $\mathcal{H}_i$.

Let us define an operator N_i by

$$N_i(h, q)(x) = -1 + \frac{\exp[-\mu(q)K(q)h(x)]}{\langle \exp[-\mu(q)K(q)h(x)], 1\rangle_i}.$$

From the properties of $K(q)$, it follows that N_i maps $\mathcal{H}_i \times \mathbb{R}$ into $\mathcal{H}_i$ and that $N_i: \mathcal{H}_i \times \mathbb{R} \to \mathcal{H}_i$ is infinitely differentiable in the sense of Fréchet. Solving the equation

$$h = N_i(h, q) \quad \text{for} \quad (h, q) \text{ in } \mathcal{H}_i \times \mathbb{R} \tag{2.1}$$

is equivalent to solving (1.1) and (1.2) for functions h which have all of the symmetries of the lattice L_i. Since $K(q)$ maps integrable functions to continuous functions, solutions of (2.1) are smooth.

Clearly $N_i(0, q) = 0$ for all q. We now establish the bifurcation of a smooth curve of nontrivial solutions of (2.1) from this line of trivial solutions. For this we must consider the linearization of N_i about $h = 0$. Now the Fréchet derivative of N_i with respect to h at $(0, q)$ is the linear operator

$$-\mu(q)K(q): \mathcal{H}_i \to \mathcal{H}_i.$$

Bifurcation from the line of trivial solutions can take place only at values of q such that $I + \mu(q)K(q)$ is noninvertible where I denotes the identity. Note, however, that $\mu(q)K(q)$ does not depend linearly on q and so to establish bifurcation we must use the full generality of Theorem 1.7 of Ref. 4 rather than the more widely known special case Theorem 2.1 on page 196 of Ref. 5 which applies only to operators having a linearization of the form qK, where K is independent of q.

3. BIFURCATION ANALYSIS

For each lattice L_i we have shown that there exists a positive constant δ_i such that $\lambda(q, \delta_i)$ is a simple eigenvalue of $K(q): \mathcal{H}_i \to \mathcal{H}_i$ for all $q \geq 1$. Furthermore, the eigenfunction $\tau_i \in \mathcal{H}_i$ of $K(q)$ corresponding to $\lambda(q, \delta_i)$ is independent of q and has maxima at and only at the positions where particles in the crystal are situated.

Hence the operator $I + \mu(q)K(q)$ is noninvertible if and only if $\mu(q)\lambda(q, G) + 1 = 0$ for some $G \in L_i^* \setminus \{0\}$. Also, if $\mu(q)\lambda(q, G) + 1 = 0$ for some $G \in L_i^* \setminus \{0\}$, then the kernel $N(I + \mu(q)K(q))$ of $I + \mu(q)K(q)$ is just the eigenspace of $K(q)$ corresponding to $\lambda(q, G)$. Hence, considering $K(q)$ as a map from $\mathcal{H}_i$ into $\mathcal{H}_i$ we have $N(I + \mu(q)K(q)) = \operatorname{span}\{\psi_i\}$ if and only if $\mu(q)\lambda(q, \delta_i) + 1 = 0$.

Now μ is known as a function of ρ (and hence of q) from the computer experiments and $\lambda(q, \delta_i)$ is given in terms of Bessel functions in Sec. 2. In Ref. 2 we plot the curves $\mu(q)$ and $-1/\lambda(q, \delta_i)$ and we find that these curves intersect at exactly one point $q_i^* > 1$. The density ρ_i^* corresponding to q_i^* lies below the density at which freezing occurs.

Theorem (*bifurcation*): For each lattice L_i, there exists an $\epsilon_i > 0$ and smooth maps $f_i: (-\epsilon_i, \epsilon_i) \to \mathcal{H}_i$ and $\gamma_i: (-\epsilon_i, \epsilon_i) \to \mathbb{R}$ such that

$$f_i(\alpha) = N_i(f_i(\alpha), \gamma_i(\alpha)),$$

$$f_i(\alpha) = \alpha\psi_i + \frac{\alpha^2}{2!} f_i''(0) + o(\alpha^2)$$

and

$$\gamma_i(\alpha) = q_i^* + \alpha\gamma_i'(0) + o(\alpha) \quad \text{for all} \quad \alpha \in (-\epsilon_i, \epsilon_i).$$

Furthermore, in a neighborhood of $(0, q_i^*)$ in $\mathcal{H}_i \times \mathbb{R}$, all of the solutions of (2.1) which do not lie on this curve $\{(f_i(\alpha), \gamma_i(\alpha)): -\epsilon_i < \alpha < \epsilon_i\}$ must be trivial solutions of the form $(0, q)$.

Proof: We shall apply Theorem 1.7 of Ref. 4. Choose and fix i. Let $X = Y = \mathcal{H}_i$ and let $F: \mathbb{R} \times X \to Y$ be defined by $F(q, h) = h - N_i(h, q_i^* - q)$ where q_i^* is the value of q such that $\mu(q)\lambda(q, \delta_i) + 1 = 0$. Clearly F has the properties (a) and (b) of Theorem 1.7 of Ref. 4. Also $N(F_h(0, 0)) = N(I + \mu(q_i^*)K(q_i^*)) = \operatorname{span}\{\psi_i\}$ and so is one-dimensional. Since $K(q)$ is self-adjoint and compact, $R(F_h(0, 0))$ is the orthogonal complement of $N(I + \mu(q)K(q))$ in $X = Y = \mathcal{H}_i$. Hence $Y/R(F_h(0, 0))$ is also one-dimensional. Thus we see that F has property (c) of Theorem 1.7 of Ref. 4.

The conclusion of our theorem will be established provided that we demonstrate that F has the transversality property (d) of Theorem 1.7 of Ref. 4. This is equivalent to showing that

$$\langle \psi_i, F_{qh}(0, 0)\psi_i \rangle_i \neq 0$$

and this in turn is equivalent to showing that

$$\left\langle \psi_i, \frac{\partial}{\partial q} \mu(q) K(q) \psi_i \right\rangle_i \neq 0 \quad \text{at} \quad q = q_i^*.$$

But

$$\left\langle \psi_i, \frac{\partial}{\partial q} \mu(q) K(q) \psi_i \right\rangle_i = \left\langle \psi_i, \frac{\partial}{\partial q} \mu(q) \lambda(q) \psi_i \right\rangle_i$$

and so the transversality condition is that

$$\frac{d}{dq} (\mu(q) \lambda(q)) \neq 0 \quad \text{at} \quad q = q_i^*, \quad \text{where } \lambda(q) \equiv \lambda(q, \delta_i).$$

That is, our condition for branching is that when we plot $\mu(q)\lambda(q)$ as a function of q, $\mu(q)\lambda(q)$ should have nonzero slope at $q = q_i^*$. We find that this is the case for the values of $\mu(q)$ given by the computer experiments. This completes the proof of the theorem. Indeed, since the requirement that $(d/dq)(\mu(q)\lambda(q)) \neq 0$ at $q = q_i^*$ is a generic property of curves $\mu(q)\lambda(q)$, and since $\mu(q)$ is determined by experiment, we may always assume that F has the transversality property (d) of Theorem 1.7 of Ref. 4.

Remarks: 1. For all sufficiently small values of α, the function $f_i(\alpha)$ has exactly the same number of maxima of ψ_i. Maxima of $f_i(\alpha)$ are associated with the sites of particles in the crystal described by $f_i(\alpha)$ and the eigenfunction has the correct number of maxima occurs correct number of particles for the structure being considered.

2. To find the values of q (equivalently ρ) at which $I + \mu(q)K(q)$ is noninvertible we must solve the equation $\mu(q)\lambda(q, G) + 1 = 0$. Writing q in terms of the density ρ and considering G as the Fourier transform variable in Kirkwood's discussion, this is exactly the equation on which the Kirkwood instability analysis is based.[3]

3. In one dimension, $\mu(q)$ can be expressed explicitly as a function of q instead of being determined by experiment as in two and three dimensions. In one dimension,

$$\mu(q) = (q-1)^{-1}.$$

But in one dimension, H is just the Hilbert space of functions h of period d such that $\int_0^d h(x)\,dx = 0$. The points G in the reciprocal lattice are $G = 2\pi k/d$, where k is an integer. Hence the eigenvalue $\lambda(q, G)$ of $K(q)$ whose eigenfunction has the correct number of maxima occurs when $k = \pm 1$. Thus,

$$\lambda\left(q, \frac{2\pi}{d}\right) = \frac{q}{\pi} \sin\frac{2\pi}{q}$$

and the corresponding eigenfunction is $\cos 2\pi x/d$. The equation

$$\mu(q)\lambda\left(q, \frac{2\pi}{d}\right) + 1 = 0$$

then becomes

$$\frac{q}{\pi(q-1)} \sin\frac{2\pi}{q} + 1 = 0$$

and it is easily seen that this equation has only the solution $q = 1.432$ in the region $q > 1$. It is also easily checked that

$$\frac{d}{dq}\left(\frac{q}{\pi(q-1)} \sin\frac{2\pi}{q}\right) \neq 0$$

at $q = 1.432$. Thus there is bifurcation of metastable or unstable crystalline solutions from the fluid curve at $q = 1.432$.

4. As in Appendix C of Ref. 1, the direction of branching of the curve of crystalline solutions can be determined by calculating the coefficients of α in the power series expansion for $\gamma_i(\alpha)$ about $\alpha = 0$. Differentiating $f_i(\alpha) = N_i(f_i(\alpha), \gamma_i(\alpha))$ with respect to α, setting α equal to zero, and taking the inner product with ψ_i yields

$$\frac{\langle \psi_i^2, \psi_i \rangle_i}{\langle \psi_i, \psi_i \rangle_i} = 2\mu'(q_i^*)\lambda(q_i^*, \delta_i) + 2\mu(q_i^*)\gamma_i'(0)\frac{\partial}{\partial q}\lambda(q_i^*, \delta_i)$$

and so

$$\gamma_i'(0) = \lambda(q_i^*, \delta_i)\left(2\frac{\partial\lambda}{\partial q}(q_i^*, \delta_i)\right)^{-1}\left(2\mu'(q_i^*)\lambda(q_i^*, \delta) - \frac{\langle \psi_i^2, \psi_i \rangle_i}{\langle \psi_i, \psi_i \rangle_i}\right).$$

In one dimension, the right-hand side of this expression can be calculated exactly because μ is known exactly. In two and three dimensions, $\mu'(q_i^*)$ must be estimated from the computer experiments. A similar procedure determines all of the subsequent coefficients.

5. The expression

$$A(h, q) = -\langle \ln(h+1), h+1 \rangle_i$$

is well defined for solutions of (1.1) since h is a continuous function of x and $h(x) > -1$ for all x. $A(h, q)$ represents the entropy of the system. Thus the quantity $A(f_i(\alpha), \gamma_i(\alpha))$, for α in a neighborhood of zero, corresponds to the entropy on the branch of crystalline solutions which bifurcates at ρ_i^*. Differentiating $B(\alpha) \equiv A(f_i(\alpha), \gamma_i(\alpha))$ with respect to α and setting α equal to zero we obtain

$$B(0) = 0,$$

$$B'(0) = 0, \quad \text{since } \langle \psi_i, 1 \rangle = 0 \text{ and } f_i'(0) = \psi_i,$$

and

$$B''(0) = -\langle \psi_i, \psi_i \rangle_i \quad \text{since } f_i''(0) \in H_i.$$

Hence $A(f_i(\alpha), \gamma_i(\alpha)) = -\|\psi_i\|_i^2\alpha^2 + O(\alpha^3)$ in a neighborhood of the bifurcation point ρ_i^*. By our normalization, the entropy of the fluid phase is zero and so we see that the crystalline solutions on the branch bifurcating at ρ_i^* have a lower entropy than the fluid solution. This means that close to the branch point the crystalline solutions are at best metastable with respect to the uniform fluid phase.

The expression

$$G(h, q) = \langle \ln(h+1), h+1 \rangle_i + \tfrac{1}{2}\mu(q)\langle K(q)h, h \rangle_i$$

can be associated with the free energy of the system. Thus the quantity $G(f_i(\alpha), \gamma_i(\alpha))$, for α in a neighborhood of zero, represents the free energy on the branch of crystalline solutions bifurcating at ρ_i^*. The coefficients in the power series expansion of $G(f_i(\alpha), \gamma_i(\alpha))$ about $\alpha = 0$ can be determined as in Appendix D of Ref. 1. We find that

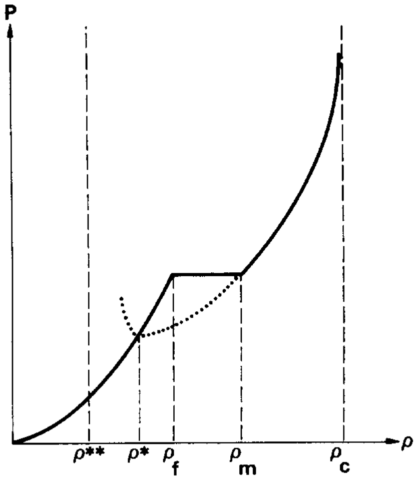

FIG. 3. Schematic of hard sphere isotherm with dotted curve representing the metastable extension of the equilibrium crystalline phase. The extension appears to intersect the fluid portion of the isotherm at a density very close to the bifurcation density, ρ^* predicted by the analyses. See Ref. 2.

$$G(f_i(\alpha), \gamma_i(\alpha)) = \xi\alpha^3 + O(\alpha^4).$$

As for the direction of branching in Remark 4, the coefficient ξ can be determined from the derivative of λ and μ at $q = q_i^*$.

6. Some comments about a similar analysis for interparticle potentials other than the hard sphere case are given in Refs. 1 and 2.

4. DISCUSSION

In two and three dimensions, the computer experiments show that for a system of hard spheres the plot of pressure P against density ρ is that shown by the unbroken curve in Fig. 1. The experiments also show that the fluid curve can be extended above the freezing density and that the solid curve can be extended below the melting density. The pressure for these metastable states is indicated by the dotted curves in Fig. 1. We show (for various lattices) that there is a density ρ^* below the freezing density at which solutions with the appropriate symmetries for crystals bifurcate from the fluid curve. The pressure for the solutions on the branch bifurcating at ρ^* is represented by the dotted line in Fig. 2. We believe that this curve joins the pressure curve for the metastable solids predicted by the computer experiments as shown in Fig. 3. Indeed, the pressure curves for the metastable solids given by the computer experiments do appear to join the pressure curve for the fluid at a value of the density close to our bifurcation point ρ^*. Consequently we believe that by continuing the branch of crystalline solutions away from the bifurcation point to higher density it will meet the curve of stable crystalline solutions at the freezing density as in Fig. 3.

Since bifurcation does not occur at the phase transition, but instead at the limit of metastability, its occurrence in hard rods, where the fluid phase is the only equilibrium state of the infinite system, is not surprising. We believe that if the branch of crystalline solutions were to be continued away from the bifurcation point to higher densities, the branch would never become stable. It is, however, of interest to further investigate this branch of solutions. If the lifetimes of crystalline arrays away from the bifurcation point are long compared to the time between collisions, then the one-dimensional case could, because of its simplicity, be useful in the study of metastability.

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A soluble dispersion relation for a three-dimensional band structure

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We show that an exact dispersion relation can be obtained for a cubic lattice made of spherically-symmetric attractive potentials. This result is obtained in a limit case where the potentials have zero range and infinite intensity.

I. INTRODUCTION

The Kronig–Penney model[1] has been extensively used as a first approximation which provides the main features of the band structure of periodic systems. Although this model has only been exactly solved in one dimension, its results are exhibited in reference textbooks[2] as an illustration of some general property of solids, such as energy gaps, crystal effective mass, etc. It has also been employed as a guide to check the accuracy of certain approximations which are, in fact, intended for application in three-dimensional problems. Since the solution for spatially-periodic potentials has proved in general to be very difficult, it would be interesting to have, at least, one exactly soluble periodic problem in three dimensions provided that this potential, even of a highly idealized type, is still flexible enough to help in the calculation and understanding of three-dimensional band structures.

The aim of this paper is to introduce a class of potentials which leads to such exact solutions.

Let us consider first the case of a one-dimensional (1-d) periodic chain of delta functions. This problem has some features in common with the 3-d potential considered in this paper, and is useful for the better understanding of the techniques employed in the future calculations.

We then have

$$V(x) = \sum_{n=-\infty}^{\infty} V_n(x)\, V_n(x) = V_0\, a\,\delta(x - na) \qquad (1)$$

with the Bloch functions

$$\psi_k(x) = \mu_k(x)\exp(ikx), \qquad (2)$$

where $\mu_k(x)$ has the same period a of the potential (1). If we substitute the Fourier expansions

$$\mu_k(x) = \sum_q C(q, k)\exp(iqx), \qquad (3)$$

and

$$V(x)\,\mu_k(x) = \sum_q B(q, k)\exp(iqx), \quad q = (2\pi/a)m, \quad m = 0, \pm 1, \pm 2\cdots, \qquad (4)$$

with

$$B(q,k) = (1/a)\int_{-a/2}^{a/2} \mu_k(x)\, V(x)\, dx = V_0\,\mu_k(na), \qquad (5)$$

into Schrödinger's equation, we get

$$\sum_q \{[(\hbar^2/2m)(q+k)^2 - E]\,C(q,k) + B(q,k)\}\exp[i(q+k)]x = 0. \qquad (6)$$

or

$$C(q,k) = -B(q,k)/[(\hbar^2/2m)(q+k)^2 - E], \qquad (7)$$

provided that the denominator of (7) does not vanish. If $(\hbar^2/2m)(q+k)^2 = E$, we must have, from (6) and (5), $\mu_k(na) = 0$. This corresponds to a particular solution in which the particle does not "see" the potential and must therefore obey a free-particle dispersion relation. Since we are interested only in wavefunctions resulting from reflection and transmission at $x = na$, $n = 0, \pm 1, \cdots$ we shall disregard those frequencies entirely.

Summing both members of (7) with respect to q, we get from (3) and (5)

$$\sum_q C(q,k) = \mu_k(0) = -\sum_q V_0\,\mu_k(0)/[(\hbar^2/2m)(q+k)^2 - E], \qquad (8)$$

or finally

$$-\frac{\hbar^2}{2mV_0a^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+k')^2 - E'}, \qquad (9)$$

where $k' = ka/2\pi$ and $E' = 2mEa/\hbar^2$.

The summation indicated in (9) can be readily performed,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+k')^2 - E} = \sum_{l=1}^{2} \mathrm{Res}\left(\frac{\pi\cot(\pi z)}{(z+k')^2 - E)}\right)_{z=z_l}, \qquad (10)$$

where $z_{1,2} = -k' \pm \sqrt{E'}$, $E' > 0$, and $z_{1,2} = -k' \pm i(|E'|)^{1/2}$, $E' < 0$. From (10) we obtain the well-known results[3]

$$\cos(ka) = \cos x_k + B\sin x_k/x_k, \qquad E > 0,$$

or

$$\cosh(ka) = \cosh x_k + B\sinh x_k/x_k, \quad E < 0,$$

where $x_k = (2mEa^2)^{1/2}/\hbar^2$ and $B = mV_0a^2/\hbar^2$.

Notice that the usefulness of this technique, in the sense that no calculation of the wavefunction is necessary, is restricted to this kind of potentials, since otherwise the coefficient $B(q,k)$ will depend on the values taken by $\mu_k(x)$ in more than one point.

In trying to extend this formalism to three dimensions we must be careful, since in higher dimensions the δ function can no longer be used to represent a limit case of a physical potential of high intensity and short range[4] (e.g., it can have states with infinite binding energy). What we shall do is to consider the attractive spherical potential

$$V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r > a, \end{cases} \qquad (11)$$

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and take the limit where $a \to 0$ and $V_0 \to \infty$ in such a way as to satisfy the relevant physical requirements: (a) There is *one* bound state with finite energy E_L (kept fixed during the limit process); (b) There is a finite (nonzero) limit for the scattering cross section. As shown in quantum mechanics text books,[5] condition (a) can be satisfied by taking

$$\left(\frac{2m}{\hbar^2} V_0 a^2\right)^{1/2} = \frac{\pi}{2} + \frac{2}{\pi} \beta_L a + \cdots \tag{12}$$

in which case there is only one bound state ($l=0$) whose binding energy $E_L = -(\hbar^2/2m)\beta_L^2$ does not depend on the "strength" $V_0 a^2$ of the potential; it actually depends on the way the limit is approached. This result must always be kept in mind when considering 3-d periodic structures. With the choice Eq. (12) our second condition is automatically satisfied; in fact, a straightforward calculation gives the partial cross sections

$$\begin{aligned} \sigma_l &\approx a^2(\beta_L a)^{4l}, \qquad l \neq 0, \\ \sigma_0 &\approx \frac{h^2}{2m} \frac{1}{E + |E_l|}, \quad l = 0, \end{aligned} \tag{13}$$

showing that for $a \to 0$, $V_0 \to \infty$, there is a finite s-wave scattering (potentials of this type without bound states produce no scattering at all). Our last (and most important) remark is related to the behavior of the wavefunctions (for fixed l and positive energy) inside the potential well (11). It is easily shown that in the limit (12), a partial wave ψ_l *which is finite for* $r > a$, behaves, for $r < a$, in the following way:

$$\text{(a)}\ \psi_0 = \frac{A}{\beta_l a} \frac{\sin(\alpha r)}{\alpha r} + O(a), \quad \alpha = \left(\frac{2mV_0}{\hbar^2}\right)^{1/2}, \tag{14}$$

for all positive values of E.

$$\text{(b)}\ \psi_l = O(\beta a)^l, \quad \beta = \left(\frac{2mE}{\hbar^2}\right)^{1/2}, \quad l \neq 0. \tag{15}$$

Here lies the main difference with the one dimensional delta potential; we can no more consider the wavefunction as a contant $\mu_k(0)$ inside the potential range. However, as shown in the next section, the problem can still be solved considering the spatial dependence of the s wave alone.

II. THE DISPERSION RELATION IN THE THREE-DIMENSIONAL CASE

Let us consider the periodic potential formed by a spatial distribution of attractive sites (11), in the limit (12), in which each site occupies the center of a cubic cell of side d.

$$V(\mathbf{x}) = \sum_{l,m,n} V(\mathbf{x} - \mathbf{x}_{l,m,n}), \quad \mathbf{x}_{l,m,n} = (l\hat{x} + m\hat{y} + n\hat{z})\, d, \tag{16}$$

with $l, m, n = 0, \pm 1, \pm 2, \cdots$.

In an entire analogy with the one-dimensional case we consider the Bloch function

$$\psi_{\mathbf{k}}(\mathbf{x}) = \mu_{\mathbf{k}}(\mathbf{x}) \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{17}$$

with

$$\mu_{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{q}} C(\mathbf{q}, \mathbf{k}) \exp(i\mathbf{q} \cdot \mathbf{x}), \quad \mathbf{q} = \frac{2\pi}{d}(l\hat{x} + m\hat{y} + n\hat{z}), \quad l, m, n = 0, \pm 1, \pm 2, \ldots, \tag{18}$$

$$V(\mathbf{x})\, \mu_{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{q}} B(\mathbf{q}, \mathbf{k}) \exp(i\mathbf{q} \cdot \mathbf{x}), \tag{19}$$

and

$$B(\mathbf{q}, \mathbf{k}) = \frac{1}{d^3} \int_{V_c} V(\mathbf{x})\, \mu_{\mathbf{k}}(\mathbf{x}) \exp(-i\mathbf{q} \cdot \mathbf{x})\, d^3x, \tag{20}$$

where V_c is the volume of one cubic cell. Substituting (17), (18), and (19) into Schrödinger's equation, we get [see (7)]

$$C(\mathbf{q}, \mathbf{k}) = \frac{-B(\mathbf{q}, \mathbf{k})}{(\hbar^2/2m)(\mathbf{q} + \mathbf{k})^2 - E}, \tag{21}$$

and, after summing both sides of (21) with respect to $\mathbf{q}$, we obtain

$$\mu_{\mathbf{k}}(0) = -\frac{2m}{\hbar^2} \sum_{\mathbf{q}} \frac{B(\mathbf{q}, \mathbf{k})}{(\mathbf{q} + \mathbf{k})^2 - \beta^2}, \quad \beta = \left(\frac{2mE}{\hbar^2}\right)^{1/2}. \tag{22}$$

The above equation for β, as a function of $\mathbf{k}$ does not look very promising at first sight, since $B(\mathbf{q}, \mathbf{k})$ contains the unknown function $\mu_{\mathbf{k}}(\mathbf{x})$. However, Eq. (22) can still be solved, taking into account the following considerations:

(a) In the case of the potential (11) the domain of integration in (20) is a sphere of radius $a \to 0$, in which $V(\mathbf{x}) = -V_0$.

(b) Consider the expansion of the wavefunction (17), for $r < a$, as a linear combination of the energy eigenstates of *one* potential well (located at the center of the sphere). According to (14) and (15) this expansion can be written as

$$\psi_{\mathbf{k}} = \frac{A_{\mathbf{k}}}{\beta_l a} \left(\frac{\sin(\alpha r)}{\alpha r} + O(a^2)\right). \tag{23}$$

(c) With $\mathbf{k}$ restricted to the first Brillouin zone, we can take $\mu_{\mathbf{k}}(\mathbf{x}) = \psi_{\mathbf{k}}(\mathbf{x})$ in the integral (20), and still obtain Eq. (22) correct up to the first order in a. In fact, the contribution of the first order term due to the expansion $\exp(i\mathbf{k} \cdot \mathbf{r}) = 1 + i\mathbf{k} \cdot \mathbf{r} - (\mathbf{k} \cdot \mathbf{r})^2/2 + \cdots$ vanishes due to the symmetry of the cubic lattice.

Substituting (23) into $B(\mathbf{q}, \mathbf{k})$ [given in (22)] and performing the angular integrations, we get

$$\frac{1}{(2m/\hbar^2) V_0 a^2} = \frac{4\pi}{d^3} \sum_{\mathbf{q}} \frac{(1/\alpha a) \int_0^1 \sin(\alpha a x)[\sin(qax)/q]\, dx}{(\mathbf{k} + \mathbf{q})^2 - \beta^2} + O(a^2). \tag{24}$$

Now we can take $a \to 0$ in both members of Eq. (24). This limit, however, must be calculated with some care, since the result, of a naive computation, will be the simple identity $4/\pi^2 = 4/\pi^2$ [see (12)]. This is a consequence of the fact that in the potential (11) and (12) the physical information (the value of the binding energy) is contained in the first order term in a. What we must do then is to compare the coefficient of the terms of order a in both sides of (24). This is best done by adding and subtracting to its second member the expression

$$\frac{4\pi}{d^3} \sum_{\mathbf{q} \neq 0} \frac{(1/\alpha a) \int_0^1 \sin(\alpha a x) \sin(qax)\, dx}{q^3} \tag{25}$$

which does not depend on β or $\mathbf{k}$. We then get

$$\frac{1}{(2m/\hbar^2)V_0 a^2} = \frac{4\pi}{d^3}\left\{\frac{1}{\alpha a}\sum_{\mathbf{q}\neq 0}\left(\int_0^1 \sin(\alpha a x)\sin(qax)\,dx\right)\frac{1}{q}\left[\frac{1}{(\mathbf{q}+\mathbf{k})^2-\beta^2}-\frac{1}{q^2}\right]+\frac{1}{\alpha a}\sum_{\mathbf{q}\neq 0}\times\int_0^1 \frac{\sin(\alpha a x)\sin(qax)\,dx}{q^3}\right\}+\frac{32}{\pi^2 d^3}\,\frac{a}{k^2-\beta^2}, \tag{26}$$

where the last term corresponds to the term $\mathbf{q}=0$ on the rhs of (24). In the first term of the rhs of (26) we can take $a\to 0$, directly inside the summation sign, since the term inside the large brackets assures the convergence, giving a term of the order of a. The second term, which is independent of $\mathbf{k}$ and β gives contributions both to the order zero [cancelling the factor $4/\pi^2$ in the expansion of the first member of (26)], and to the order a. It can be calculated by adding and subtracting to the second member of (26) a similar term, with $\sin(\pi/2x)$ in the place of $\sin(\alpha a x)$. This last contribution to (26) can then be written as

$$\frac{4\pi}{d^3}\left(\frac{1}{\alpha a}\right)\left[\sum_{q\neq 0}\frac{1}{q^3}\int_0^1\left(\sin(\alpha a x)-\sin\frac{\pi x}{2}\right)\sin(qax)\,dx + \sum_{q\neq 0}\frac{1}{q^3}\int_0^1 \sin\frac{\pi}{2}\,\sin(qax)\,dx\right]. \tag{27}$$

The first term in (27) is of the order of a, since

$$\sin(\alpha a x)-\sin\left(\frac{\pi}{2}x\right)\approx\frac{2}{\pi}\quad\beta_l\, a x\cos\frac{\pi}{2}x\quad+\cdots\quad\text{[see (12)]}.$$

After this substitution, the summation in $\mathbf{q}$ is readily transformed into the following integral:

$$4\pi\,\frac{(2/\pi)\beta_l a}{2a}\int_0^\infty\frac{1}{q^3}\left[\int_0^1\cos\left(\frac{\pi}{2}x\right)\sin(qx)\,x\,dx\right]\frac{4\pi q^2\,dq}{(2\pi)^3} = \frac{8}{\pi^3}\left(1-\frac{2}{\pi}\right)\beta_l a. \tag{28}$$

The last term in (27) can also be transformed into an integral whose value is just the zero order term $4/\pi^2$. Here, however, we must remember that we need the right-hand side of (26) calculated up to the *first order in* a; that is, the *difference* between the series and the integral in the limit $a\to 0$ must be taken into account. Numerical calculations give the following result:

$$\frac{4\pi}{d^3}\left(\frac{1}{\alpha a}\right)\sum_{\mathbf{q}\neq 0}\frac{1}{q^3}\int_0^1 \sin\frac{\pi}{2}x\sin(qax)\,dx = \left(\frac{1}{\alpha a}\right)\left[4\pi\int_0^\infty\frac{1}{q^3}\int_0^1\sin\,\frac{\pi}{2}x\,\sin qx\,dx\right]\frac{4\pi q^2\,dq}{(2\pi)^3} - \frac{71.2}{2\pi^3}\,\frac{a}{d}+\cdots = \frac{4}{\pi^2}\{1-17.8\,a/d\pi^2-4\beta_l a/\pi^2+\cdots\}. \tag{29}$$

Substituting into (26) the information contained in (27), (28), and (29), we obtain the final result,

$$-\beta_l+\frac{8.9}{\pi d}=\frac{4\pi}{d^3}\left[\frac{1}{k^2-\beta^2}+\sum_{\mathbf{q}\neq 0}\left(\frac{1}{(\mathbf{q}+\mathbf{k})^2-\beta^2}-\frac{1}{q^2}\right)\right], \tag{30}$$

which gives an exact expression for the dispersion relation of the problem. If we consider negative energies $(\beta\to i\beta)$ and take the limit $d\to\infty$ it is easy to verify that the single well solution is regained. Really, taking $d\to\infty$ in (30) we get

$$-\beta_l=\frac{1}{(2\pi)^2}\int\left(\frac{1}{(\mathbf{q}+\mathbf{k})^2+\beta^2}-\frac{1}{q^2}\right)d^3q=-\beta \tag{31}$$

for all values of $\mathbf{k}$.

In conclusion: We have obtained an exact dispersion relation for a three-dimensional periodic structure made out of attractive square wells, in the limit case of infinite depth and zero range (with one bound s-state). It is our hope that the present work will be useful in the study of periodic potentials. Here, a thorough computer calculation of (30) (beyond the possibilities of the authors) exploring its dependence on the directionality of $\mathbf{k}$ and in the parameters d and β_l will provide the surfaces of constant energy, the density of states, etc.

This problem can certainly be extended in more than one direction: We can, for example, consider the same potentials forming lattices of different symmetries, or we can try a different limit process in order to obtain new bound states. A good deal can also possibly be done in the calculation of impurity states.

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On zeroes of the pion electromagnetic form factor

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We develop a general procedure for the location of possible zeroes of the pion form factor, which relies on interpolation theory for analytic functions. The zeroes are confined (in the unit disk) to regions bounded by (real) roots of algebraic equations and by algebraic curves. These regions depend both on the interpolation data and the class of functions, which is suitable for the physical problem.

1. INTRODUCTION

During the last few years considerable work has been performed on the rigorous phenomenology of the electromagnetic form factor of the pion, i. e., on the clarification of the exact implications and the significance of experimental data for the pion structure. This work has shown that the information implied by data depends on the theoretical framework in which they are considered and analyzed. This dependence follows from the fact, that one is usually lead to extremal problems in interpolation theory for analytic functions and that their solution generally depends on the class of functions used. The choice of a certain class means assuming some *a priori* restrictions on the pion form factor.

The question one has to answer is then whether certain numerical information is able to determine uniquely a function out of the chosen class; this function is then to be considered as the form factor. As a rule this does not happen, and so one has a whole subclass of functions which agree with the information, out of which one cannot select uniquely the form factor. Rigorous phenomenology therefore considers this whole subclass of functions, which means the whole class of models compatible with a certain experimental situation instead of only one model. The predictive power of experimental data is reflected in the structure of this subclass and increases generally with an increasing volume of data.

The quantities of main interest in the phenomenology of the pion form factor are, on the one side, the values allowed for the form factor and for its derivative and, on the other side, the values of certain integrals appearing through vacuum polarization in atomic level shifts and (anomalous) magnetic moments, in which the form factor enters quadratically.

In the first case one faces the mathematical problem of determining the region covered by the values assumed by the analytic functions of a certain class or by their derivatives at a given point of the complex plane. Sometimes it is, however, of interest to know precisely that part of the complex plane in which the functions of this considered class may take a preassigned value. This problem appears, for instance, in connection with the location of possible zeroes of the pion form factor, which are formally important for the use of the Omnès type representations. In relatively simple situations the problem of the location of zeros can be solved analytically, in the more complicated cases the analytic solutions are only the starting point of further numerical work.

The second case, i. e., the range of values of integrals as in the magnetic moment problem, can be correlated with the first one. These correlations turn out, however, to be increasingly difficult to study even numerically as the experimental data becomes more numerous and diverse.

In this paper we investigate the domains in the complex plane allowed for zeroes of the pion form factor in various experimental situations and theoretical frames. It turns out that if spacelike experimental data are exact and compatible with an entirely local or entirely global bound on the modulus in the timelike region, the region of location is bounded by algebraic curves. If spacelike data are not exact, the zeroes are confined to unions of such regions. Further, if the timelike bound is given locally on part of the cut and globally on the other, the analytic location of zeroes becomes an extremely difficult task, which so far remains, apart from the simplest situation, unsolved.

There have been previous considerations[1-5] related to zeroes of the pion form factor, but only part of them gave optimal rigorous results. We try to include all this work into a general point of view, which we hope to be both simple and effective.

2. ZEROES OF THE FORM FACTOR

The mathematical problems arising in the rigorous phenomenology of the pion form factor may be considered as interpolations with analytic functions of a bounded norm. The norm may be taken in the sense of the space H^∞ or H^2, or even in a more complicated manner.[6] We shall be interested in that situation where the values c_i, $i=1,\ldots,n$, assumed by the form factor $f(z)$ at some fixed points z_i and the value N of the norm do not determine uniquely the interpolating functions. All

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these functions can then be explicitly parametrized (in H^∞ and H^2) in terms of the interpolation data and an arbitrary bounded (in the chosen norm) analytic function $\psi(z)$, i.e.,

$$f(z)=\phi(z_1,\ldots,z_n;c_1,\ldots,c_n,z,\psi(z)),\qquad \|\psi(z)\|\le N. \tag{2.1}$$

In this paper we shall always assume that the complex t plane is mapped onto the unit disk of the z plane (with $t=0\rightarrow z=0$), so $F_\pi(t)\equiv f(z)$.

The requirement $f(z_0)=0$ at a certain point z_0 fixes $\psi(z)$ at $z=z_0$ (in fact uniquely, see Ref. 7),

$$\psi(z_0)=R(z_1,\ldots,z_n;c_1,\ldots,c_n;z_0). \tag{2.2}$$

On the other hand, the values which the bounded analytic function $\psi(z)$ can assume at z_0 are restricted to a definite region D_N in the complex plane for $\psi(z_0)$. (In H^∞ this is essentially due to the Schwarz lemma; see Ref. 1.) This region depends on z_0 and the norm N and will be described in Secs. 3 and 5. So, finally, we conclude that the form factor $f(z)$ may have a zero at $z=z_0$, if the value of R in (2.2) belongs to the region D_N.

We will solve this problem first for the H^∞ norm and then for H^2, since the mathematical techniques are different although the spirit of the procedures is in both cases the one outlined above.

3. ZEROES IN H^∞

It has been shown in Ref. 7 how the problem of interpolation for (real) analytic functions of the (Smirnov) class D[8] can be reduced due to a theorem of Szegö[1] to interpolation for functions $w(z)$ real analytic in the unit disc $|z|<1$ and bounded by $|w(z)|\le 1$. In short, this is done as follows: let $m(z=\exp(i\theta))$ be an upper bound for the modulus of the form factor $f(z)$ along the unit circle. Then $f(z)$ is bounded everywhere inside the unit disk by the outer function $E(z)$ with

$$\log E(z)=\frac{1-z^2}{\pi}\int_0^\pi d\theta\frac{\ln m(\exp(i\theta))}{1-2z\cos\theta+z^2}. \tag{3.1}$$

The form factor $f(z)$ can then be represented by

$$f(z)=w(z)E(z), \tag{3.2}$$

where $|w(z)|\le 1$ for $|z|\le 1$. In Ref. 7 the procedure of interpolation for the bounded functions $w(z)$ is given. It is based on an explicit parametrization of the interpolating functions $w(z)$ in terms of an arbitrary bounded real function.

It is known[1] that a function $\bar{w}(z)$, analytic in $|z|<1$ and obeying

$$|\bar{w}(z)|\le 1, \tag{3.3}$$

$$\bar{w}^*(z)=\bar{w}(z^*), \tag{3.4}$$

can take at a point z only values in (a lens representing) the intersection of the two disks bounded by the circles going through the points $(1,-1,z)$ and $(1,-1,-z)$, respectively, i.e., in the region described by the inequalities

$$-1\le\bar{w}(z)\le 1\quad(z=z^*) \tag{3.5}$$

and

$$\begin{aligned}&|\bar{w}|^2-\frac{\bar{w}-\bar{w}^*}{z-z^*}(|z|^2-1)-1\le 0\quad(z\ne z^*),\\ &|\bar{w}|^2+\frac{\bar{w}-\bar{w}^*}{z-z^*}(|z|^2-1)-1\le 0\quad(z\ne z^*).\end{aligned} \tag{3.6}$$

A derivation of (3.6) (simpler than that given in Ref. 1) goes as follows: One starts with interpolating functions obeying (3.3) fulfilling the requirement $\bar{w}(z_0)=b$,

$$\bar{w}(z)=\frac{b+[(z-z_0)/(1-zz_0^*)]w_1(z)}{1+[(z-z_0)/(1-zz_0^*)]b^*w_1(z)}. \tag{3.7}$$

Analyticity of $\bar{w}(z)$ for $|z|\le 1$ implies $|w_1(z)|\le 1$. Consistency with (3.4) requires $\bar{w}(z_0^*)=b^*$, which fixes $w_1(z)$ at z_0^*,

$$w_1(z_0^*)=\frac{b^*-b}{1-b^{*2}}\frac{1-z^{*2}}{z^*-z_0}. \tag{3.8}$$

The condition $|w_1(z_0^*)|\le 1$ has the solutions (3.6) for $b=\bar{w}(z)$. One observes that the functions $w(z)=\pm(z-c)/(1-cz)$ with c real, $-1\le c\le 1$, form the boundary of the lens (3.6).

It is noteworthy that the region (3.6) is not only symmetric with respect to the origin, but also with respect to the real axis.

As an illustration of our procedure for locating the zeroes we rederive the domain given in Ref. 1 from the normalization of the form factor. Normalization is equivalent to imposing

$$w(0)=a\quad(0<a\le 1) \tag{3.9}$$

on $w(z)$. If $a<1$, then form factors normalized at $z=0$ have a factor [see (3.2)]

$$w(z)=\frac{a+z\bar{w}(z)}{1+az\bar{w}(z)},\quad |\bar{w}(z)|\le 1,\quad \bar{w}^*(z)=\bar{w}(z^*) \tag{3.10}$$

[compare (3.7) with $z_0=0$]. If z is a zero of the form factor, then $w(z)=0$ and

$$\bar{w}(z)=-a/z, \tag{3.11}$$

which can be true only if $(-a/z)$ inserted for $\bar{w}(z)$ obeys (3.6). One of these inequalities is trivially satisfied, whereas the other gives

$$|z|\ge a^{1/2}\quad(z\ne z^*). \tag{3.12}$$

Real zeroes can come, according to (3.5), closer to the origin,

$$|z|\ge a\quad(z=z^*). \tag{3.13}$$

If we now go over to several interpolation points, then the rational function $-a/z$ changes into a more complicated one. We shall not treat the general case, but only the simplest one which already shows the general features, namely that with one interpolation point $w(x)=b$ besides the origin $[w(0)=a]$.

In this case, the interpolation of $w(z)$ is obtained by iterating (3.7) with the result that[7,9]

$$\bar{w}(z)=\left[\frac{1}{x}\frac{b-a}{1-ab}+\frac{z-x}{1-xz}\bar{\bar{w}}(z)\right]\left[1+\frac{1}{x}\frac{b-a}{1-ab}\frac{z-x}{1-xz}\bar{\bar{w}}(z)\right]^{-1}. \tag{3.14}$$

One must require

$$|\bar{w}(x)| = \left|\frac{1}{x}\frac{b-a}{1-ab}\right| \leq 1 \tag{3.15}$$

in order that $\bar{w}(z)$ should not have poles for $|z| \leq 1$. The interpolation is unique, only if the equal sign in (3.15) applies.

If z is a zero of $w(z)$ $[w(z)=0]$ and if

$$z \neq x, \quad -\frac{a}{x}\frac{b-a}{1-ab},$$

then

$$\bar{\bar{w}}(z) = -\left[\left(a+\frac{z}{x}\frac{b-a}{1-ab}\right)\bigg/\left(z+\frac{a}{x}\frac{b-a}{1-ab}\right)\right]\frac{1-xz}{z-x}. \tag{3.16}$$

The point

$$z = -\frac{a}{x}\frac{b-a}{1-ab},$$

of modulus smaller than a, cannot be a zero because of (3.13) and (3.15); $z=x$ is a zero only if $b=0$. The position of real zeroes is delimited by the inequalities

$$(z-x)\left(z+\frac{a}{x}\frac{b-a}{1-ab}\right)\left[(z-x)\left(z+\frac{a}{x}\frac{b-a}{1-ab}\right) - (1-xz)\left(a+\frac{z}{x}\frac{b-a}{1-ab}\right)\right] \geq 0, \tag{3.17}$$

$$(z-x)\left(z+\frac{a}{x}\frac{b-a}{1-ab}\right)\left[(z-x)\left(z+\frac{a}{x}\frac{b-a}{1-ab}\right) + (1-xz)\left(a+\frac{z}{x}\frac{b-a}{1-ab}\right)\right] \geq 0,$$

following from (3.5). The location of complex zeroes is obtained by inserting (3.16) into (3.6) applied to $\bar{\bar{w}}(z)$. This leads to the inequalities

$$x(1-bx)|z|^4 + x(b-x)(z+z^*)|z|^2 - [(1+ax^3)b - (a+x^3)]|z|^2 - ax(1-bx)(z+z^*) - ax(b-x) \geq 0, \tag{3.18}$$

$$x(1+bx)|z|^4 - x(b+x)(z+z^*)|z|^2 + [(1-ax^3)b - (a-x^3)]|z|^2 + ax(1+bx)(z+z^*) - ax(b+x) \geq 0.$$

It thus turns out that the position of real zeroes is essentially determined by the roots of two equations of the second degree, whereas the complex zeros lie in a region bounded by algebraic curves of the fourth degree. This reflects the general situation: For n interpolation points $\bar{w}(z)$ in (3.5), (3.6) is generally the quotient of two polynomials of the nth degree. Therefore, (3.5) leads essentially to inequalities of degree n and (3.6) leads to inequalities of degree $2n$.[10] The location of zeroes then depends (in the real case) on the position of real roots of two algebraic equations of degree n and (in the complex case) on the topology of two algebraic curves of degree $2n$. Their qualitative discussion is a problem of algebraic geometry[11–13] which we shall, however, not pursue.

As a further illustration of the method we give in the following the solution to the problems considered by Cronström[3] and by Bonneau and Martin,[4] which are in fact limiting cases of interpolations in one or two points.

4. SOME LIMITING SITUATIONS

A. Example 1 (Cronström[3])

In Ref. 3, Cronström uses the p wave threshold condition to write a sum rule, which tells (in principle) whether zeroes may exist or not. In our formulation the sum rule is obtained by writing the p wave condition for $f(z)=w(z)E(z)$ [Eq. (3.2)], $0=f'(z=1)=w'(1)E(1)+w(1)E'(1)$. Therefore,

$$-\frac{E'(1)}{E(1)} = +\frac{w'(1)}{w(1)} \equiv +\kappa. \tag{4.1}$$

Using (3.1) and (4.1) for functions with $m(\exp(i\theta)) = |f(\theta)| = |F(t)|$ we obtain the condition $\kappa \geq 0$ and the expression of κ in terms of the modulus of the form factor

$$\kappa = \frac{1}{\pi}\int_0^\pi \frac{d\theta}{(1-\cos\theta)}\ln\left|\frac{f(\theta)}{f(0)}\right| = \frac{m}{\pi}\int_{4m^2}^\infty \frac{dt}{(t-4m^2)^{3/2}}\ln\left|\frac{F(t)}{F(4m^2)}\right|, \tag{4.2}$$

where we used $t=4m^2/\cos^2(\theta/2)$. If $\kappa=0$, then the form factor has no zeros. The condition $\kappa \geq 0$ for (4.2) together with the significance of $\kappa=0$ are the sum rule of Ref. 3.

We now inquire about the allowed domain for zeroes if $\kappa > 0$. This can be regarded as a limiting case of the interpolation with $w(x)=b$ in the limit that b approaches ± 1 as $x \to 1$ so that

$$b \approx w(1) + w'(1)(x-1) = \pm 1 \pm \kappa(x-1). \tag{4.3}$$

To this end, we use the representation (3.7) applied to $w(z)$,

$$w(z) = \left(b + \frac{z-x}{1-zx}\bar{w}(z)\right)\bigg/\left(1+\frac{z-x}{1-zx}b\bar{w}(z)\right). \tag{4.4}$$

The requirement of a zero at z, $w(z)=0$, yields first

$$\bar{w}(z) = -b\frac{1-xz}{z-x}. \tag{4.5}$$

The inequality (3.5) gives then in the limit $x \to 1$

$$z \leq \frac{\kappa-1}{\kappa+1} \tag{4.6}$$

for real zeroes, whereas complex zeroes are delimited by

$$(2+\kappa)|z|^2 - \kappa(z+z^*) + (\kappa-2) \geq 0 \tag{4.7}$$

[see (3.6)]. Equation (4.7) represents a disk bounded by a circle situated symmetric with respect to the real axis and going through the points $z=(\kappa-2)/(\kappa+2)$ and $z=1$.[14]

B. Example (Bonneau-Martin[4])

The authors of Ref. 4 also use, in addition to Cronström's assumptions, the normalization at $z=0$. Their problem may be considered as the interpolation with two points $z=0, x$, $w(0)=a$, $w(x)=b$, in the limit (4.3) for x and b. The constraint on b is in this case stronger than in Example 1 (i.e., $b \leq 1$, $\kappa \geq 0$). Namely, if we take the upper sign in (4.3), it follows from (3.15) that

$b \leq (a+x)/(1-ax)$, $x>0$; this leads (in the limit $x \to 1$) to the additional constraint for κ,

$$\kappa \geq \frac{1-a}{1+a}. \tag{4.8}$$

In the limit, the (nontrivial) inequalities for the real zeroes become

$$(z+a)(\kappa z^2 + [2(1-a) - (1+a)\kappa]z + a\kappa) \leq 0 \tag{4.9}$$

and those for complex zeroes become

$$(1+\kappa)|z|^4 + (1-\kappa)(z+z^*)|z|^2 - [3(1-a) - (1+a)\kappa]|z|^2$$
$$- a(1+\kappa)(z+z^*) - a(1-\kappa) \geq 0, \quad |z|^2 - a \geq 0. \tag{4.10}$$

For the lower sign in (4.3) we obtain new results (because the sign of a is fixed). The limitation on κ from (3.15) is $b \geq (a-x)/(1-ax)$, i.e.,

$$\kappa \geq \frac{1+a}{1-a} \tag{4.11}$$

and the (nontrivial) inequalities are formally obtained by the substitution $a \to -a$ in (4.9) and (4.10).

As long as one has no information on the sign of b and κ obeys $\kappa \geq (1+a)/(1-a)$, one has to take the union of the domains described by (4.9)—(4.10) and their counterparts with $a \to -a$ as the region allowed for zeroes.

5. ZEROES IN H^2

Due to the inclusion of the class H^∞ into H^2, one expects that the region allowed for zeroes of (real) analytic functions $h(z) \in H^2$, in $|z|<1$, with $\|h\| \leq 1$,

$$\|h\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(\tau)|^2 d\theta, \quad \tau = \exp(i\theta), \tag{5.1}$$

is larger than the corresponding region for functions obeying $|h(z)| \leq 1$, $|z|<1$. The quantitative determination of the new region again follows the procedure outlined in Sec. 2. One first determines [in analogy to (3.5) and (3.6)] the values allowed for a real analytic function $h(z)$ at a fixed point z, if its norm is bounded by

$$\|h\|^2 \leq h^2. \tag{5.2}$$

In order to do this, we consider the interpolation problem (in H^2) for prescribed values $h(\xi)$, $h(\xi^*)$, where $|\xi|<1$. The interpolating functions can be determined according to a technique applied in Ref. 15. They are

$$h(z) = \frac{1-\xi^2}{1-\xi z} h(\xi) + \frac{z-\xi}{1-\xi z} h_2(z) \quad (\xi = \xi^*), \tag{5.3}$$

$$h(z) = \frac{1-|\xi|^2}{(\xi-\xi^*)(1-\xi z)(1-\xi^* z)}$$
$$\times[(z-\xi^*)(1-\xi^2)h(\xi) - (z-\xi)(1-\xi^{*2})h(\xi^*)]$$
$$+ \frac{z-\xi}{1-\xi z}\frac{z-\xi^*}{1-\xi^* z} h_2(z) \quad (\xi \neq \xi^*), \tag{5.4}$$

where $h_2(z) \in H^2$ is arbitrary. Equations (5.3) and (5.4) are analogous to (3.7). Requiring $h(\xi^*) = h^*(\xi)$, $h_2^*(z) = h_2(z^*)$ ensures real analyticity.

The decompositions (5.3) and (5.4) are orthogonal[15] in the sense that

$$\|h\|^2 = (1-\xi^2)h^2(\xi) + \|h_2\|^2 \quad (\xi = \xi^*), \tag{5.5}$$

$$\|h\|^2 = \|h_2\|^2 + \frac{1-|\xi|^2}{|\xi-\xi^*|^2}[2(1-\xi^2)(1-\xi^{*2})|h(\xi)|^2$$
$$- (1-|\xi|^2)(1-\xi^2)h^2(\xi) - (1-|\xi|^2)(1-\xi^{*2})h^{*2}(\xi)]$$
$$(\xi \neq \xi^*). \tag{5.6}$$

Applying (5.2) to (5.5) and (5.6) under the condition that $\|h_2\|^2 \geq 0$ gives the constraints

$$h^2(\xi) \leq \frac{h^2}{1-\xi^2} \quad (\xi = \xi^*), \tag{5.7}$$

$$2(1-\xi^2)(1-\xi^{*2})|h(\xi)|^2 - (1-|\xi|^2)(1-\xi^2)h^2(\xi)$$
$$- (1-|\xi|^2)(1-\xi^{*2})h^{*2}(\xi) \leq \frac{|\xi-\xi^*|^2}{1-|\xi|^2}h^2 \quad (\xi \neq \xi^*). \tag{5.8}$$

The extremal functions [which realize equality in (5.7) and (5.8)] are obtained by putting $h_2(z) \equiv 0$ in (5.3) and (5.4). Equations (5.7) and (5.8) are analogous to (3.5) and (3.6).

The boundary of the region (5.8) is a curve of the second degree, which is an ellipse symmetric with respect to the origin, but not with respect to the real axis, and with both semiaxes strictly smaller than $h(1-|\xi|^2)^{-1/2}$. Consequently, region (5.8) is strictly contained inside the set of admissible values at ξ of all possible functions (not only real), holomorphic in the unit disk, with $\|h\|$ bounded by h. The latter set is obtained from a representation similar to (5.3) and is the disk of radius $h(1-|\xi|^2)^{-1/2}$.

As before, we first consider the interpolation in a single point $h(0) = \alpha$. From (5.7) we get $\alpha^2 \leq h^2$. If $\alpha^2 < h^2$, the interpolating functions are [see (5.3) with $\xi = 0$]

$$h(z) = \alpha + z h_1(z). \tag{5.9}$$

If $z = z_0$ is now a zero of $h(z)$, then $h_1(z)$ is constrained at $z = z_0$,

$$h_1(z_0) = -\alpha/z_0, \tag{5.10}$$

but the parameter $h_1(z_0)$ cannot take any value. It has to obey the constraints (5.7) or (5.8) [with $h(z)$ replaced by $h_1(z)$, ξ by z_0, and h^2 by $h_1^2 = h^2 - \alpha^2$]. Thus (5.7) gives for real z_0's,

$$-a \leq z_0 \leq a, \quad a = |\alpha|/h \tag{5.11}$$

and (5.8) for complex z_0's,

$$|z_0| \geq a^{1/2}. \tag{5.12}$$

These inequalities represent a region of the same form as (3.13) and (3.12).[16]

The general interpolation problem in $n+1$ real points x_k is solved by an expansion of the form[15]

$$h(z) = \sum_{k=0}^{n} a_k r_k(z) + B_{n+1}(z)h_{n+1}(z), \tag{5.13}$$

with suitably chosen functions $r_k(z)$ and $B_{n+1}(z)$ [see Ref. 15 or for $n=1$ the discussions after (5.14)]. The expansion is orthogonal in the sense that

$$||h||^2 = \sum_{k=0}^{n} a_k^2 + ||h_{n+1}||^2 \le h^2. \tag{5.14}$$

As long as $\sum_{k=0}^n a_k^2 < h^2$, $h_{n+1}(z)$ is an arbitrary function with the norm,

$$||h_{n+1}||^2 \le h_{n+1}^2 \equiv h^2 - \sum_{k=0}^{n} a_k^2. \tag{5.15}$$

A zero $z = z_0$ of $h(z)$ requires that

$$h_{n+1}(z_0) = -B_{n+1}^{-1}(z_0) \sum_{k=0}^{n} a_k r_k(z_0). \tag{5.16}$$

The regions allowed for zeroes are obtained by again using (5.7) and (5.8) [with $h(z)$ replaced by $h_{n+1}(z_0)$, ξ by z_0, and h^2 by h_{n+1}^2]. Again we get the boundary of the region allowed for zeroes from algebraic equations and curves of degrees which increase with the number of interpolation points.

Finally we discuss in this section the interpolation problem with two points $h(0)=\alpha$, $h(x)=\beta$. Equations (5.13) and (5.14) read, in this case,

$$h(z) = \alpha + \frac{z}{x}\frac{1-x^2}{1-xz}(\beta-\alpha) + z\frac{z-x}{1-xz}h_2(z), \tag{5.17}$$

$$||h||^2 = \alpha^2 + \frac{1-x^2}{x^2}(\beta-\alpha)^2 + ||h_2||^2 \le h^2. \tag{5.18}$$

At a zero $z = z_0$, $h_2(z_0)$ assumes the value

$$h_2(z_0) = \frac{1}{z_0(z_0-x)}\frac{1}{x}[\alpha(z_0-x) - \beta(1-x^2)z_0]. \tag{5.19}$$

This value must be consistent with functions $h_2(z)$ with

$$||h_2||^2 \le h_2^2 = h^2 - \alpha^2 - \frac{1-x^2}{x^2}(\beta-\alpha)^2. \tag{5.20}$$

The general inequalities (5.7) and (5.8) yield

$$[x^2(h^2-\alpha^2) - (1-x^2)(\beta-\alpha)^2]z^2(z-x)^2 - (1-z^2)[\alpha(z-x) - \beta(1-x^2)z]^2 \ge 0 \tag{5.21}$$

for real zeroes, and

$$\begin{aligned}(1-|z|^2)\{(\gamma|z|^2 - \alpha x(z+z^*) + x^2)[\gamma|z|^4 - \gamma x(z+z^*)|z|^2 \\ + (\gamma+\alpha x^2)|z|^2 - \alpha x(z+z^*) + \alpha x^2] + \beta x(1-x^2)|z|^2 \\ \times[\beta x(1-x^2)|z|^4 + \gamma(z+z^*)|z|^2 - (\gamma-\alpha)x|z|^2 \\ - \alpha x(z+z^*)^2 - \alpha x^2(z+z^*)]\} - |z|^4|z-x|^4 \\ \times[x^2(h^2-\alpha^2) - (1-x^2)(\beta-\alpha)^2] \le 0,\end{aligned} \tag{5.22}$$

$$\gamma = \alpha - \beta(1-x^2)$$

for complex zeroes.

6. ZEROES FOR AN INTERMEDIATE CLASS ($H^{2,\infty}$) OF FUNCTIONS

By $H^{2,\infty}$ we denote the class of functions $h(z)$, real and analytic in the unit disk $|z|<1$, belonging to the class D and obeying the conditions

$$|h(\tau)| \le s(\theta), \quad \theta\in\Gamma. \tag{6.1}$$

$$\frac{1}{2\pi}\int_{C\Gamma}|h(\tau)|^2 d\theta \le h_C^2\omega, \quad \omega = \frac{1}{2\pi}\int_{C\Gamma} d\theta \tag{6.2}$$

on the arc Γ of the unit circle and on its complement $C\Gamma$, with $\ln s(\theta)\in L^1$. For this class, the solutions to the interpolation problem are not known analytically. Therefore, we cannot apply the same scheme of reasoning as before. In fact, we are only able to solve the problem in the simplest case of interpolation in a single point, $h(0)=\alpha$ by a rather special procedure.

First we again use the decomposition (3.2) for $h(z)$, i.e., $h(z) = w(z)E(z)$. From $h(0)=\alpha$ follows

$$w(0) = \alpha E^{-1}(0) \tag{6.3}$$

for the function $w(z)$ to which we shall apply the results developed in Sec. 3. Interpolation is possible if conditions (6.1) and (6.2) are compatible with $w^2(0)\le 1$. On the other hand, $E(0)$ is also bounded from above.[5,15] Indeed, according to (3.1), (6.1), and (6.2),

$$\begin{aligned}\ln E(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|h(\exp(i\theta))|\,d\theta \\ &\le \frac{1}{2\pi}\int_\Gamma \ln s(\theta)\,d\theta + \frac{1}{2\pi}\int_{C\Gamma}\ln|h(\exp(i\theta))|\,d\theta \\ &\le \frac{1}{2\pi}\int_\Gamma \ln s(\theta)\,d\theta + \omega\ln h_C \equiv \ln E_0(0).\end{aligned} \tag{6.4}$$

In the last step we used Jensen's inequality

$$\frac{1}{2\pi}\int_{C\Gamma}\ln|h(\exp(i\theta))|^2\frac{d\theta}{\omega} \le \ln\int_{C\Gamma}|h(\exp(i\theta))|^2\frac{d\theta}{2\pi\omega}. \tag{6.5}$$

Equality in (6.4) is attained for an outer function $E_0(\theta)$ with $|E_0(\theta)| = s(\theta)$ on Γ and $|E_0(\theta)| = h_C = \text{const}$ on $C\Gamma$.

From (6.3) follows that zeroes may be present if $E_0(0) > |\alpha|$; these lie in regions given by (3.12) and (3.13), with

$$a = |\alpha|E^{-1}(0). \tag{6.6}$$

It is, of course, not essential that the interpolating point is $x=0$, but as in the case of H^2 [16] the forbidden region for zeroes is not simply the conformal transformation of the disk $|z|\le a^{1/2}$. Rather, one has to take the disk $|z| < a_x^{1/2}$, where

$$a_x = |\alpha|F^{-1}(x), \tag{6.7}$$

and $F(z)$ is the outer function with boundary values

$$|F(\exp(i\theta))| = \begin{cases} s(\theta), & \theta\in\Gamma, \\ h_C\left(\dfrac{\omega}{\omega(x)}\right)^{1/2} p^{1/2}(\theta), & \theta\in C\Gamma,\end{cases} \tag{6.8}$$

$$p(\theta) = \frac{1-x^2}{1+x^2-2x\cos\theta}, \qquad \omega(x) = \frac{1}{2\pi}\int_{C\Gamma} p(\theta)\,d\theta. \tag{6.9}$$

7. COMPARISON OF THE REGIONS OF ZEROES FOR THE THREE CLASSES OF FUNCTIONS

In this section we shortly compare the allowed regions for zeroes for the three classes of functions considered. To this end we have to relate the bounds $m(\theta)$ (Sec. 3, H^∞ problem), $s(\theta)$, h_C^2 (Sec. 6, $H^{2,\infty}$ problem), and h^2

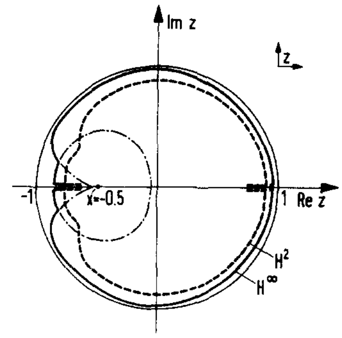

FIG. 1. Allowed domains for zeroes in the H^∞ and H^2 case.

(Sec. 4, H^2 problem) by

$$m(\theta)=s(\theta), \quad \theta\in\Gamma, \tag{7.1}$$

$$h_C^2\omega=\frac{1}{2\pi}\int_{C\Gamma} m^2(\theta)\,d\theta, \tag{7.2}$$

$$h^2=\frac{1}{2\pi}\int_{-\pi}^{\pi} m^2(\theta)\,d\theta. \tag{7.3}$$

Then the classes of functions are ordered as

$$H^\infty\subset H^{2,\infty}\subset H^2 \tag{7.4}$$

and the allowed regions for zeroes obey the same ordering. For interpolation in one point $h(x)=\alpha$ this is equivalent to

$$\alpha^2E^{-2}(x)\geqslant\alpha^2F^{-2}(x)\geqslant\alpha^2\frac{1-x^2}{h^2} \tag{7.5}$$

[where $E(x)$ and $F(x)$ are given by (3.1) and (6.8), respectively]. Equality on the left-hand side of (7.5) is reached only if $m^2(\theta)=h_C^2p(\theta)\omega/\omega(x)$, $\theta\in C\Gamma$; the right-hand side becomes an equality only if $m^2(\theta)=h^2p(\theta)$, $\theta\in\Gamma$ with $h^2=h_C^2\omega/\omega(x)$.

It may be worth mentioning that in the H^∞ and H^2 case the zeroes on the complex boundary of the region allowed (for zeroes by interpolations in n points), come from functions having $n+1$ zeroes; the zeroes at the endpoints of the real intervals come from functions with n zeroes. [In the H^∞ case this follows directly from the remark made after Eq. (3.8).]

8. COMMENTS

The procedure developed in this paper permits one in principle to determine the region of zeroes allowed for interpolating functions (in H^∞), compatible with a certain highest value χ_M^2 of χ^2, in the sense of Ref. 7b. The practical difficulty, which can, however, be ultimately overcome on a numerical level if there is a strong enough motivation for performing the computation, lies in the determination of all interpolation problems compatible with χ_M^2 (i.e., of the points of the set D_n of Ref. 7b lying in the ellipsoid $\chi^2\leqslant\chi_M^2$). The situation in H^2 is similar, maybe to a certain extent simpler.

The results of this paper do not imply statements on the actual existence of zeroes, only on the fact that the existence cannot be excluded by a certain amount of (experimental) information. One can, of course, ask if one can manage to interpolate with functions having no zeroes and thereby answer the question if zeroes are necessary. In the case of H^∞ one has then to allow [for an arbitrary $m(\theta)$] for the appearance of so called singular inner functions[17] in order to have enough flexibility. In H^2 or $H^{2,\infty}$, one can even interpolate reasonably well with outer functions.[5]

The procedures presented in this paper also remain true in the situation where information is available on the phase of the form factor, if one applies the trick of the Omnês function.[18,19]

Figure 1 shows the allowed domain for zeroes in the cases H^∞ and H^2, if the value of the form factor is known at one experimental point $z=-0.4991$, apart from normalization [Eqs. (3.17), (3.18), (5.21), (5.22)]. The bound $m(\theta)$, needed in the H^∞ problem, and also for the computation of h^2 [Eq. (7.3)] was taken from Ref. 20.

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On intelligent spin states

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In this paper we give a more compact representation of the intelligent spin states defined by Aragone, Guerri, Salamó, and Tani. Using this new representation, we discuss the differences between minimum uncertainty states, coherent Bloch spin states and intelligent states. The evolution of these states under a particular time dependent Hamiltonian is studied, showing the relevance of the noncompact subgroup K of the Lorentz group. Finally we analyze the radiative properties connected with the intelligent states for a pointlike medium. The main results are: (I) they have a nonvanishing dipole moment (as the Bloch states) and (II) the proper intelligent states give a spontaneous emission intensity which is different from the one provided by the Bloch states.

1. INTRODUCTION

In a recent paper, Aragone, Guerri, Salamó and Tani,[1] constructed the intelligent spin states as those which satisfy the Heisenberg equality for the angular momentum operators. Many questions of physical interest were not discussed there.

The purpose of this work is threefold: (a) to give a clear distinction between intelligent states, minimum uncertainty states, and Bloch states; (b) to show a more compact representation of intelligent states; and (c) to determine the time evolution and some radiative properties of two different systems initially set in an intelligent state.

This article is organized as follows: In the next section we give a more compact expression for the intelligent states than the original, and we discuss the connection between intelligent states and coherent spin states.[2-4] We will show the difference between the $2j+1$ intelligent states and the $2j+1$ states obtained by applying the two-parameter rotation $R(\tau)$, defined by Arecchi, Courtens, Gilmore, and Thomas (ACGT),[4] to the standard Wigner states $|j,m\rangle$.

Section 3 is devoted to analyzing the difference between minimum uncertainty states, atomic coherent spin states, and intelligent states. We calculate the expectation values of J_x, J_y, J_z and their quadratic deviations for intelligent states, using the technique of generating functionals, whose details are presented in Appendix A.

In Sec. 4 we present the explicit evolution of a nonrelativistic high spin system, initially set in an intelligent state, immersed in a magnetic atmosphere.

We also estimate the macroscopic dipole and emission rates of a pointlike laser.

In the last section we make some comments and remarks.

2. COHERENT SPIN STATES AND INTELLIGENT STATES

The SU(2) algebra is defined by the usual commutation relations,

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad i,j,k=1,2,3, \tag{1a}$$

or, in terms of the ladder operators $J_\epsilon \equiv J_1 + i\epsilon J_2$ ($\epsilon = +1, -1$) and J_3, by

$$[J_\epsilon, J_{-\epsilon}] = 2\epsilon J_3, \quad [J_3, J_\epsilon] = \epsilon J_\epsilon. \tag{1b}$$

The $(2j+1)$-dimensional Hilbert space spanned by the eigenvectors of J^2 and J_3 (labeled by $|j,m\rangle$ or by $|m\rangle$)

$$J^2|j,m\rangle = j(j+1)|j,m\rangle, \quad J_3|j,m\rangle = m|j,m\rangle, \tag{2}$$

is denoted by H_j.

A useful formula for computation is

$$(j+\epsilon m)!\,|m\rangle = \binom{2j}{j+\epsilon m}^{-1/2} J_\epsilon^{j+\epsilon m}|-\epsilon j\rangle. \tag{3}$$

The ladder operators are useful in order to construct[4] the atomic coherent spin states or Bloch states $|\tau\rangle$,

$$\begin{aligned}|\tau\rangle &\equiv (1+|\tau|^2)^{-j}\exp(\tau J_+)|-j\rangle \\ &= \exp(\tau J_+)\exp[\ln(1+|\tau|^2)J_3]\exp(-\tau^* J_-|-j\rangle \\ &\equiv R(\tau)|-j\rangle,\end{aligned} \tag{4}$$

where $\tau \equiv \tan\frac{1}{2}\theta\exp(-i\varphi)$, $\theta \in [0, 2\pi)$. $R(\tau)$ represents a rotation through an angle θ about the axis $\hat{a} \equiv \sin\varphi\,\hat{e}_1 - \cos\varphi\,\hat{e}_2$.

Two different Bloch states are not necessarily orthogonal. In fact their inner product is

$$\langle\tau_1|\tau_2\rangle = (1+|\tau_1|^2)^{-j}(1+|\tau_2|^2)^{-j}(1+\tau_1^*\tau_2)^{2j}. \tag{5}$$

The expression of the atomic coherent spin given in Eq. (4) is analogous to that for Glauber states, $|z\rangle = N(z)\exp(za+)|0\rangle$, where the operator $\exp(za^+)$ is applied to the ground state of the harmonic oscillator.[5,6]

The Glauber states satisfy the Heisenberg equality $\Delta x\,\Delta p = \frac{1}{2}$. Therefore, one could also enquire whether the states $|\tau\rangle$ satisfy the Heisenberg equality for the SU(2) algebra,

$$\Delta J_1^2\,\Delta J_2^2 = \tfrac{1}{4}\langle J_3\rangle^2 \tag{6}$$

or, what are all the states $|w\rangle$ which verify Eq. (6)?

For a careful analysis of Eq. (6), let us define two homogeneous functionals of zeroth order, the uncertainty functional $I(\psi)$,

$$I(\psi) \equiv \langle\psi|\Delta J_1^2|\psi\rangle\langle\psi|\Delta J_2^2|\psi\rangle\langle\psi|\psi\rangle^{-2}, \tag{7a}$$

and the half-commutator squared functional $C(\psi)$,

$$C(\psi) \equiv 4^{-1}|\langle\psi|[J_1, J_2]|\psi\rangle|^2\langle\psi|\psi\rangle^{-2}. \tag{7b}$$

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In terms of these functionals the Heisenberg equality looks like

$$I(\psi)=C(\psi). \tag{6'}$$

We shall refer to $|u\rangle$ as a minimum (maximum, stationary) uncertainty state if $I(\psi)$ has a local minimum (maximum, stationary point) at $|\psi\rangle=|u\rangle$. Moreover, $|w\rangle$ shall be called an intelligent state if $I(w)=C(w)$.

Therefore, in principle we have three different kind of states related to the angular momentum algebra: the Bloch states $|\tau\rangle$, the intelligent states $|w\rangle$, and the minimum uncertainty states $|u\rangle$.

It is worthwhile to stress that, in the case of the Heisenberg algebra $\{x,p,[x,p]=i\}$, the corresponding functional $C(\psi)=4^{-1}\langle\psi|[x,p]|\psi\rangle^2\cdot\langle\psi|\psi\rangle^{-2}$ has a constant value: $\frac{1}{4}$. Therefore, any intelligent state of this algebra must be a minimum uncertainty state too.

However, this property does not necessarily hold for other algebras where $C(\psi)$ is not a constant number, as in the case of SU(2).

It is a well established property of quantum mechanics[7] that all the intelligent spin states are given by the set of all the states that satisfy the linear equation,

$$J_\alpha|w\rangle\equiv(J_1-i\alpha J_2)|w\rangle=(\langle J_1\rangle_w-i\alpha\langle J_2\rangle_w)|w\rangle\equiv w|w\rangle, \tag{8a}$$

where α is a real number. Defining $\gamma_\epsilon\equiv\frac{1}{2}(1-\epsilon\alpha)$, $\epsilon=\pm1$, J_α can also be written as a linear combination of the ladder operators,

$$J_\alpha=\gamma_+J_++\gamma_-J_-=\gamma_\epsilon J_\epsilon, \tag{8b}$$

leading to the explicit expression of the intelligent spin states shown in Ref. 1. With the present notation they can be written as

$$\begin{aligned}&|w_N(\tau_\alpha)\rangle=\hat{a}_N\sum_{l=0}^{N}\binom{N}{l}(2j-l)!\,(-2\tau_\alpha J_+)^l|\tau_\alpha\rangle,\quad 0\le N\le 2j,\\&\tau_\alpha^2=\gamma_+\gamma_-^{-1},\quad w_N\equiv2\gamma_+\tau_\alpha^{-1}(j-N),\end{aligned} \tag{9}$$

where $\hat{a}_N$ is a normalizing factor which shall be determined later on.

We note that for a given τ_α we have $2j+1$ different eigenvalues w_N, as we see from the explicit form of w_N. Therefore, the set $\{|w_N(\tau_\alpha)\rangle\}$ is for a given α, $|\alpha|\neq1$, a basis of H_j.[8]

It is also worthwhile to point out that, due to the fact that α must be real (therefore $\gamma_+\gamma_-^{-1}$ is real too), $\tau_\alpha=\pm(\gamma_+/\gamma_-)^{1/2}$ can only be real or pure imaginary.[9]

However, we could think of enlarging the definition (9) for $|w_N(\tau)\rangle$ to any complex number without giving raise to any mathematical inconsistency. In this case one has to stress that for complex τ not on the real or imaginary axis, $|w_N(\tau)\rangle$ does not represent a solution of the Heisenberg equation anymore. We shall call these states the generalized intelligent states.

There are two special cases of N, the extremes 0 and $2j$. In fact $|w_0(\tau)\rangle=|\tau\rangle$ and (it shall be shown in this section) $|w_{2j}(\tau)\rangle=|-\tau\rangle$. Actually these are the simpler cases of the general law relating intelligent states corresponding to opposite complex numbers,

$$|w_{N_1}(\tau_1)\rangle=|w_{N_2}(\tau_2)\rangle,\quad N_1+N_2=2j,\quad \tau_1+\tau_2=0. \tag{10}$$

This relation is easily seen after having established the value of the inner product $\langle\rho|w_{N_1}(\tau)\rangle=\langle w_0(\rho)|w_{N_1}(\tau)\rangle$ given in Appendix A.[10]

In order to perform calculations of physical interest, it is convenient to have a simpler expression than Eq. (9) to describe the intelligent states. Fortunately this can be done just by ordinary straightforward algebra. It turns out that $|w_N(\tau)\rangle$ can be written as

$$\begin{aligned}&|w_n(\tau)\rangle=a_nY_1\partial_y^n y^{2j}\exp(\tau_yJ_+)|-j\rangle,\\&n=0,\ldots,2j,\end{aligned} \tag{11}$$

where

$$\begin{aligned}&a_n\equiv\hat{a}_NN!\,(1+|\tau|^2)^{-j},\quad n,\ Y_1,\ \tau_y \text{ given by}\\&n\equiv2j-N,\quad Y_1f(y)\equiv f(1),\\&(\partial_y^n)f(y)\equiv\partial^nf/\partial_y^n,\quad \tau_y\equiv\tau(1-2y^{-1}),\end{aligned} \tag{12}$$

and the corresponding eigenvalue w_n is given by $w_n\equiv2\gamma_+\tau^{-1}(j-n)$. Taking into account definition (4) and introducing the auxiliary polynomials $p_j(y,z,|\tau|)$,

$$p_j(y,z,\tau)\equiv(yz+\tau\tau^*(y-2)(z-2))^j, \tag{13}$$

one can write down the intelligent states as

$$\begin{aligned}|w_n(\tau)\rangle&=a_nY_1\partial_y^n\exp(\tau_yJ_+)\exp(-2\ln yJ_3)|-j\rangle\\&=a_nY_1\partial_y^np_j(y,y,\tau)|\tau_y\rangle,\end{aligned} \tag{14a}$$

where the normalizing factor a_n is shown to be (see Appendix A)

$$a_n=\{Z_1Y_1\partial_y^n\partial_z^np_{2j}(y,z,\tau)\}^{-1/2}\equiv(p_{2j}^{nn})^{-1/2} \tag{14b}$$

and $|\tau_y\rangle$ means the Bloch state corresponding to the complex number $\tau(1-2y^{-1})=\tau_y$.

We note that in the expression given in Eq. (14a) for the intelligent spin states the operator $Y_1\partial_y^n$ occurs. Therefore, one has to know the behavior of $p_j(y,y,\tau)|\tau_y\rangle$ in a neighborhood of $y=1$, in order to obtain the corresponding derivatives.

States having the structure $p_j(y,y,\tau)|\tau_y\rangle=\exp(\tau_yJ_+)\times\exp(-2\ln yJ_3)|-j\rangle$ are not atomic coherent, since the group parameters τ_y, y do not verify the condition for a Bloch-type rotation $R(\tau)$ $(y\neq1+|\tau_y|^2)$.

However, the structure (14a) proves to be very useful in order to deduce many properties of intelligent states from the corresponding properties of the associated Bloch states $|\tau_y\rangle$.

One can also ask if an intelligent state $|w_n(\tau)\rangle$ coincides with some Bloch state $|\mu\rangle$. In order to answer this question, one can prove that[11]

$$|w_n(\tau)\rangle=|\mu\rangle\rightleftarrows n=0,\quad \tau=-\mu\quad\text{or}\quad n=2j,\quad \tau=\mu, \tag{15}$$

which shows that proper intelligent states ($|w_n(\tau)\rangle$, $n\neq0,2j$) are not Bloch states, but a refinement of them.

Moreover, since for each τ we have $2j+1$ different intelligent states, it is natural to enquire whether they could be obtained through some operation applied to the

Wigner basis $|m\rangle$. In other words: Are the $2j+1$ states $|\tau, m\rangle \equiv R(\tau)|m\rangle$ $(m=-j+1,\ldots,j-1,j)$ intelligent?

Straightforward calculation yields (notice that $\tau = \tan\frac{1}{2}\theta \exp(-il\pi/2)$, l integer)

$$J_\alpha|\tau, m\rangle = -m\sin\theta|\tau, m\rangle + \cos\theta(j+m)^{1/2} \times (j-m+1)^{1/2}|\tau, m-1\rangle. \tag{16}$$

The second term in the right-hand side shows that $|\tau, m\rangle$ is not an eigenvector of J_α, unless $\cos\theta(j+m)=0$.

As in general $|\tau|\neq 1$, the only possibility we are left with is $m=-j$, which means that in the set $\{|\tau m\rangle\}$, only $|\tau, -j\rangle = |\tau\rangle$ is intelligent. In the particular case where $\cos\theta = 0$ $(\theta=\pi/2+k\pi$, k integer), it is immediate to see that such a situation corresponds to $\alpha=0$, ∞, i.e., $J_\alpha = J_1$ or J_2, respectively. In that case it is easy to understand why $|\tau = \exp(-il\pi/2), m\rangle$ is an eigenstate of J_1 (or J_2): $R(\tau=\exp(-il\pi/2)$ corresponds to $\pi/2$ rotations about J_2 (or J_1), therefore the states $|\tau\rangle = \exp(-il\pi/2), m\rangle$ are nothing else but the Wigner basis with respect to the x (or y) axis.

3. EXPECTATION VALUES FOR INTELLIGENT STATES AND MINIMUM UNCERTAINTY STATES

In order to define calculations of physical quantities for systems prepared in an intelligent state, one has to develop a suitable technique to handle the corresponding matrix elements. As ACGT have shown for the Bloch states, the technique of the generating functions has been proved to be very useful. In Appendix A we present with some details how the technique due to ACGT is extended to deal with intelligent spin states.

If we define the operators $(\cdot)^{n_1 n_2}$ as

$$f^{n_1 n_2} \equiv Y_1 Z_1 \partial_y^{n_1}\partial_z^{n_2} f(y,z) \equiv \left[\frac{\partial^{n_1}}{\partial_y^{n_1}}\frac{\partial^{n_2}}{\partial_z^{n_2}} f(y,z)\right]_{y=z=1}, \tag{17}$$

one finds (see Appendix A) for the expectation values of J_i for a system in an intelligent state,

$$\begin{aligned}
&\langle w_n(\tau)|J_1|w_n(\tau)\rangle \\
&\quad\equiv \langle J_1\rangle_{n\tau} = 2j\,\mathrm{Re}\tau[y(z-2)p_{2j-1}(y,z,\tau)]^{nn}(p_{2j}^{nn})^{-1} \\
&\langle w_n(\tau)|J_2|w_n(\tau)\rangle \\
&\quad\equiv \langle J_2\rangle_{n\tau} = -2j\,\mathrm{Im}\tau[y(z-2)p_{2j-1}(y,z,\tau)]^{nn}(p_{2j}^{nn})^{-1} \\
&\langle w_n(\tau)|J_3|w_n(\tau)\rangle \\
&\quad\equiv \langle J_3\rangle_{n\tau} = j[(\tau\tau^*(y-2)(z-2)-zy)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}
\end{aligned} \tag{18}$$

Further on, by taking second-order derivatives of the generating function X_A, defined in Eq. (A8), we evaluate the quadratic deviations $\langle \Delta J_i^2\rangle_{n\tau}$,

$$\begin{aligned}
\langle \Delta J_1^2\rangle_{n\tau} = {}& \tfrac{1}{2}j(2j-1)(\tau^2+\tau^{*2})[y^2(z-2)^2 p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1} \\
&+ j[(y^2z^2 + 2j\tau\tau^* yz(y-2)(z-2))p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1} \\
&+ \tfrac{1}{2}j[(\tau\tau^*(y-2)(z-2)-zy)p_{2j-1}]^{nn} \\
&- 4j^2(\mathrm{Re}\tau)^2\{[y(z-2)p_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2},
\end{aligned} \tag{19}$$

$$\begin{aligned}
\langle \Delta J_2^2\rangle_{n\tau} = {}& -\tfrac{1}{2}j(2j-1)(\tau^2+\tau^{*2})[y^2(z-2)^2 p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1} \\
&+ j[(y^2z^2 + 2j\tau\tau^* yz(y-2)(z-2))p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1} \\
&+ \tfrac{1}{2}j[((y-2)(z-2)\tau\tau^* - zy)P_{2j-1}]^{nn}(p_{2j}^{nn})^{-1} \\
&- 4j^2(\mathrm{Im}\tau)^2\{[y(z-2)P_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2},
\end{aligned}$$

$$\begin{aligned}
\langle \Delta J_3^2\rangle_{n\tau} = {}& -4j^2\{[zyp_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2} + 2j[zyp_{2j-1}]^{nn} \\
&\times (p_{2j}^{nn})^{-1} + 2j(2j-1)[y^2z^2p_{2j-2}]^{nn}(p_{2j}^{nn})^{-1}.
\end{aligned}$$

In a similar way, the mean values of monomials of the type $J_{i_1}^{a_1}J_{i_2}^{a_2}J_{i_3}^{a_3}$ can also be calculated by an appropriate number of derivatives of the generating functions, one of which is $X_A(\alpha\beta\gamma)$, defined in Eq. (A8).

Once we have obtained the values of $\langle \Delta J_{1,2}^2\rangle_{n\tau}$ and $\langle J_3\rangle_{n\tau}$ we are in a position to discuss more precisely what are the differences between minimum uncertainty states $|\mu\rangle$, and intelligent states $|w\rangle$ of the SU(2) algebra. As we know this algebra has commutators which are not numbers, it is a good candidate to find out explicit examples of intelligent states which are not minimum uncertainty states.

Actually, in order to determine all the minimum uncertainty states, one should have to parametrize H_j and thereafter calculate $I(\psi)$ and $C(\psi)$ for this H_j parametrization. Proceeding in that way, one obtains two functions depending upon $4j+1$ independent real parameters and it is a standard task to find both the local minimums of $I(\psi)$ and the subvariety where $I(\psi)=C(\psi)$.

If we restrict ourselves to a subset B of H_j, we can explore what happens on B. Evidently, any intelligent state that belongs to B is an intelligent state in H_j. On the contrary, that $|u_B\rangle$ is a minimum uncertainty state on B does not necessarily imply that $|u_B\rangle$ shall be a minimum uncertainty state on the large variety H_j.

For $B\equiv\{|\tau\rangle,\ \tau=\tan\frac{1}{2}\theta\exp(-i\varphi)\}$, the uncertainty functional $I(\psi)$ has, on B, the value[12]

$$I(\tau) = \tfrac{1}{4}j^2(1-\sin^2\theta\sin^2\varphi)(1-\sin^2\theta\cos^2\varphi), \tag{20}$$

while for $C(\psi)$, we have

$$C(\tau) = 4^{-1}j^2\cos^2\theta. \tag{21}$$

Due to the simplicity of both $I(\tau)$ and $C(\tau)$, it is immediate to solve the Heisenberg equation $I(\tau)=C(\tau)$. That gives

$$j^2\sin^4\theta\sin^2 2\varphi = 0, \tag{22}$$

or equivalently

$\theta=0, \varphi$ arbitrary, θ arbitrary, $\varphi = n\pi/2$ (n integer).

Because of the degeneracy at the origin in the polar representation (θ,φ) of the complex plane, the solution given in Eq. (22) is exactly the set of the two axes of the complex plane. That corresponds to the fact already mentioned: The only intelligent Bloch states are those contained in the two axes. Of course, as we have shown before, there are intelligent states which are not Bloch states.

In connection with the possible minimum uncertainty states located on B, one has to find the local minimums of $I(\tau)$. $I(\tau)$ has nine stationary points τ_s,

$$\tau_s = \tan(m\pi/4)\exp(-in\pi/4), \quad m=0,1, \quad n=0,1,\ldots,6,7. \tag{23}$$

It is straightforward to verify that $\tau_s=0$ gives a maximum of $I(\tau)$, and that $\tau_s=\exp(-in\pi/2)$ give the four minimums while the remaining four points $\tau_s=\exp[-i(\pi/4$

$+n\pi/2)$] give saddle points of $I(\tau)$ in the subset B. That means that only the four points of $B(\tau_s = \exp(-in\pi/2))$ can be minimum uncertainty states on H_j.

Nevertheless we have a lot of intelligent states defined on B (τ any real or pure imaginary number) which shall proceed to be intelligent states when we enlarge the calculations to the whole H_j.

4. DYNAMICAL PROPERTIES OF THE INTELLIGENT STATES

The first situation that we want to consider is the time evolution of a nonrelativistic spin j system (of magnetic moment γ), in a magnetic environment $\mathbf{B}(t)$ of the type considered by Gilmore[13]:

$$\mathbf{B}(t) \equiv 2B_\perp(\cos 2\omega_1 t\hat{x} + \sin 2\omega_1 t\hat{y}) + 2B_\parallel \hat{z}, \tag{24}$$

where $2B_\parallel\hat{z}$ is a constant magnetic field along a fixed direction and B is the strength of a perpendicular field of proper frequency $2\omega_1$.

The corresponding time-dependent Hamiltonian is

$$H(t) = -\hbar\gamma\mathbf{J}\cdot\mathbf{B}(t) = -\hbar\gamma(B_\perp \exp(-2i\omega_1 t)J_+ + B_\perp \exp(2i\omega_1 t)J_- + 2B_\parallel J_3), \tag{25}$$

with $\mathbf{J}$ represented in the $(2j+1)$-dimensional space H_j. By going to the two-dimensional representation of SU(2), Gilmore has evaluated the time evolution operator $U(t)$ which satisfies the Schrödinger equation $i\hbar\dot{U} = HU$,

$$U(t) = \begin{Bmatrix} \cos^2\psi \exp(i\omega_- t) + \sin^2\psi \exp(-i\omega_+ t) & i\sin 2\psi \sin\omega_2 t \exp(-i\omega_1 t) \\ i\sin 2\psi \sin\omega_2 t \exp(i\omega_1 t) & \cos^2\psi \exp(-i\omega_- t) + \sin^2\psi \exp(i\omega_+ t) \end{Bmatrix}, \tag{26a}$$

where ω_+, ω_-, and ψ are given by

$$\omega_\pm \equiv \omega_2 \pm \omega_1, \quad \omega_2 \equiv [\gamma^2 B_\perp^2 + (\gamma B_\parallel + \omega_1)^2]^{1/2},$$
$$\sin 2\psi \equiv \gamma B_\perp \omega_2^{-1}, \quad \cos 2\psi \equiv (\gamma B_\parallel + \omega_1)\omega_2^{-1}. \tag{26}$$

Let us assume that our system has been initially prepared in an intelligent state $|w_n(\tau)\rangle$. Therefore, in any other subsequent instant t, the system shall be in a certain state $|w_n(t,\tau)\rangle$ determined by the evaluation operator $U(t)$; namely, $|w_n(t,\tau)\rangle = U(t)|w_n(\tau)\rangle$. We want to investigate whether $|w_n(t,\tau)\rangle$ is an intelligent state or, at least, how close to an intelligent state it is while it evolves. We know, after ACGT, that a Bloch state remains a Bloch state along its evolutions under the Hamiltonian (25).

Moreover, as both $|w_0(\tau)\rangle$ and $|w_{2j}(\tau)\rangle$ are Bloch states, it might happen that any proper intelligent state could evolve remaining in the subset of the intelligent states too.

In order to give an answer to this question, let us briefly mention some useful facts concerning SU(2) and $|w_n(\tau)\rangle$, as has been given in Eq. (14a).

The first property we want to point out concerns the structure of $|w_n(\tau)\rangle$ itself; $|w_n(\tau)\rangle$ can be written

$$|w_n(\tau)\rangle = a_n Y_1 \partial_y^n k(y,\tau)|-j\rangle,$$
$$k(y,\tau) \equiv \exp(\tau_y J_+)\exp(-J_3 \ln y^2), \tag{27}$$

where $k(y,\tau)$ belongs to $SL(2,C)$,[4] the analytic continuation of SU(2).[14] In the two-dimensional representation of $SL(2,C)$, $k(y,\tau)$ has the form

$$k(y,\tau) = \exp(\tau_y J_+)\exp(-(\ln y^2)J_3) = \begin{Bmatrix} 1 & \tau_y \\ \cdot & 1 \end{Bmatrix}\begin{Bmatrix} y^{-1} & \cdot \\ \cdot & y \end{Bmatrix} = \begin{Bmatrix} y^{-1} & \tau(y-2) \\ \cdot & y \end{Bmatrix}, \tag{28}$$

showing that it belongs to the well-known four parameter subgroup K of $SL(2,C)$,[15] as reviewed in Appendix B. We prove in this appendix that for $y \neq 1$, $k(y,\tau)$ contains a Lorentz boost and, therefore, $k(y,\tau)$ does not represent a proper rotation.

The operator $U(t)k(y,\tau) \equiv \hat{l}(t,y,\tau)$ has also been explicitly evaluated in Appendix B, Eq. (B8). This allows us to write the state $|w_n(t,\tau)\rangle$ as follows:

$$|w_n(t,\tau)\rangle = a_n(\tau)Y_1\partial_y^n \hat{l}_4^{2j}\exp(\hat{l}_2\hat{l}_4^{-1}J_+)|-j\rangle, \tag{29a}$$

where

$$\hat{l}_2 \equiv [\tau(y-2)\cos^2\psi + y\sin\psi\cos\psi]\exp(i\omega_- t) + [\tau(y-2)\sin^2\psi - y\sin\psi\cos\psi]\exp(-i\omega_+ t),$$
$$\hat{l}_4 \equiv [\tau(y-2)\sin\psi\cos\psi + y\sin^2\psi]\exp(i\omega_+ t) + [y\cos^2\psi - \tau(y-2)\sin\psi\cos\psi]\exp(-i\omega_- t).$$

Although the structure of the state $|w_n(t,\tau)\rangle$ seems complicated, it is proved in Appendix B that this state becomes, up to a phase factor, an intelligent state if the transverse magnetic field vanishes, i.e., $B_\perp = 0$. Only in this case the evolution of an intelligent state of order n determined by the complex number τ is a generalized intelligent state, of the same order n, corresponding to the complex t-dependent number $\tau' = \tau\exp(2i\gamma B_\parallel t)$. If $n=0$ we recover the result of ACGT: $|w_0(t,\tau)\rangle = \exp(2i \times \arg\hat{l}_4)|\hat{l}_2\hat{l}_4^{-1}\rangle$. That is, the evolution of a Bloch state keeps being a Bloch state, up to a phase factor.

The second situation we want to treat here is the relevance of the intelligent states in connection with the pointlike laser,[16] either with a semiclassical or a fully quantized representation of the laser field.

By a semiclassical pointlike laser we mean a collection of identical atoms, each with two effective energy levels (with $\hbar\omega$ the energy gap) interacting with a classical field $\mathbf{E}(t) = 2\mathrm{Re}\{\mathbf{E}_0\exp(i\omega t)\}$, which has the resonant mode of frequency ω.

The Hamiltonian corresponding to this system is, following ACGT,

$$H = H_A + H_{AF} \equiv \hbar\omega J_3 - (\mathbf{p}\cdot\mathbf{E}_0^*)J_+\exp(-i\omega t) - (\mathbf{p}^*\cdot\mathbf{E}_0)J_-\exp(i\omega t), \tag{30}$$

where the vector $\mathbf{p}$ is the complex dipole moment associated to each atom giving rise to the total dipole moment

$$\mathbf{D} \equiv \mathbf{p} J_+ + \mathbf{p}^* J_-. \qquad (31)$$

For a system of N_0 atoms, the cooperation number j must satisfy the inequality

$$j \leq \tfrac{1}{2} N_0. \qquad (32)$$

We are assuming either that $\mathbf{p}$ verifies $\mathbf{p} \cdot \mathbf{E}_0 = 0 = \mathbf{p}^* \cdot \mathbf{E}_0^*$ or, if this selection rule does not apply, that we are working in the rotating wave approximation.

If one neglects the interaction term H_{AF} between matter and the electromagnetic field, namely $H_{AF}=0$ in Eq. (30), it is possible to give an estimate of the expectation value of $\mathbf{D}$. For the system initially in an intelligent state $|w_n(\tau)\rangle$, the state $|w_n(t,\tau)\rangle$ becomes $|w_n(t,\tau)\rangle = \exp(-ij\omega t)\,|w_n(\tau\exp(-ij\omega t))\rangle$. Therefore,

$$\begin{aligned}\langle w_n(t,\tau)|\mathbf{D}|w_n(t,\tau)\rangle &= \mathbf{p}\langle J_+\rangle_{n\tau}(t) + \mathbf{p}^*\langle J_-\rangle_{n\tau}(t) \\ &= 2j(\mathbf{p}\tau^*\exp(i\omega t) + \mathbf{p}^*\tau\exp(-i\omega t)) \\ &\quad\times[y(z-2)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}. \end{aligned} \qquad (33)$$

This result is a refinement of the corresponding one for the Bloch state, which is reobtained here by taking $n=0$. (It is worthwhile to remind the reader that for the Wigner—Dicke states the expectation value of $\mathbf{D}$ vanishes.)

As the macroscopic dipole of the system does not vanish, there exists a nonvanishing classical radiation intensity I_c generated by this oscillating dipole, which in the wave zone is

$$I_c = I_0 \cdot 4j^2\tau\tau^*\{[y(z-2)p_{2j-1}]^{nn}\}^2(p_{2j}^{nn})^{-2}. \qquad (34)$$

Introducing the fully quantized Hamiltonian

$$H = H_A + H_F + H_{AF} \equiv \hbar\omega J_3 + \hbar\omega a^+ a + \gamma a J_+ + \gamma a^+ J_-, \qquad (35)$$

we can calculate the emission rate for the pointlike laser.[17]

The spontaneous emission intensity can be calculated for an initial intelligent state $|w_n(\tau)\rangle$, in a way similar to what ACGT did for this model,

$$\begin{aligned} I_n^{sp} &= I_0 \sum_{m=-j}^{j} |\langle m|J_-|w_n(\tau)\rangle|^2 \\ &= I_0\langle w_n(\tau)|J_+J_-|w_n(\tau)\rangle \\ &= I_0\langle J_-J_+\rangle_{n\tau} + 2I_0\langle J_3\rangle_{n\tau}. \end{aligned} \qquad (36a)$$

The matrix elements occurring in this relation are easily evaluated by means of the generating function $X_A(\alpha,\beta,\gamma)$, given by Eq. (A9),

$$\begin{aligned} I_{n\tau}^{sp} &= I_0 2j\tau\tau^*\{(2j-1)[yz(y-2)(z-2)p_{2j-2}]^{nn} \\ &\quad\times(p_{2j}^{nn})^{-1} + [(y-2)(z-2)p_{2j-1}]^{nn}(p_{2j}^{nn})^{-1}\}, \end{aligned} \qquad (36b)$$

an expression which reduces for $n=0$ to the results founc by ACGT for Bloch states.

In the case of a Dicke—Wigner initial state $|m\rangle$, the spontaneous emission intensity is $I_D^{sp} = I_0(j+m)(j-m+1)$. In order to compare the spontaneous emission intensities between intelligent states and Dicke—Wigner states we have to evaluate I_D^{sp} for a Dicke—Wigner state having the same energy expectation value that $|w_n(\tau)\rangle$.[18] Therefore, introducing $m = \langle J_3\rangle_{n\tau}$ in I_D^{sp}, we get

$$\begin{aligned}[I_D^{sp}]_{m=\langle J_3\rangle_{n\tau}} &= I_0 \cdot 2j[1-(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}] \\ &\quad\times[1+2j(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}] \neq I_{n\tau}^{sp}. \end{aligned} \qquad (37)$$

A similar calculation for the stimulated intensity I^{st} leads to:

$$\begin{aligned} I_{n\tau}^{st} &= I_0\sum_m\{|\langle m|J_-|w_n(\tau)\rangle|^2 - |\langle m|J_+|w_n(\tau)\rangle|^2\} \\ &= 2\langle J_3\rangle_{n\tau}\cdot I_0. \end{aligned} \qquad (38)$$

Consequently, using the value given in Eq. (18) of $\langle J_3\rangle_{n\tau}$ we have

$$I_{n\tau}^{st} = I_0 \cdot 2j[1-2(yzp_{2j-1})^{nn}(p_{2j}^{nn})^{-1}] = I_D^{st}, \qquad (39)$$

which is identical to the stimulated intensity emitted for an initial Dicke state with quantum number $m = \langle J_3\rangle_{n\tau}$.

Just for completeness, one can explicitly calculate $\langle J_3\rangle_{0\tau}$, $\langle J_3\rangle_{1\tau}$, and $\langle J_3\rangle_{2\tau}$. It happens that, for $j \geq 3$, the three values decrease for $0 \leq \theta < \pi/2$, and increase for $\pi/2 < \theta < \pi$,

$$\langle J_3\rangle_{2\tau} < \langle J_3\rangle_{1\tau} < \langle J_3\rangle_{0\tau} \quad (j \geq 3), \qquad (40a)$$

$$\langle J_3\rangle_{0\tau} = -2j\cos\theta, \quad \langle J_3\rangle_{1\tau} = -2j\cos\theta\left[1 + \frac{(2-j^{-1})\sin^2\theta}{2j\cos^2\theta + \sin^2\theta}\right], \qquad (40b)$$

$$\langle J_3\rangle_{2\tau} = -2\cos\theta\,\frac{[(j-1)(j-2)(2j-3)\cos^4\theta + 2(j-1)(4j-5)\cos^2\theta + (5j-4)]}{[(j-1)(2j-3)\cos^4\theta + 4(j-1)\cos^2\theta + 1]} \qquad (40c)$$

However, as we have not been able to proceed a step further we are not allowed to claim a general property from Eqs. (40). The only statement we are making is that the stimulated emission intensity (and also the energy expectation value) of the proper intelligent states $(n=1,2)$ is greater than the stimulated emission intensity arising from the Bloch state corresponding to the same value of the parameter τ.

The last point we want to mention concerning the different behavior of intelligent states in comparison with Bloch states is the following: Suppose we have initially prepared a system of spin j in an intelligent state $|w_n(\tau)\rangle$ and we want to know what is the probability that, under the magnetic Hamiltonian (25), the system could be found in $t>0$ in a Wigner state $|m\rangle$. Making use of the results of Appendices A and B, we obtain the transition probabilities

$$p_{(n,\tau)\to|m\rangle} = |\langle m|w_n(t\tau)\rangle|^2 = (n!)^2 a_n^2(\tau)\binom{2j}{j+m}$$
$$\times\left\{\sum_{(l_1,l_2)=(0,0)}^{(n,n)} a_2^{l_1} a_2^{*l_2} a_4^{n-l_1} a_4^{*n-l_2}\right.$$
$$\times c_2^{j+m-l_1} c_2^{*j+m-n+l_2} c_4^{j-m-n+l_1} c_4^{*j-m-n+l_2}$$
$$\times\binom{j+m}{l_1}\binom{j+m}{l_2}\binom{j-m}{n-l_1}\binom{j-m}{n-l_2}, \tag{41a}$$

where a_i and c_i, $i=2,4$ are

$$\begin{aligned}
a_2 &\equiv \exp(-i\omega_1 t)[\tau\cos\omega_2 t + i\sin\omega_2 t(\tau\cos 2\psi + \sin 2\psi)],\\
a_4 &\equiv \exp(i\omega_1 t)[\cos\omega_2 t + i\sin\omega_2 t(\tau\sin 2\psi - \cos 2\psi)],\\
c_2 &\equiv \exp(-i\omega_1 t)[(\tau-2)\cos\omega_2 t + i\sin\omega_2 t((\tau-2)\cos 2\psi\\
&\quad + \sin 2\psi)],\\
c_4 &\equiv \exp(i\omega_1 t)[\cos\omega_2 t - i\sin\omega_2 t(\tau\sin 2\psi + \cos 2\psi)].
\end{aligned} \tag{41b}$$

In order to see how a pure intelligent state behaves, one can take a particular case of Eqs. (41). For instance, let us choose $|m\rangle = |-j\rangle$. Making use of the above result, it turns out that ($\tau = \tan\frac{1}{2}\theta\exp(+in\pi/2)$)

$$\Gamma \equiv \frac{p_{(1,\tau)\to|-j\rangle}}{p_{(0,\tau)\to|-j\rangle}} = (2j\cos^2\theta + \sin^2\theta)^{-1}$$
$$\times\frac{1+\sin^2\omega_2 t\cdot[\sin^2 2\psi(\tau^2-1) - \tau\sin 4\psi]}{1+\sin^2\omega_2 t\cdot[\sin^2 2\psi(\tau^2-1) + \tau\sin 4\psi]}. \tag{42}$$

This ratio Γ is finite for any ψ, τ, and t unless τ takes the value $\tau'_\psi = -\operatorname{cotan}2\psi$. In that case, Eq. (42) becomes

$$\Gamma \equiv \frac{p_{(1,\tau'_\psi)\to|-j\rangle}}{p_{(0,\tau'_\psi)\to|-j\rangle}} = [1 + (2j-1)\cos^2 4\psi]^{-1}$$
$$\times(1 + \sin 2\psi^{-1}\sin 6\psi\sin^2\omega_2 t)\cos^{-2}\omega_2 t, \tag{43}$$

showing that, for $t_n = (n+\frac{1}{2})\pi/\omega_2$ the value of Γ is infinite. Consequently we see that the behavior of the proper intelligent state $|w_1(t,\tau'_\psi)\rangle$ is qualitatively different from the behavior of the Bloch state $|w_0(t,\tau'_\psi)\rangle$.

Further, as for τ'_ψ, the function $c_4(t)$ appearing in Eq. (41) has the value

$$c_4(\tau'_\psi) = \exp(i\omega_1 t)\cos\omega_2 t. \tag{44}$$

It is clear that for instants $t_n = (2n+1)\pi/2\omega_2$ and for numbers n, m (which have to verify $n+m \leqslant j-1$)[19] the transition probability $p_{(n,\tau'_\psi)\to|m\rangle}$ vanishes with period $T_2 = \pi/\omega_2$.

Looking at the structure of the probability $p_{(n,\tau)\to|m\rangle}$, one gets two other special values of τ,

$$\tau''_\psi = 2 - \tan 2\psi, \quad \tau'''_\psi = 2. \tag{45}$$

These values cause the periodic vanishing of $p_{(n,\tau)\to|m\rangle}$ too, now because $c_2(t)$ vanishes with the same period as above, for each instant $t'_n = n\pi/\omega_2$ and for quantum numbers n, m such that $n+1 \leqslant j+m$.

5. DISCUSSION AND COMMENTS

We have been able to establish a clear distinction between intelligent spin states, minimum uncertainty states, and Bloch states. We have shown that the generalized intelligen states constitute a refinement of the Bloch states containing them as extreme cases.

We also pointed out in Eq. (10) the symmetry in the definition of intelligent states which allow us to restrict the analysis of $|w_n(\tau)\rangle$ to any half-plane containing the origin of the whole complex plane.

Thereafter we evaluated, through the technique of the generating functions, the expectation values of both the components of the angular momentum vector and of their mean square deviations. They turned out to be rational functions of $\tau\tau^* = \tan^2\frac{1}{2}\theta$.

Moreover, by making use of some algebraic properties of the noncompact subgroup K of $\mathrm{SL}(2,C)$ we studied some dynamical properties of the intelligent states valid both for a reasonable time dependent model of a spin-j particle in a magnetic atmosphere and for a pointlike laser.

One important result found is that for a permanent magnetic field $\mathbf{B} = 2B_\parallel\hat{e}_3$, proper intelligent states evolve continuously in the set of generalized intelligent states. Of course, the two extreme states ($n = 0, 2j$) which are Bloch and intelligent evolve in the assembly of the complex Bloch states.

The transition probabilities, for a system prepared in an intelligent state, of becoming in time t a Wigner–Dicke state, have been computed. It turned out that there exist three values of the real parameter τ defining an intelligent state for which $p_{(n,\tau)\to|m\rangle}$ vanishes periodically.

In the case of the pointlike laser, the spontaneous and stimulated emission intensities and the macroscopic dipole of the system have also been evaluated showing again a refinement of the results obtained using Bloch states.

We have also proved that, in general, an intelligent state is not a minimum uncertainty state and we pointed out where the noncoincidence of both kind of states stems.

It is also worthwhile to note that, contrary to what has recently been asserted by Kolodziejczyk,[20] the coherent states defined by Mikhailov[21] cannot be used to explain the relationship between coherent and intelligent states, essentially because the only Mikhailov coherent state which is intelligent is, trivially, the ground state.

Finally, let us remark that Vetri's comment[22] that Radcliffe states which do not point in the z direction and are labeled "intelligent" in Ref. 1 are actually those oriented in such a way that the $\hat{n}$ axis is along $\hat{x}$ or $\hat{y}$ is precisely what Aragone, Guerri, Salamó, and Tani meant when they said that "only those Radcliffe states located on the real line or the imaginary axis are intelligent states."

APPENDIX A

In this Appendix we are going to show the details concerning some of the calculations whose results have been used in the text.

Let us recall that the states we are dealing with have been written in the form [Eqs. (13)].

$$|w_n\rangle = a_{n_1}\partial_y^{n_1}\{p_j(y, y, |\tau|)\,|\tau_y\rangle\}_{y=1}, \tag{A1}$$

where

$$p_j(y, z, |\tau|) \equiv [yz + |\tau|^2(y-2)(z-2)]^j. \tag{A2}$$

Suppose we are interested in computing the value of $\langle w_{n_2}|w_{n_1}\rangle$, where $|w_{n_1}\rangle$ remains as in (A1). $|w_{n_2}\rangle$ may be written

$$|w_{n_2}\rangle = a_{n_2}\partial_z^{n_2}\{p_j(z, z, |\tau|)\,|\tau_z\rangle\}_{z=1}, \tag{A3}$$

where instead of y we use a different variable z, in order to avoid confusion. Making the scalar product we have (p_j is real for y, z real numbers)

$$\langle w_{n_2}|w_{n_1}\rangle = a_{n_2}^* a_{n_1}\partial_z^{n_2}\partial_y^{n_1}\{p_j(y, y, \tau) \times p_j(z, z, \tau)\langle\tau_z|\tau_y\rangle\}_{z=1} \tag{A4}$$

but, by virtue of Eq. (5),

$$\begin{aligned}\langle\tau_z|\tau_y\rangle &= (1+|\tau_y|^2)^{-j}(1+|\tau_z|^2)^{-j}(1+\tau_z^*\tau_y)^{2j}\\ &\equiv y^{2j}p_j(y, y, |\tau|)^{-1}z^{2j}p_j(z, z, |\tau|)^{-1}\\ &\quad\times[1+|\tau|^2(1-2/y)(1-2/z)]^{2j}\\ &= p_j(y, y, |\tau|)^{-1}p_j(z, z, |\tau|)^{-1}p_{2j}(y, z, |\tau|).\end{aligned} \tag{A5}$$

Introducing this value of $\langle\tau_{z_2}|\tau_y\rangle$ into Eq. (A4) we get the final value of $\langle w_{n_2}|w_{n_1}\rangle$,

$$\begin{aligned}\langle w_{n_2}|w_{n_1}\rangle &= a_{n_2}^* a_{n_1}\{\partial_z^{n_2}\partial_y^{n_1}p_{2j}(y, z, |\tau|)\}_{y=1=z}\\ &= a_{n_2}^* a_{n_1}p_{2j}^{n_1 n_2}.\end{aligned} \tag{A6}$$

If we take here $n_2 = n_1$ and impose that the result found must be 1, we get the modulus of the normalizing factor a_n, as was mentioned in Eq. (14). Once we get the value of the a_n, the scalar product (A6) is completely defined,

$$\langle w_{n_2}|w_{n_1}\rangle = \frac{p_{2j}^{n_1 n_2}}{(p_{2j}^{n_1 n_1})^{1/2}(p_{2j}^{n_2 n_2})^{1/2}}. \tag{A7}$$

In order to calculate expected values of observables contained in the SO(3) algebra, it is of crucial importance to evaluate the generator function $X_A(\alpha, \beta, \gamma)$, defined in the ACGT paper as

$$X_A(\alpha, \beta, \gamma) \equiv \langle w_n|\exp(\gamma J_-)\exp(\beta J_3)\exp(\alpha J_+)|w_n\rangle. \tag{A8}$$

Introducing the form (A3) of $|w_n\rangle$ and applying the Baker–Campbell–Haussdorff formula we have that

$$X_A(\alpha, \beta, \gamma) = |a_n|^2\partial_z^n\partial_y^n\{zy\exp(-\beta/2) + [\tau(y-2)+\alpha y][\tau^*(z-2)+\gamma z]\exp(\beta/2\}^{2j} \tag{A9}$$

which, if we define the auxiliary function q in y, z, α, β, γ, by

$$q_{2j}(\alpha, \beta, \gamma, y, z, \tau) \equiv \{zy\exp(-\beta/2) + \exp(\beta/2) \times[\tau(y-2)+\alpha y][\tau^*(z-2)+\gamma z]\}^{2j}, \tag{A10}$$

can be rewritten in the shorter form

$$X_A(\alpha\beta\gamma) = |a_n|^2 q_{2j}^{nn}(\alpha, \beta, \gamma). \tag{A11}$$

Once we have evaluated X_A, it is very simple to estimate the expected values of, for instance, J_1, J_2, J_3, and $(\Delta J_1)^2$, $(\Delta J_2)^2$ for the intelligent states $|w_n\rangle$.

In fact,

$$\begin{aligned}\langle w_n|J_1|w_n\rangle &= \tfrac{1}{2}\langle w_n|J_+|w_n\rangle + \tfrac{1}{2}\langle w_n|J_-|w_n\rangle\\ &= \tfrac{1}{2}(\partial_\alpha X_A)_{\alpha=\beta=\gamma=0} + \tfrac{1}{2}(\partial_\gamma X_A)_{\alpha=\beta=\gamma=0},\end{aligned} \tag{A12}$$

$$\langle w_n|J_3|w_n\rangle = (\partial_\beta X_A)_{\alpha=\beta=\gamma=0}, \tag{A13}$$

and

$$4(\Delta J_1)^2 = \langle J_+^2\rangle + \langle J_-^2\rangle + 2\langle J_- J_+\rangle + 2\langle J_3\rangle - 4\langle J_1\rangle^2.$$

Consequently,

$$\begin{aligned}4(\Delta J_1)^2 &= (\partial_{\alpha\alpha}^2 X_A)_{\alpha=\beta=\gamma=0} + (\partial_{\gamma\gamma}^2 X_A)_{\alpha=\beta=\gamma=0}\\ &\quad+ 2(\partial_{\alpha\gamma}^2 X_A)_{\alpha=\beta=\gamma=0} + 2(\partial_\beta X_A)_{\alpha=\beta=\gamma=0}\\ &\quad- [(\partial_\alpha X_A)_{\alpha=\beta=\gamma=0} + (\partial_\gamma X_A)_{\alpha=\beta=\gamma=0}]^2,\end{aligned} \tag{A14}$$

and in the same way the value of $4(\Delta J_2)^2$ can be given,

$$\begin{aligned}4(\Delta J_2)^2 &= -(\partial_{\alpha\alpha}^2 X_A)_0 - (\partial_{\gamma\gamma}^2 X_A)_0 + 2(\partial_{\alpha\gamma}^2 X_A)_0\\ &\quad+ 2(\partial_\beta X_\alpha)_0 + [(\partial_\alpha X_A)_0 - (\partial_\gamma X_A)_0]^2.\end{aligned} \tag{A15}$$

It is interesting to observe that $q_{2j}(0,0,0) = p_{2j}(y, z, |\tau|)$.

APPENDIX B

In this Appendix we shall give some group results concerning SL$(2, C)$ and its subgroup K.

The four-parameter subgroup K has been extensively used in connection with the irreducible representations of the Lorentz group (see for instance Ref. 15). K is defined as the set of all the elements k of SL$(2, C)$ of the form

$$K = k(p, q) = \begin{pmatrix} p^{-1} & q\\ 0 & p\end{pmatrix}, \quad p, q \text{ complex numbers.} \tag{B1}$$

The importance of K lies in the fact that any element l of SL$(2, C)$ can uniquely be decomposed in the form

$$l = kz, \quad k = \begin{pmatrix} p^{-1} & q\\ 0 & p\end{pmatrix}, \quad z = \begin{pmatrix} 1 & \cdot\\ z & 1\end{pmatrix}. \tag{B2}$$

Moreover, as any $k(pq)$ can uniquely be factorized in the form

$$\begin{aligned}k &= \begin{pmatrix} 1 & qp^{-1}\\ \cdot & 1\end{pmatrix}\begin{pmatrix} p^{-1} & \cdot\\ \cdot & p\end{pmatrix} = \begin{pmatrix} p^{-1} & q\\ \cdot & p\end{pmatrix}\\ &= \exp(qp^{-1}J_+)\exp(-2\ln p J_3),\end{aligned} \tag{B3}$$

l can be uniquely decomposed as a product of three exponentials,

$$l = \exp(qp^{-1}J_+)\exp(-2\ln p\cdot J_3)\exp(zJ_-).$$

Let an arbitrary $l \in$ SL$(2, C)$ be given,

$$l = \begin{pmatrix} l_1 & l_2\\ l_3 & l_4\end{pmatrix}. \tag{B4}$$

It is easy to check that

$$l = \exp(l_2 l_4^{-1}J_+)\exp(-2\ln l_4 J_3)\exp(l_3 l_4^{-1}J_-). \tag{B5}$$

The elements $k(y,\tau)$ defined in Eq. (28) have the structure (B1), therefore the convenience of dealing with K (even if the restriction one could make of keeping p real could suggest that the three-dimensional subgroup $K' \equiv \{k \in K : p \text{ real}\}$ should play some specific role, more centrally than K itself).

Just for the sake of completeness it is possible to write down the four-dimensional Lorentz transformation $\Lambda(k)$ represented by $k(y,\tau)$. Following Gel'fand, Graev, and Vilenkin[23] it is straightforward to prove that $\Lambda(k) = \Lambda_1\Lambda_2$, where Λ_1 is the standard Lorentz boost ($|y| \neq 1$) and Λ_2 is a distortion of the $\{x^2, x^-\}$ two-dimensional plane (or of the $\{x^1, x^-\}$ two-plane accordingly to whether τ is a real or an imaginary number, respectively). The distortion Λ_2 turns out to be

$$\begin{aligned}(\Lambda_2 x)^+ &= x^+,\\ (\Lambda_2 x)^- &= x^- + \tau_y\tau_y^* x^+ + 2^{1/2}\mathrm{Re}\tau_y x^2 - 2^{1/2}\mathrm{Im}\tau_y x^1,\\ (\Lambda_2 x)^1 &= x^1 - 2^{1/2}\mathrm{Im}\tau_y x^+, \quad (\Lambda_2 x)^2 = x^2 + 2^{1/2}\mathrm{Re}\tau_y x^+,\end{aligned} \tag{B6}$$

while the boost Λ_1 applied to $\hat{x} \equiv \Lambda_2 x$ gives

$$\begin{aligned}&(\Lambda_1\hat{x})^+ = y^{-2}\hat{x}^+, \quad (\Lambda_1\hat{x})^- = y^2\hat{x}^-,\\ &(\Lambda_1\hat{x})_1 = \hat{x}_1, \quad (\Lambda_1\hat{x})_2 = \hat{x}_2,\end{aligned} \tag{B7}$$

where we denoted by $x^\pm \equiv 2^{-1/2}(x^0 \mp x^3)$ the usual two null coordinates.

We are interested in the decomposition (B5) for the operator $U(t)k(y,\tau)$ in order to have $|w_n(t,\tau)\rangle$ written in a way resembling an intelligent state. Calculating the matrix product, we get

$$\hat{l} \equiv U(t)k(y,\tau) \equiv \begin{pmatrix}\hat{l}_1 & \hat{l}_2\\ \hat{l}_3 & \hat{l}_4\end{pmatrix} \equiv \begin{cases} y^{-1}\cos^2\psi\exp(i\omega_- t) + y^{-1}\sin^2\psi\exp(-i\omega_+ t), & [\tau(y-2)\cos^2\psi + y\sin\psi\cos\psi]\exp(i\omega_- t)\\ & \quad + [\tau(y-2)\cos^2\psi - y\sin\psi\cos\psi]\exp(i\omega_+ t),\\ y^{-1}\sin\psi\cos\psi[\exp(i\omega_+ t) - \exp(-i\omega_- t)], & [y\sin^2\psi + \tau(y-2)\sin\psi\cos\psi]\exp(i\omega_+ t)\\ & \quad + [y\cos^2\psi - \tau(y-2)\sin\psi\cos\psi]\exp(-i\omega_- t).\end{cases} \tag{B8}$$

With this result, one obtains for $|w_n(t,\tau)\rangle = U(t)|w_n(\tau)\rangle$,

$$\begin{aligned}|w_n(t,\tau)\rangle &= a_n(\tau)Y_1\partial_y^n \hat{l}(y,t,\tau)|-j\rangle\\ &= a_n Y_1 \partial_y^n \hat{l}_4^{2j}\exp(\hat{l}_2\hat{l}_4^{-1}J_+|-j\rangle,\end{aligned} \tag{B9}$$

or what is the same,

$$\begin{aligned}|w_n(t,\tau)\rangle = a_n(\tau)Y_1\partial_y^n\{&\exp(2ij\arg\hat{l}_4)\\ &\times(|\hat{l}_2|^2 + |\hat{l}_4|^2)^j|\hat{l}_2\hat{l}_4^{-1}\rangle\},\end{aligned} \tag{B10}$$

in terms of the Bloch state $|\hat{\tau}\rangle = |\hat{l}_2\hat{l}_4^{-1}\rangle$. In the case where $n=0$ (and consequently the term has been prepared in a Bloch state), we have for $|w_n(t,\tau)\rangle$,

$$|w_0(t,\tau)\rangle = \exp(2ijY_1\arg\hat{l}_4)\cdot Y_1|\hat{l}_2\hat{l}_4^{-1}\rangle, \tag{B11}$$

a state which differs by a phase factor $2jY_1\arg l_4$ from the standard Bloch state corresponding to the complex number $\tau(t) = Y_1(\hat{l}_2\hat{l}_4^{-1})$.

The explicit expression shown in Eq. (B10) for $|w_n(t,\tau)\rangle$ allows an easy calculation of the transition number $\langle\mu|w_n(t,\tau)\rangle$ for an arbitrary coherent spin state $|\mu\rangle$,

$$\begin{aligned}\langle\mu|w_n(t,\tau)\rangle &= a_n(\tau)a_0(\mu)Y_1\partial_y^n\{(\hat{l}_4 + \mu^*\hat{l}_2)^{2j}\}\\ &= a_n(\tau)a_0(\mu)(!n)\binom{2j}{n}[\sin\psi(\sin\psi + \tau\cos\psi)\\ &\quad\times\exp(i\omega_+ t) + \cos\psi(\cos\psi - \tau\sin\psi)\exp(-i\omega_- t)\\ &\quad + \mu^*\sin\psi(\sin\psi\tau - \cos\psi)\exp(-i\omega_+ t)\\ &\quad + \mu^*\cos\psi(\tau\cos\psi + \sin\psi)\exp(i\omega_- t)]^n\\ &\quad\times[\sin\psi(\sin\psi - \tau\cos\psi)\exp(i\omega_+ t)\\ &\quad + \cos\psi(\cos\psi + \tau\sin\psi)\exp(-i\omega_- t)\\ &\quad - \mu^*\sin\psi(\tau\sin\psi + \cos\psi)\exp(-i\omega_+ t)\\ &\quad - \mu^*\cos\psi(\tau\cos\psi - \sin\psi)\exp(i\omega_- t)]^{2j-n'}.\end{aligned} \tag{B12}$$

This expression is very useful in order to investigate under what conditions $|w_n(t,\tau)\rangle$ could be an intelligent state. It is sufficient to calculate $\langle\mu|w_n'(\tau')\rangle$ and to compare its value with (B12). If we prove that there exists (n',τ') such that for any complex μ, $\langle\mu|w_{n'}(\tau')\rangle = \langle\mu|w_n(t,\tau)\rangle$, then the state $|w_n(t,\tau)\rangle$ keeps being intelligent along its evolution under the influence of the Hamiltonian given in Eq. (26). Since

$$\begin{aligned}&\langle\mu|w_{n'}(\tau')\rangle\\ &= a_{n'}(\tau')a_0(\mu)(!n')\binom{2j}{n''}(1+\mu^*\tau')^{n'}(1-\mu^*\tau')^{2j-n'},\end{aligned} \tag{B13}$$

and both polynomials in the variable μ^* (B12) and (B13) must be identical, they have to contain the same roots with the same multiplicity. Therefore, n' has to be equal to n. Moreover, if we proceed with the analysis, one can immediately recognize that they are going to coincide iff $\sin 2\psi = 0$. That implies $\cos 2\psi = (-1)^p$ or, equivalently, $\psi = n\pi/2$. The condition $\psi = n\pi/2$ [see Eq. (26b)] is equivalent to saying that $B_\perp = 0$. Thus, after Eq. (B8), we have

$$|w_n(t,\tau)\rangle = \exp(-2ij\gamma B_\parallel t)|w_n(\tau\exp(2i\gamma B_\parallel t))\rangle. \tag{B14}$$

Of course, if $\tau = |\tau|\exp(in\pi/2)$, $\tau' = \tau\exp(2i\gamma B_\parallel t) = |\tau|\exp(i(n\pi/2 + 2\gamma B_\parallel t))$ we get a generalized intelligent state, which is strictly intelligent for t such that $2\gamma B_\parallel t = m\pi/2$, i.e., it is periodically intelligent.

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[8]In the sense that they are $2j+1$ nonvanishing linearly independent vectors. Of course we do not know whether they are orthogonal. Their inner product is given in Appendix A.

[9]Actually it can be easily seen that τ_α ranges over all the points different from the origin of the two axes of the complex plane while α takes any real value $\alpha : |\alpha| \neq 1$. It is easy to see that $|\alpha|=1$ only gives a trivial solution to Eq. (8a): If $\alpha=+1$, $w=0$, $|w_N\rangle = |-j\rangle$, and if $\alpha=-1$, $w=0$, $|w_N\rangle = |j\rangle$. If one wants to extend the definition of $|w_N(\tau)\rangle$ to $\tau=0$, it turns out that $|w_N(0)\rangle = |-j\rangle$ for all N.

[10]Even directly, it is enough to realize that $|w_{N_1}(\tau_1)\rangle$ and $|w_{N_2}(\tau_2)\rangle$ are normalized eigenvectors corresponding to the same eigenvalue of $J\alpha$ and that both of them have the same signature for their projection along $|-j\rangle$.

[11]In fact, take from Eq. (9), $|w_n(\tau)\rangle = a_n(2j-n)!^{-1}(1+\tau\tau^*)^j \times \sum_{l=0}^{2j-n} \binom{2j-n}{l} (2j-l)!\,(-2\tau J_+)^l\,|\tau|\rangle = |\mu\rangle = (1+\mu\mu^*)^{-j}\exp(\mu J_+)\,|-j\rangle$. Then, if we multiply both sides times $\exp(\tau J_+)$ and compare them, we see that (n,τ) has to be either $(0,-\mu)$ or $(2j,\mu)$.

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Higher indices of group representations*

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The nth order index of an irreducible representation of a semisimple compact Lie group, n a nonnegative even integer, is defined as the sum of nth powers of the magnitudes of the weights of the representation. It is shown, in many situations, to have additivity properties similar to those of the dimension under reduction with respect to a subgroup and under reduction of a direct product. The second order index is shown to be Dynkin's index, multiplied by the rank of the group. Explicit formulas are derived for the fourth order index. A few reduction problems are solved with the help of higher indices as an illustration of their utility.

1. INTRODUCTION

Many years ago Dynkin[1,2] defined the *index of the irreducible representation* (IR) λ of a simple Lie group (all groups in this article are compact) by the formula

$$j_\lambda = N_\lambda(K_\lambda{}^2 - R^2)/r. \tag{1}$$

N_λ is the dimension of the IR, and r is the order of the group; $\mathbf{R}$ is half the sum of the positive weights of the adjoint representation and

$$\mathbf{K}_\lambda = \mathbf{M}_\lambda + \mathbf{R}, \tag{2}$$

where $\mathbf{M}_\lambda$ is the highest weight of the IR. The scale in weight space is fixed by giving the highest weight of the adjoint representation the magnitude $\sqrt{2}$. Dynkin's index is closely related to the second order Casimir operator whose eigenvalue[3] is $K_\lambda{}^2 - R^2$.

Dynkin showed that his index has additivity properties similar to those of the dimension under reduction of an IR with respect to a simple subgroup or under reduction of the direct product of two IR's. If the IR λ decomposes into the subgroup IR's μ, then

$$j_\lambda = \rho \sum_\mu j_\mu, \tag{3}$$

where ρ depends on the group and subgroup but not on the IR λ. Similarly, if the direct product of IR's 1 and 2 decomposes into the IR's λ, then

$$N_2 j_1 + N_1 j_2 = \sum_\lambda j_\lambda. \tag{4}$$

We generalize Dynkin's index by defining the nth order index, n a nonnegative even integer, as the sum of nth powers of the magnitudes of the weights of the IR:

$$I_\lambda^{(n)} = \Big(\sum_{\mathbf{x}} x^n\Big)_\lambda. \tag{5}$$

The sum is over all weights $\mathbf{x}$ belonging to the IR λ, each occurring a number of times equal to its multiplicity. The zeroth order index is just the dimension and in Sec. 3 we show that the second order index is just Dynkin's index (1), multiplied by the rank of the group.

In Sec. 2 relations analogous to (3) and (4) are shown to hold for certain higher indices, provided that the weight diagrams satisfy appropriate symmetry conditions. The definition (5) extends readily to semisimple groups so that (3) and its analogs for higher indices hold also when the subgroup is not simple.

In Sec. 4 formulas for the fourth order index are derived for all the simple groups.

Section 5 contains the solution of some reduction problems with the help of higher indices as an illustration of their utility.

2. PROPERTIES OF THE INDICES

The useful properties of the indices derive from the symmetry of the weight diagrams of the simple groups. We say that a weight system is rotationally symmetric (in l-dimensional weight space) to second order if

$$\sum_{\mathbf{x}} x_i = 0 \tag{6}$$

and

$$\sum_{\mathbf{x}} x_i x_j = \frac{\delta_{ij}}{l} \sum_{\mathbf{x}} x^2 \tag{7}$$

and that it is rotationally symmetric in third order if

$$\sum_{\mathbf{x}} x_i x_j x_k = 0 \tag{8}$$

and in fourth order if

$$\sum_{\mathbf{x}} x_i x_j x_k x_m = \frac{\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}}{l(l+2)} \sum_{\mathbf{x}} x^4; \tag{9}$$

x_i are the Cartesian coordinates of the weight $\mathbf{x}$. The conditions (6–9) would hold for a rotationally symmetric distribution.

The weight diagrams of all simple groups are rotationally symmetric to second order, and those of the exceptional groups G_2, F_4, E_6, E_7, and E_8 are rotationally symmetric to fourth order. SU(3) weights satisfy (9) but not (8). Trivially, SU(2) diagrams are symmetric to all orders. We indicate in Sec. 4 how to determine which of these conditions hold for a particular group.

When an IR of a group is reduced according to a subgroup, the effect on the weight diagram is a change of scale which may be different in different directions in weight space. Since the weight diagrams of all simple groups are rotationally symmetric in second order, the additivity property

$$I_\lambda^{(2)} = \rho_2 \sum_\mu I_\mu^{(2)} \tag{10}$$

follows immediately. The factor ρ_2 can be computed from the known scale changes. Equation (10) is just Dynkin's relation (3) [the relation between $I_\lambda{}^{(2)}$ and j_λ is derived in Sec. 3]. When the group whose IR is to be reduced is rotationally symmetric in fourth order [i.e. is SU(3) or an exceptional group], $I_\lambda^{(4)}$ has an additivity property like $I_\lambda{}^{(2)}$:

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$$I_\lambda{}^{(4)} = \rho_4 \sum_\mu I_\mu{}^{(4)}. \tag{11}$$

The subgroup is unrestricted.

Formulas (10), (11) hold equally when the subgroup is a direct product of simple groups. The second and fourth order indices for a direct product of n simple groups are naturally defined as

$$I^{(2)}_{\lambda_1\cdots\lambda_n} = \prod_i N_{\lambda_i} \sum_j I^{(2)}_{\lambda_j}/N_{\lambda_j}, \tag{12}$$

$$I^{(4)}_{\lambda_1\cdots\lambda_n} = \left(\prod_i N_{\lambda_i}\right)\left(\sum_j I_{\lambda_j}{}^{(4)}/N_{\lambda_j} + 2\sum_{j>k} I^{(2)}_{\lambda_j} I^{(2)}_{\lambda_k}/N_{\lambda_j}N_{\lambda_k}\right). \tag{13}$$

In case there is no change in scale in any direction under the subgroup reduction a simple additivity property holds for indices of all orders

$$I^{(n)}_\lambda = \sum_\mu I^{(n)}_\mu. \tag{14}$$

Equation (14) holds, for example, for $G_2 \supset \mathrm{SU}(3)$, $F_4 \supset \mathrm{O}(9)$, $E_6 \supset \mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3)$, $E_7 \supset \mathrm{SU}(8)$, $E_8 \supset \mathrm{O}(16)$, $\mathrm{O}(2n+1)\supset \mathrm{O}(2n)$. In some cases where the subgroup is a direct product of simple groups the sum of whose ranks equals the rank of the original group, Eq. (14) holds if the scale of weight space is suitably adjusted for the different groups comprising the direct product. An example is $G_2 \supset \mathrm{SU}(2)\times\mathrm{SU}(2)$.

We now turn to the additivity properties of the indices under reduction of direct products. When IR's 1 and 2 of a simple group are multiplied, the weights in the direct product are $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y}$ and $\mathbf{z}$ are the weights of the respective IR's 1 and 2.

Hence, if λ are the IR's in the direct product, we have

$$\sum_\lambda I^{(2)}_\lambda = \sum_{\mathbf{yz}} (y^2 + z^2) = N_2 I^{(2)}_1 + N_1 I^{(2)}_2 \tag{15}$$

in agreement with Dynkin's result (4). The fourth order index has a similar property:

$$\sum_\lambda I^{(4)}_\lambda = \sum_{\mathbf{yz}} (y^4 + z^4 + 2y^2z^2 + 4(\mathbf{y}\cdot\mathbf{z})^2)$$

$$= N_2 I_1{}^{(4)} + N_1 I_2{}^{(4)} + [2(l+2)/l] I^{(2)}_1 I^{(2)}_2. \tag{16}$$

Equations (15), (16) hold for all simple groups.

The analogous result for the sixth order index is

$$\sum_\lambda I^{(6)}_\lambda = N_2 I^{(6)}_1 + N_1 I^{(6)}_2 + [3(l+4)/l](I^{(4)}_2 I^{(2)}_1 + I^{(2)}_2 I^{(4)}_1). \tag{17}$$

In deriving (17) the condition (8) was assumed; accordingly, it holds for all simple groups except O(6) and SU(n) with $n \geq 3$.

Provided that (8) and (9) hold, we can deduce a similar relation for $I^{(8)}_\lambda$:

$$\sum_\lambda I^{(8)}_\lambda = N_2 I^{(8)}_1 + N_1 I^{(8)}_2 + \frac{4(l+6)}{l}(I^{(6)}_2 I^{(2)}_1 + I^{(2)}_2 I^{(6)}_1) + \frac{6(l+4)(l+6)}{l(l+2)} I^{(4)}_2 I^{(4)}_1, \tag{18}$$

(18) is valid for SU(2) and the exceptional groups.

3. THE SECOND ORDER INDEX

We now relate the second order index $I^{(2)}_\lambda$ to Dynkin's index (1). The character χ_λ of the IR λ of a simple group may be defined as

$$\chi_\lambda = \left(\sum_{\mathbf{x}} e^{\mathbf{x}\cdot\boldsymbol{\varphi}}\right)_\lambda. \tag{19}$$

The l components φ_i of φ are Cartesian coordinates in weight space. The indices may be expressed in terms of the character

$$I^{(2n)}_\lambda = \left(\sum_i \partial^2/\partial\varphi_i^2\right)^n \chi_\lambda \bigg|_{\varphi=0}. \tag{20}$$

The character may be written[4] in terms of the characteristic ξ_λ,

$$\chi_\lambda = \xi_\lambda/\Delta; \tag{21}$$

ξ_λ is defined by

$$\xi_\lambda = \sum_S (-1)^S \exp(\boldsymbol{\varphi}\cdot S\mathbf{K}_\lambda). \tag{22}$$

$\mathbf{K}_\lambda$ is given by (2); the sum is over Weyl reflections S; $(-1)^S$ is ± 1 according to whether S is even or odd. Δ in (21) is the characteristic of the scalar IR:

$$\Delta = \sum_S (-1)^S \exp(\boldsymbol{\varphi}\cdot S\mathbf{R}). \tag{23}$$

Using (20), (21), we can write for the second order index

$$I^{(2)}_\lambda = \frac{1}{\Delta}\sum_i \frac{\partial^2 \xi_\lambda}{\partial\varphi_i^2}\bigg|_{\varphi=0} - \frac{\chi_\lambda}{\Delta}\sum_i \frac{\partial^2\Delta}{\partial\varphi_i^2}\bigg|_{\varphi=0} - \frac{2}{\Delta}\sum_i \frac{\partial\chi_\lambda}{\partial\varphi_i}\frac{\partial\Delta}{\partial\varphi_i}\bigg|_{\varphi=0}.$$

From (22), (23) we see $\sum_i \partial^2\xi_\lambda/\partial\varphi_i^2 = K_\lambda^2 \xi_\lambda$ and $\sum_i \partial^2\Delta/\partial\varphi_i^2 = R^2\Delta$; hence, using

$$N_\lambda = \chi_\lambda|_{\varphi=0}, \tag{24}$$

we get

$$I^{(2)}_\lambda = N_\lambda(K_\lambda^2 - R^2) - \frac{2}{\Delta}\sum_i \frac{\partial\chi_\lambda}{\partial\varphi_i}\frac{\partial\Delta}{\partial\varphi_i}\bigg|_{\varphi=0}. \tag{25}$$

Expanding χ_λ, Eq. (19), to second degree in φ and using (6), (7), we find

$$\frac{\partial\chi_\lambda}{\partial\varphi_i} = \frac{1}{l} I^{(2)}_\lambda \varphi_i + \text{higher terms}. \tag{26}$$

Weyl has shown[4] [see Eq. (33)] that the leading term in Δ is of degree $\frac{1}{2}(r-l)$ in $\boldsymbol{\varphi}$; hence, by Euler's theorem on homogeneous functions,

$$\frac{1}{\Delta}\sum_i \frac{\partial\chi_\lambda}{\partial\varphi_i}\frac{\partial\Delta}{\partial\varphi_i}\bigg|_{\varphi=0} = \frac{r-l}{2l} I^{(2)}_\lambda.$$

It follows that

$$I^{(2)}_\lambda = l N_\lambda (K_\lambda^2 - R^2)/r. \tag{27}$$

The second order index is just Dynkin's index (1), multiplied by the rank l.

4. FOURTH ORDER INDEX

According to (20) we need the fourth degree term $\chi^{(4)}_\lambda$ in the character in order to determine the fourth order index. To this end we expand the characteristic

$$\xi_\lambda = (\xi_\lambda)_0(1 + \xi_\lambda^{(2)} + \xi_\lambda^{(4)} + \cdots); \tag{28a}$$

$(\xi_\lambda)_0$ is the leading term, of degree $\frac{1}{2}(r-l)$ in φ, and $\xi_\lambda^{(p)}$ is of degree p. We may ignore a possible term $\xi_\lambda^{(3)}$ because it cannot contribute to $\chi_\lambda^{(4)}$. Similarly

$$\Delta=\Delta_0(1+\Delta^{(2)}+\Delta^{(4)}+\cdots). \tag{28b}$$

Then, according to (21), the second and fourth degree terms in the character are

$$\chi_\lambda^{(2)}=N_\lambda(\xi_\lambda^{(2)}-\Delta^{(2)}), \tag{29}$$

$$\chi_\lambda^{(4)}=N_\lambda[\xi_\lambda^{(4)}-\Delta^{(4)}-\Delta^{(2)}(\xi_\lambda^{(2)}-\Delta^{(2)})]. \tag{30}$$

We can use (29) to help evaluate $\xi_\lambda^{(2)}$. From (19) and the rotational symmetry (7) in second order of the weights χ, we know that

$$\chi_\lambda^{(2)}=I_\lambda^{(2)}\varphi^2/2l=N_\lambda(K_\lambda^2-R^2)\varphi^2/2r. \tag{31}$$

Now Weyl has shown[4] that under the substitution

$$\varphi\to\mathbf{R}\theta \tag{32}$$

the characteristic becomes

$$\xi_\lambda\to\prod_{\boldsymbol\alpha}{}^{+}\sinh\frac{\theta}{2}\mathbf{K}_\lambda\cdot\boldsymbol\alpha. \tag{33}$$

The product is over the positive roots $\boldsymbol\alpha$. Incidentally, (33) illustrates that the degree of the leading term in ξ_λ (and in Δ) is $\frac{1}{2}(r-l)$, the number of positive roots.

Expanding ξ_λ and invoking the rotational symmetry of $\xi_\lambda^{(2)}$, we find

$$\xi_\lambda^{(2)}=\frac{\varphi^2}{24R^2}\sum_{\boldsymbol\alpha}{}^{+}(\mathbf{K}_\lambda\cdot\boldsymbol\alpha)^2=\frac{\varphi^2K_\lambda^2}{24lR^2}\sum_{\alpha}{}^{+}\alpha^2. \tag{34a}$$

The last step in (34a) follows from the rotational symmetry in second order of the roots $\boldsymbol\alpha$. Putting $\mathbf{K}_\lambda=\mathbf{R}$, we deduce

$$\Delta^{(2)}=\frac{\varphi^2}{24l}\sum_{\alpha}{}^{+}\alpha^2. \tag{34b}$$

Comparing (29), (31), and (34) shows that

$$\sum_{\alpha}{}^{+}\alpha^2=12lR^2/r \tag{35}$$

and hence that

$$\xi_\lambda^{(2)}=K_\lambda^2\varphi^2/2r,\quad \Delta^{(2)}=R^2\varphi^2/2r. \tag{36}$$

Weyl[4] has given formulas for the characteristics of the classical groups. We use them later in this section to determine their $\xi_\lambda^{(4)}$ and hence their fourth order indices. Because of the rotational symmetry of their weight diagrams in fourth order, the exceptional groups are actually easier to treat. Hence we deal with them first.

A. Exceptional groups

Using Eqs. (19) and (9), we find for the fourth degree term in the character of an exceptional group

$$\chi_\lambda^{(4)}=I_\lambda^{(4)}\varphi^4/8l(l+2) \tag{37}$$

or, under the substitution (32),

$$\chi_\lambda^{(4)}\to I_\lambda^{(4)}R^4\theta^4/8l(l+2). \tag{38}$$

It is straightforward to expand (33) and thus determine $\xi_\lambda^{(4)}$; $\Delta^{(4)}$ is then obtained by putting $\mathbf{K}_\lambda=\mathbf{R}$. Substituting for $\xi_\lambda^{(4)}$, $\Delta^{(4)}$, $\xi_\lambda^{(2)}$, $\Delta^{(2)}$ in (30) and comparing with (38), we find

$$I_\lambda^{(4)}=\frac{l+2}{N_\lambda l}\{I_\lambda^{(2)}\}^2-\frac{N_\lambda l(l+2)}{360R^4}\sum_{\boldsymbol\alpha}{}^{+}[(\mathbf{K}_\lambda\cdot\boldsymbol\alpha)^4-(\mathbf{R}\cdot\boldsymbol\alpha)^4]. \tag{39}$$

Because of the rotational symmetry in fourth order of the roots α, we can use (9) to derive

$$\sum_{\boldsymbol\alpha}{}^{+}(\mathbf{A}\cdot\boldsymbol\alpha)^4=\frac{3A^4}{l(l+2)}\sum_{\boldsymbol\alpha}{}^{+}\alpha^4, \tag{40}$$

where $\mathbf{A}$ in an arbitrary vector in weight space. Equation (40) can be used to simplify the second term in (39). The final result is

$$I_\lambda^{(4)}=\frac{l+2}{N_\lambda l}\{I_\lambda^{(2)}\}^2-\frac{N_\lambda}{120R^4}\{K^4-R^4\}\sum_{\boldsymbol\alpha}{}^{+}\alpha^4. \tag{41}$$

Equation (41) is valid for all five exceptional groups G_2, F_4, E_6, E_7, E_8 and for SU(2) and SU(3). The values of $\sum_{\boldsymbol\alpha}^{+}\alpha^4$ for these seven groups are SU(2): 4, SU(3): 12, G_2: 40/3, F_4: 60, E_6: 144, E_7: 252, E_8: 480.

B. Symplectic groups

According to Weyl[4] the characteristic of the symplectic group $\mathrm{Sp}(2n)=C_n$ is

$$\xi_\lambda=|\sinh(l_j\varphi_i/\sqrt{2})| \tag{42}$$

Here $|a_{ij}|$ means the $n\times n$ determinant whose ij element is a_{ij}. The l_j are representation labels. They are related to the Cartan labels λ_j by

$$l_j=\sum_{k=j}^{n}\lambda_k+n-j+1 \tag{43}$$

The Cartan labels are certain components of the highest weight, $\lambda_k=2\mathbf{M}_\lambda\cdot\boldsymbol\alpha_k/\alpha_k^2$, where α_k are the simple roots, ordered as in Table I of Ref. 5.

To expand ξ_λ in powers of φ, we first expand $\sinh(l_j\varphi_i/\sqrt{2})$

$$\xi_\lambda=2^{-n/2}\left(\prod_i l_i\varphi_i\right)\left|\sum_{\alpha=0}^{\infty}\frac{(l_j\varphi_i)^{2\alpha}}{2^\alpha(2\alpha+1)!}\right|. \tag{44}$$

Now repeat the following operation $n-1$ times, giving j in succession the values $1,\ldots,n-1$. Subtract the jth column from each column k for which $k>j$ and remove a factor $(l_k^2-l_j^2)/2(2j+1)!$ from the column k. Omitting a factor $(\prod_{i=1}^n\varphi_i)[2^{n^2/2}\prod_{\alpha=1}^n(2\alpha+1)!]^{-1}$ from ξ_λ (and Δ) we get finally

$$\xi_\lambda=\left(\prod_i l_i\right)\prod_{k>j}(l_k^2-l_j^2)|a_{ij}|, \tag{45}$$

where

$$a_{ij}=\sum_{\alpha=0}^{\infty}\frac{(2j-1)!\,p_\alpha(l_1^2,\ldots,l_j^2)\varphi_i^{2j-2+2\alpha}}{2^\alpha(2j+2\alpha-1)!}. \tag{46}$$

Here $p_\alpha(\epsilon_1,\ldots,\epsilon_j)$ is the symmetric function[4] of degree α defined by

$$\prod_{i=1}^{j}(1-z\epsilon_i)^{-1}=\sum_\alpha p_\alpha(\epsilon_1,\ldots,\epsilon_j)z^\alpha. \tag{47}$$

We will need $p_\alpha(\epsilon_1,\ldots,\epsilon_n)$ only for $\alpha\leqslant 4$. Explicitly, these functions are

$$p_0(\epsilon)=1,\quad p_1(\epsilon)=\sum_i\epsilon_i$$

$$p_2(\epsilon)=\sum_i\epsilon_i^2+\sum_{i>j}\epsilon_i\epsilon_j,$$

$$p_3(\epsilon)=\sum_i \epsilon_i^3+\sum_{i\neq j}\epsilon_i^2\epsilon_j+\sum_{i>j>k}\epsilon_i\epsilon_j\epsilon_k, \tag{47'}$$

$$p_4(\epsilon)=\sum_i \epsilon_i^4+\sum_{i\neq j}\epsilon_i^3\epsilon_j+\sum_{i>j}\epsilon_i^2\epsilon_j^2$$
$$+\sum_{i\neq j,k}\sum_{j>k}\epsilon_i^2\epsilon_j\epsilon_k+\sum_{i>j>k>l}\epsilon_i\epsilon_j\epsilon_k\epsilon_l.$$

In deriving (46) we used the identity

$$p_\alpha(\epsilon_1,\ldots,\epsilon_j,\epsilon)-p_\alpha(\epsilon_1,\ldots,\epsilon_j,\epsilon')$$
$$=(\epsilon-\epsilon')p_{\alpha-1}(\epsilon_1,\ldots,\epsilon_j,\epsilon,\epsilon'). \tag{48}$$

Retaining only the $\alpha=0$ terms in (45), we get

$$(\xi_\lambda)_0=\left(\prod_i l_i\right)\prod_{k>j}(l_k^2-l_j^2)\,|\varphi_i^{2j-2}|. \tag{49}$$

From (21) we get the well-known dimension formula for $\mathrm{Sp}(2n)$,[4]

$$N_\lambda=\left(\prod_i \frac{l_i}{l_i^0}\right)\prod_{k>j}\frac{(l_j-l_k)(l_j+l_k)}{(l_j^0-l_k^0)(l_j^0+l_k^0)}. \tag{50}$$

To get the fourth order term $\xi_\lambda^{(4)}$ we must in $|\varphi_i^{2j-2}|$

1. replace the last column φ_i^{2n-2} by

$[16n(n+1)(2n+1)(2n+3)]^{-1}p_2(l_1^2,\ldots,l_n^2)\varphi_i^{2n+2}$,

2. replace the second last column φ_i^{2n-4} by

$[16n(n-1)(2n-1)(2n+1)]^{-1}p_2(l_1^2,\ldots,l_{n-1}^2)\varphi_i^{2n}$,

3. replace the last column φ_i^{2n-2} by

$[4n(2n+1)]^{-1}p_1(l_1^2,\ldots,l_n^2)\varphi_i^{2n}$ *and* the second last column φ_i^{2n-4} by

$[4(n-1)(2n-1)]^{-1}p_1(l_1^2,\ldots,l_{n-1}^2)\varphi_i^{2n-2}$

and then add the three contributions.

The result is

$$\xi_\lambda^{(4)}=[16n(n+1)(2n+1)(2n+3)p_2(l_1^2,\ldots,l_n^2)$$
$$\times p_2(\varphi_1^2,\ldots,\varphi_n^2)+[16n(n-1)(2n-1)(2n+1)]^{-1}$$
$$\times\left(\sum_{i>j}l_i^2l_j^2\right)\left(\sum_{i>j}\varphi_i^2\varphi_j^2\right). \tag{51}$$

To get (51), we used Weyl's formula

$$|\epsilon_i^{l_j}|=|\epsilon_i^{j-1}|\cdot|p_{l_j-n+j}(\epsilon_1,\ldots,\epsilon_n)| \tag{52}$$

as well as the identity

$$p_1(\epsilon_1,\ldots,\epsilon_{n-1})p_1(\epsilon_1,\ldots,\epsilon_n)-p_2(\epsilon_1,\ldots,\epsilon_{n-1})=\sum_{i>j}\epsilon_i\epsilon_j. \tag{53}$$

In a similar manner we find

$$\xi_\lambda^{(2)}=[4n(2n+1)]^{-1}\left(\sum_j l_j^2\right)\left(\sum_i\varphi_i^2\right). \tag{54}$$

$\Delta^{(4)}$ and $\Delta^{(2)}$ are obtained from $\xi_\lambda^{(4)}$ and $\xi_\lambda^{(2)}$ by the replacement $l_i\to l_j^0$. Finally we get $I_\lambda^{(4)}$ by substituting for $\xi_\lambda^{(4)},\Delta^{(4)},\xi_\lambda^{(2)}$, and $\Delta^{(2)}$ in (30) and applying $\sum_i\partial^2/\partial\varphi_i^2$. The result is

$$I_\lambda^{(4)}=N_\lambda\left(\frac{(n+5)[p_2(l^2)-p_2((l^0)^2)]}{4(n+1)(2n+1)(2n+3)}\right.$$
$$\left.+\frac{\sum_{i>j}[(l_il_j)^2-(l_i^0l_j^0)^2]}{4(2n-1)(2n+1)}-\frac{(n+2)\sum_i(l_i^0)^2\sum_j[l_j^2-(l_j^0)^2]}{2n(2n+1)^2}\right). \tag{55}$$

For $I_\lambda^{(2)}$ we get similarly

$$I_\lambda^{(2)}=\frac{N_\lambda}{2(2n+1)}\sum_j[l_j^2-(l_j^0)^2]. \tag{56}$$

C. Orthogonal groups

We turn next to the orthogonal group in an odd number of dimensions $O(2n+1)=B_n$. The formula[4] for ξ_λ is the same as for $\mathrm{Sp}(2n)$, except that the scale is changed in weight space $\varphi_i\to\sqrt{2}\,\varphi_i$ [$\varphi\to2\varphi$ for O(3)] and the interpretation of the representation labels is different; instead of (43) we now have

$$l_j=\sum_{k=j}^{n-1}\lambda_k+\tfrac12\lambda_n+n-j+\tfrac12. \tag{57}$$

Because of the scale change the right-hand side of (55) should be multiplied by 4 [16 for O(3)] and the right-hand side of (56) should be multiplied by 2 [4 for O(3)]. The dimension N_λ is still given by (50) but with the l's defined by (57).

For the rotation group in an even number of dimensions, $O(2n)=D_n$, the characteristic is given by the sum of two $n\times n$ determinants[4]

$$\xi_\lambda=\tfrac12|\cosh l_j\varphi_i|+\tfrac12|\sinh l_j\varphi_i| \tag{58}$$

with

$$l_j=\sum_{k=j}^{n-2}\lambda_k+\tfrac12(\lambda_{n-1}+\lambda_n)+n-j,\quad 1\leqslant j\leqslant n-1,$$
$$l_n=\tfrac12(\lambda_{n-1}-\lambda_n). \tag{59}$$

The second determinant in (58) can be ignored for the purpose of calculating second and fourth order indices. For O(6) it contributes to $\xi^{(3)}$ but not to $\xi^{(2)}$ or $\xi^{(4)}$. For O(8) it contributes a term proportional to $\varphi_1\varphi_2\varphi_3\varphi_4$ to $\xi^{(4)}$ and nothing to $\xi^{(2)}$. For higher even orthogonal groups $O(2n)$ its lowest contribution is to $\xi^{(n)}$, with $n\geqslant5$. We do not consider O(4), since it is not simple anyway.

The determinant $|\cosh l_j\varphi_i|$ can be expanded just as $|\sinh l_j\varphi_i|$ was for $\mathrm{Sp}(2n)$ and $O(2n+1)$. For the dimension we get Weyl's result[4]

$$N_\lambda=\prod_{k>j}\frac{(l_j-l_k)(l_j+l_k)}{(l_j^0-l_k^0)(l_j^0+l_k^0)} \tag{60}$$

and for the second and fourth order indices we find

$$I_\lambda^{(2)}=(2n-1)^{-1}N_\lambda\sum_j(l_j+l_j^0)(l_j-l_j^0) \tag{61}$$

and

$$I_\lambda^{(4)}=N_\lambda\frac{(n+5)[p_2(l^2)-p_2((l^0)^2)]}{(n+1)(2n-1)(2n+1)}+\frac{\sum_{i>j}[(l_il_j)^2-(l_i^0l_j^0)^2]}{(2n-1)(2n-3)}$$
$$-\frac{2(n+2)\sum_i(l_i^0)^2\sum_j[l_j^2-(l_j^0)^2]}{(2n-1)(2n-3)}. \tag{62}$$

Formulas (60), (61) are valid for all $O(2n)$, (62), for $O(2n)$, $n\geqslant3$.

D. Special unitary groups

For the special unitary group $SU(n)=A_{n-1}$ the characteristic is the $n\times n$ determinant[4]

$$\xi_\lambda=|\exp l_j\eta_i|. \tag{63}$$

The representation labels l_j are given in terms of Cartan labels λ_j by

$$l_j=\sum_{k=j}^{n-1}\lambda_k+n-j \tag{64}$$

with $l_n=0$. The weight space coordinates η_i are not independent but satisfy

$$\sum_i \eta_i=0. \tag{65}$$

They may be expressed in terms of orthogonal coordinates φ_i:

$$\eta_i=-[(i-1)/i]^{1/2}\varphi_{i-1}+\sum_{j=i}^{n-1}[j(j+1)]^{-1/2}\varphi_j, \tag{66}$$

where $\varphi_0=0$. The gradient operator may be expressed in terms of η:

$$\frac{\partial}{\partial\varphi_i}=-\left(\frac{i}{i-1}\right)^{1/2}\frac{\partial}{\partial\eta_{i+1}}+[i(i+1)]^{-1/2}\sum_{j=i}^{i}\frac{\partial}{\partial\eta_j}$$

and similarly

$$\sum_i\frac{\partial^2}{\partial\varphi_i^2}=\frac{n-1}{n}\sum_i\frac{\partial^2}{\partial\eta_i^2}-\frac{2}{n}\sum_{i>j}\frac{\partial^2}{\partial\eta_i\partial\eta_j}. \tag{67}$$

The determinant (63) can be expanded in powers of η_i as before by expanding the exponential, subtracting columns, and removing factors. After cancelling an irrelevant factor from ξ_λ (and from Δ) we get

$$\xi_\lambda=\prod_{k<m}(l_k-l_m)|a_{ij}|, \tag{68}$$

where

$$a_{ij}=\sum_{\alpha=0}^{\infty}\frac{(j-1)!p_\alpha(l_1,\ldots,l_j)}{(j-1+\alpha)!}\eta_i^{j-1+\alpha}. \tag{69}$$

Weyl's dimension formula[4] for SU(n) may be written down

$$N_\lambda=\prod_{i<j}(l_i-l_j)/(l_i^0-l_j^0). \tag{70}$$

The quadratic term $\xi_\lambda^{(2)}$ is a sum of three parts which may be denoted by (0,2), (1,1), (2,0). Here (α_{n-1},α_n) means $|a_{ij}|$ with only the $\alpha=0$ term of the expansion (69) retained in every column except the last two, where only the α_{n-1} and α_n terms respectively are retained. We find for the second order index

$$I_\lambda^{(2)}=N_\lambda[n(n+1)]^{-1}\sum_{i<j}[(l_i-l_j)^2-(l_i^0-l_j^0)^2]. \tag{71}$$

The quartic term $\xi_\lambda^{(4)}$ is more complicated. It is the sum of seventeen terms which we denote by (0004), (0013), (0040), (0022), (0031), (0112), (0202), (0400), (0130), (1111), (1120), (2011), (2020), (1201), (1300), (3001), (4000). Here $(\alpha_{n-3}\alpha_{n-2}\alpha_{n-1}\alpha_n)$ implies the determinant $|a_{ij}|$ but with only the $\alpha=0$ term of the expansion retained, except for the last four columns, where only the $\alpha_{n-3},\alpha_{n-2},\alpha_{n-1},\alpha_n$ terms respectively are retained. The result for $I_\lambda^{(4)}$ is

$$\begin{aligned}I_\lambda^{(4)}=N_\lambda\Bigg[&[p_4(l)-p_4(l^0)]\frac{(n-1)(n^2+7n-6)}{n^2(n+1)(n+2)(n+3)}+[p_4(l)-p_1(l)p_3(l)-p_4(l^0)+p_1(l^0)p_3(l^0)]\frac{n^2+7n-6}{n^2(n+1)(n+2)}\\&+\{3[p_2(l)]^2-3[p_1(l)]^2p_2(l)+[p_1(l)]^4-p_4(l)-3[p_2(l^0)]^2+3[p_1(l^0)]^2p_2(l^0)-[p_1(l^0)]^4+p_4(l^0)\}\cdot\frac{1}{n}\\&+\{[p_2(l)]^2+p_1(l)p_3(l)-p_4(l)-[p_1(l)]^2p_2(l)-[p_2(l^0)]^2-p_1(l^0)p_3(l^0)+p_4(l^0)\\&+[p_1(l^0)]^2p_2(l^0)\}\frac{n-3}{n^2(n+1)}-\frac{1}{6}\sum_{i<j}\{(l_i-l_j)^2-(l_i^0-l_j^0)^2\}\Bigg].\end{aligned} \tag{72}$$

E. Symmetry of weight diagrams

In this and the preceding section, we have assumed that Eqs. (6), (7) are satisfied by the weight diagrams of all simple groups, and that, in addition, Eqs. (8), (9) hold for weight diagrams of the exceptional groups and SU(2). Equation (9) also holds for SU(3) diagrams. It remains to indicate how these properties may be verified.

In subsections B, C, D of this section, expansions of the characteristic ξ_λ for the classical groups are obtained to terms of degree 4 in φ. The absence of a term $\xi_\lambda^{(1)}$ and the fact that $\xi_\lambda^{(2)}$ is proportional to φ^2 proves the validity of Eqs. (6), (7) (rotational symmetry to second order) for these groups. The absence of a term $\xi_\lambda^{(3)}$ for the symplectic groups $Sp(2n)=C_n$, the odd orthogonal groups $O(2n+1)=B_n$, and the even orthogonal groups $O(2n)=D_n$ for $n\geq 4$ verifies the validity of (8) for those groups.

The additional symmetries claimed, Eqs. (8), (9) (rotational symmetry to fourth order) for the exceptional groups and Eq. (9) for SU(3) may be checked by explicit use of Weyl reflections. We illustrate the procedure for F_4. The point (x_1,x_2,x_3,x_4) in weight space is associated under Weyl reflections with the points $\frac{1}{2}(x_1-x_2-x_3-x_4, x_2-x_1-x_3-x_4, x_3-x_1-x_2-x_4, x_4-x_1-x_2-x_3)$ and $\frac{1}{2}(x_1+x_2+x_3+x_4, x_1+x_2-x_3-x_4, x_1+x_3-x_2-x_4, x_1+x_4-x_2-x_3)$; each of these three points is associated with $4!2^4-1=383$, others being obtained by permuting and reversing signs of components (1152 points in all, the member of F_4 Weyl reflections). It is easy to verify that Eqs. (6)–(9) hold for these points. For E_8, which has over two thirds of a billion Weyl reflections only five essentially different points are associated with an arbitrary point; the rest are obtained from them by permutations of components and by reversing signs of components in pairs.

5. EXAMPLES

The purpose of this section is to illustrate the validity and use of our results. The 2- and 4-indices serve for decomposing rather complicated direct products of representations and for finding branching rules. In each case we present two examples. The first one involving A_2 is elementary and results of our computation are well known; our second example concerning E_7 is highly nontrivial. As a third example one can take the content of a separate paper (Ref. 5) where some branching rules and Clebsch–Gordan series for E_8 are calculated.

In subsequent examples a reducible representation is determined entirely from equalities of dimensions, 2- and 4-indices. Practically one needs the values of N, $I^{(2)}$, and $I^{(4)}$ for any representation which could be relevant to the problem. Each of these quantities is given by an explicit algebraic expression; hence it is easily programmed and computed.

A. Clebsch-Gordan series

First we decompose the direct product of two representations (11) of A_2. In order to make our point about equality of dimensions and indices $I^{(2)}$ and $I^{(4)}$, we arrange the corresponding quantities as follows.

$$\begin{array}{lrcl} & (11)\times(11) &=& (22)+(30)+(03)+(11)+(11)+(00) \\ N: & 64 &=& 27+10+10+8+8+1 \\ I^{(2)}: & 192 &=& 108+30+30+12+12+0 \\ I^{(4)}: & 960 &=& 648+132+132+24+24+0 \end{array} \tag{73}$$

As our second example we decompose a product of three lowest representations of E_7. First consider the product of two only. One has

$$\begin{array}{lrcl} & (0000010)\times(0000010) &=& (0000020)+(0000100)+(1000000)+(0000000) \\ N: & 3136 &=& 1463+1539+133+1 \\ I^{(2)}: & 9408 &=& 4620+4536+252+0 \\ I^{(4)}: & 32256 &=& 16632+15120+504+0 \end{array} \tag{74}$$

In order to complete our example, we have to multiply each term in the direct sum in (74) by (0000010). One gets

$$\begin{array}{lrcl} & (0000010)\times(0000020) &=& (0000030)+(0000110)+(1000010)+(0000010) \\ N: & 81928 &=& 24320+51072+6480+56 \\ I^{(2)}: & 381612 &=& 120960+237888+22680+84 \\ I^{(4)}: & 2113650 &=& 713664+1308384+91476+126 \end{array} \tag{75}$$

$$\begin{array}{lrcl} & (0000010)\times(0000100) &=& (0000110)+(0001000)+(1000010)+(0000001)+(0000010) \\ N: & 86184 &=& 51072+27664+6480+912+56 \\ I^{(2)}: & 383292 &=& 237888+120120+22680+2520+84 \\ I^{(4)}: & 2020410 &=& 1308384+612612+91476+7812+126 \end{array} \tag{76}$$

$$\begin{array}{lrcl} & (0000010)\times(1000000) &=& (1000010)+(0000001)+(0000010) \\ N: & 7448 &=& 6480+912+56 \\ I^{(2)}: & 25284 &=& 22680+2520+84 \\ I^{(4)}: & 99414 &=& 91476+7812+126 \end{array} \tag{77}$$

B. Branching rules

In order to find branching rules (BR), one needs first to determine the factors ρ_2 and ρ_4 of (11) and (12) for every algebra–subalgebra pair. The most convenient is to use the BR for the lowest nontrivial representation which usually is the way that an embedding is specified.

We choose $SU(3)\supset O(3)$ as our first example. The fact that (10) of A_2 contains the representation (2) of A_1 gives

$$(10)\supset(2)$$

$$N:\quad 3=3$$

TABLE I. Representations of subalgebras A_7, E_6, $A_1 \times F_4$, $G_2 \times C_3$, and $A_2 \times A_5$ contained in the representation (0000010) of E_7.

subalgebra	representation	N	$I^{(2)}$	$I^{(4)}$
A_7	(0100000)	28	42	63
	(0000010)	28	42	63
E_6	(100000)	27	36	48
	(000010)	27	36	48
	2(000000)	2	0	0
$A_1 \times F_4$	(1) (0001)	52	74	109
	(3) (0000)	4	10	41
$G_2 \times C_3$	(01) (100)	42	45	$50\frac{1}{2}$
	(00) (001)	14	15	$19\frac{1}{2}$
$A_2 \times A_5$	(00) (00100)	20	30	45
	(10) (10000)	18	27	$40\frac{1}{2}$
	(01) (00001)	18	27	$40\frac{1}{2}$

$$I^{(2)}: 2 = \rho_2 \cdot 4 \tag{78}$$

$$I^{(4)}: \tfrac{4}{3} = \rho_4 \cdot 8$$

Hence $\rho_2 = \frac{1}{2}$ and $\rho_4 = \frac{1}{6}$. Then for (11) of A_2 it holds that

$$\begin{aligned} &(11) \supset (4) \quad + (2) \\ N:\ & 8 \quad = 5 \quad + 3 \\ I^{(2)}:\ & 12 \quad = \rho_2 (20 \ + 4) \\ I^{(4)}:\ & 24 \quad = \rho_4 (136 + 8) \end{aligned} \tag{79}$$

where ρ_2 and ρ_4 are the same as before.

The lowest representation (0000010) of E_7 has $N = 56$, $I^{(2)} = 84$, and $I^{(4)} = 126$. Let us consider the subalgebras A_7, E_6, $A_1 \times F_4$, $G_2 \times C_3$, and $A_2 \times A_5$. The embedding into E_7 is specified by their representations contained in (0000010) of E_7. Arranging vertically irreducible components of representations of subalgebras, one has Table I. From there we find ρ_2 and ρ_4. Thus for A_7, $\rho_2 = 1$, $\rho_4 = 1$; for E_6, $\rho_2 = \frac{7}{6}$, $\rho_4 = \frac{21}{16}$; for $A_1 \times F_4$, $\rho_2 = 1$, $\rho_4 = \frac{21}{25}$; for $G_2 \times C_3$, $\rho_2 = \frac{7}{5}$, $\rho_4 = \frac{9}{5}$; for $A_2 \times A_5$, $\rho_2 = 1$, $\rho_4 = 1$. For the present example we choose the representation (0000020) of E_7. It has $N = 1463$, $I^{(2)} = 4620$, and $I^{(4)} = 16632$. Inspecting representations of the subalgebras together with their $N, I^{(2)}$, and $I^{(4)}$, one concludes that only the representations shown in Table II satisfy the constraints imposed by equality of dimensions and indices. More precisely, for each subalgebra the column N must add to the dimension 1463 of (0000020) of E_7, columns $I^{(2)}$ and $I^{(4)}$ must give $4620 \cdot \rho_2$ and $16632 \cdot \rho_4$, respectively, where ρ_2 and ρ_4 were determined above.

6. REMARKS AND CONCLUSIONS

It should be remarked that the additivity properties of the indices, which are the subject of this paper, arise from the similar properties of the character, of which the indices are the rotationally symmetric moments.

Under reduction of an IR λ of a group with respect to a subgroup, each point ϕ undergoes a linear transformation $\phi \to \phi' = \rho\phi$, where ϕ' is the corresponding subgroup point. The matrix of the transformation ρ may be singular (its rank is the rank of the subgroup). The character $\chi_\lambda(\phi)$ contains all information about the branching rule:

$$\chi_\lambda(\phi) \to \chi_\lambda(\rho\phi) = \chi_\lambda(\phi') = \sum_\mu \chi_\mu(\phi'); \tag{a}$$

a similar property holds for direct products:

$$\chi_1(\phi)\chi_2(\phi) = \sum_\lambda \chi_\lambda(\phi). \tag{b}$$

The characters are notoriously difficult to work with. Explicit expressions for them do not exist, to our knowledge, for the exceptional groups, except[6] for G_2. The moments of the weights of an IR

$$(I^{(p)}_{i_1, \ldots, i_p})_\lambda = \left(\prod_{j=1}^{p} \frac{\partial}{\partial \phi_{i_j}} \right) \chi_\lambda(\phi) \Big|_{\phi=0} \tag{c}$$

together carry the same information as the character. This suggests that it may be advantageous to compute the low moments of IR's of the simple groups. To distinguish contragredient IR's, it is necessary to consider at least one odd-dimensional moment. In Sec. 4 all moments up to dimension four are given, in essence, for all simple groups. For many reduction problems of practical interest, they are all that are required.

TABLE II. Representations of subalgebras A_7, E_6, $A_1 \times F_4$, $G_2 \times C_3$, and $A_2 \times A_5$ contained in (0000020) of E_7.

subalgebra	representation	N	$I^{(2)}$	$I^{(4)}$
A_7	(0200000)	336	1120	4256
	(0100010)	720	2240	7840
	(0000020)	336	1120	4256
	(0001000)	70	140	280
	(0000000)	1	0	0
E_6	(200000)	351	1008	3360
	(100010)	650	1800	5760
	(000020)	351	1008	3360
	2(100000)	54	72	96
	2(000010)	54	72	96
	3(000000)	3	0	0
$A_1 \times F_4$	(2) (0002)	972	3240	12744
	(0) (0010)	273	504	1176
	(4) (0001)	130	640	4616
	(2) (0001)	78	176	472
	(6) (0000)	7	56	784
	(2) (0000)	3	4	8
$G_2 \times C_3$	(02) (200)	567	1404	4212
	(01) (101)	490	1120	$3078\frac{2}{3}$
	(10) (010)	196	392	$925\frac{1}{3}$
	(00) (002)	84	216	768
	(01) (010)	98	140	$217\frac{1}{3}$
	(00) (200)	21	24	36
	(01) (000)	7	4	$2\frac{2}{3}$
$A_2 \times A_5$	(00) (00200)	175	600	2400
	(00) (10001)	35	60	120
	(20) (20000)	126	450	1836
	(01) (01000)	45	90	180
	(02) (00002)	126	450	1836
	(10) (00010)	45	90	180
	(10) (10100)	315	990	3420
	(01) (00101)	315	990	3420
	(11) (10001)	280	900	3240
	(00) (00000)	1	0	0

Our interest in the present paper was devoted to the indices of representations and their potential use in computations. However, the factors $\rho_2, \rho_4, \cdots$ are of fundamental interest too. Being independent of a particular representation, they characterize the subalgebra embedding. In fact, a quantity equivalent to ρ_2, called index of subalgebra, has been extensively used in classification of subalgebras of the exceptional Lie algebras. The ρ_2 alone does not allow to distinguish all nonconjugate but isomorphic subalgebras inside of the same algebra. It would be of interest to find whether or not thus ambiguity is completely eliminated by using, for instance, $\rho_2, \rho_4, \rho_6, \cdots$ for characterizing the subalgebras.

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Conservation of charge and the Einstein–Maxwell field equations

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In a space of four dimensions I determine all possible second-order vector–tensor field equations which are derivable from a variational principle, compatible with the notion of charge conservation and in agreement with Maxwell's equations in a flat space. The general solution to this problem contains the Einstein–Maxwell field equations (with cosmological term) as a special case.

1. SECOND-ORDER VECTOR-TENSOR FIELD THEORIES

In Einstein's theory of gravitation, the field equations governing the symmetric Lorentzian metric tensor, g_{ab}, and the antisymmetric electromagnetic field tensor, F_{ab}, in regions devoid of sources are

$$G^{ij} - 2(F^{ia}F^{j}{}_{a} - \tfrac{1}{4}g^{ij}F^{ab}F_{ab}) = 0 \qquad (1.1)$$

and

$$F^{ij}{}_{|j} = 0, \qquad (1.2)$$

where F_{ab} is defined by

$$F_{ab} = \psi_{a,b} - \psi_{b,a} \qquad (1.3)$$

and ψ_a denotes the vector potential of the electromagnetic field.[1] The above equations are referred to as the source-free Einstein–Maxwell field equations. It is well known[2] that these *second-order* field equations can be derived from a variational principle, in the sense that there exists a Lagrange scalar density L of the form

$$L = L(g_{ab}; g_{ab,i_1}; \ldots ; g_{ab,i_1\cdots i_\alpha}; \psi_a; \psi_{a,i_1}; \ldots ; \psi_{a,i_1\cdots i_\beta}), \qquad (1.4)$$

which is such that its associated Euler–Lagrange equations,[3] $E^{ij}(L) = 0$ and $E^i(L) = 0$, are equivalent to Eqs. (1.1) and (1.2), respectively.

Within the context of Einstein's theory when sources of the gravitational and electromagnetic field are present, Eqs. (1.1) and (1.2) are modified through the addition of $8\pi T^{ij}$ and $-4\pi J^i$ to the right-hand side of these equations respectively, where T^{ij} and J^i denote the energy–momentum tensor and charge–current vector of the sources. Now *in general* the law of conservation of charge is equivalent to the demand that J^i be divergence-free; i.e., $J^i{}_{|i} = 0$. Due to the fact that $F^{ij}{}_{|ji} \equiv 0$ we see that in the presence of sources the Einstein–Maxwell field equations are compatible with charge conservation.

In view of the accuracy to which the law of conservation of charge has been tested in physics to date it seems reasonable to require that any attempted generalization of the Einstein–Maxwell field equations should be consonant with this principle. The problem is, do such generalizations exist? More exactly, is the Einstein–Maxwell field theory (with cosmological term included[4]) unique among all possible vector–tensor field theories of gravitation and electromagnetism which satisfy the following three conditions:

(a) there exists a Lagrange scalar density of the form (1.4) which is such that in the absence of sources the field equations are given by $E^{ij}(L) = 0$ and $E^i(L) = 0$;

(b) the source-free field equations are at most of second-order in the derivatives of both g_{ij} and ψ_i and actually do contain terms involving either $g_{ij,hk}$ or $\psi_{i,hk}$;

(c) in the presence of sources the field equations assume the form $E^{ij}(L) = 8\pi\sqrt{g}\, T^{ij}$ and $E^i(L) = 16\pi\sqrt{g}\, J^i$, where $E^i(L)$ is such that $E^i(L)_{|i} \equiv 0$.

It is easily seen that the Einstein–Maxwell field theory is *not* uniquely determined by (a), (b), and (c). For if L is any Lagrange scalar density of the form $L = L(g_{ab}; F_{ab})$, then its associated Euler–Lagrange tensors are given by $E^{ij}(L) = \partial L/\partial g_{ij}$ and $E^i(L) = -2d/dx^j(\partial L/\partial F_{ij})$. Since $E^i(L)$ is a contravariant vector density and $\partial L/\partial F_{ij}$ is antisymmetric in i and j, it is clear that $E^i(L)_{|i} = 0$. Due to this observation we see that it is quite easy to construct vector–tensor field theories which satisfy the above three conditions and are distinct from the Einstein–Maxwell field theory.

Now we are all well aware of the success of Maxwell's equation, $F^{ij}{}_{|j} = 0$, in describing the behavior of the electromagnetic field in regions which are devoid of sources and such that gravitational effects are negligible. Thus it seems necessary to require that any attempted modification of the Einstein–Maxwell field theory must satisfy conditions (a)–(c) above, along with

(d) $E^i(L) = \gamma\sqrt{g}\, F^{ij}{}_{|j}$ when evaluated for a flat metric tensor, where γ is some nonzero real constant.

As a result of this last restriction we see that any vector–tensor field theory which satisfies conditions (a)–(d) will always be consistent with the principle of conservation of charge and will be compatible with Maxwell's equations in a flat space.

The purpose of this paper is to prove that in a space of four dimensions the Einstein–Maxwell field theory (with cosmological term included) is *not* the only vector–tensor field theory of gravitation and electromagnetism which satisfies conditions (a)–(d). This result will be an immediate consequence of the following theorem which we shall establish in the next section.

Theorem: In a space of four dimensions the most

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general pair of tensorial concomitants

$$A^{ij}=A^{ij}(g_{ab};g_{ab,c};g_{ab,cd};\psi_a;\psi_{a,b};\psi_{a,bc}) \tag{1.5a}$$

and

$$B^i=B^i(g_{ab};g_{ab,c};g_{ab,cd};\psi_a;\psi_{a,b};\psi_{a,bc}) \tag{1.5b}$$

(both of which are tensor densities) which satisfy the following assumptions:

(i) there exists a Lagrange scalar density

$$L=L(g_{ab};g_{ab,i_1};\cdots;g_{ab,i_1\cdots i_\alpha};\psi_a;\psi_{a,i_1};\cdots;\psi_{a,i_1\cdots i_\beta})$$

(where α and β are nonnegative integers) of class C^∞ for which

$$A^{ij}=E^{ij}(L) \quad\text{and}\quad B^i=E^i(L); \tag{1.6}$$

(ii) B^i is divergence-free; i.e.,

$$B^i{}_{|i}=0; \tag{1.7}$$

and

(iii) when evaluated for a flat metric

$$B^i=\gamma\sqrt{g}F^{ij}{}_{|j}, \tag{1.8}$$

where γ is a real constant, is given by

$$\begin{aligned}A^{ij}=&\lambda\sqrt{g}\,G^{ij}+\tau\sqrt{g}\,\delta^{iabc}_{defk}\,g^{dj}F_{al}F^{el}R_{bc}{}^{fk}\\&+\tau\sqrt{g}\,\delta^{iabc}_{defk}\,g^{dj}F_{ab}{}^{|k}F^{ef}{}_{|c}\\&+(\gamma/2)\sqrt{g}\,(F^{ia}F^j{}_a-\tfrac14 g^{ij}F^{ab}F_{ab})+\mu\sqrt{g}\,g^{ij}\end{aligned} \tag{1.9}$$

and

$$B^i=2\tau\sqrt{g}\,\delta^{iabc}_{defk}F^{de}{}_{|a}R_{bc}{}^{fk}+\gamma\sqrt{g}\,F^{ij}{}_{|j}, \tag{1.10}$$

where λ, τ, and μ are arbitrary real constants. Furthermore, a Lagrangian which yields A^{ij} and B^i as its Euler–Lagrange expressions is

$$\begin{aligned}L=&-\lambda\sqrt{g}\,R-(\tau/2)\sqrt{g}\,\delta^{abcd}_{efkl}\,F_{ab}F^{ef}R_{cd}{}^{kl}\\&-(\gamma/4)\sqrt{g}\,F^{ab}F_{ab}+2\mu\sqrt{g}.\end{aligned} \tag{1.11}$$

Now it is customary to assume that the tensorial concomitants (and *not* the field variables) appearing in the field equations governing physical field theories are of class C^∞. Consequently, the above theorem provides us with the form of the field equations of *all* vector–tensor field theories of gravitation and electromagnetism which satisfy assumptions (a)–(d). In fact the source-free field equations of any such field theory are given by $A^{ij}=0$ and $B^i=0$ for a suitable choice of λ, τ, μ, and γ. Thus, we see that due to the terms with coefficient τ in Eqs. (1.9) and (1.10) there do exist vector–tensor field theories which satisfy conditions (a)–(d) and yet are quite distinct from the Einstein–Maxwell field theory (with cosmological term). Moreover, it should be noted that the field equations of those vector–tensor field theories for which $\tau\neq 0$, involve a highly nonlinear interaction between the metric tensor and the electromagnetic field.

In Eqs. (1.9) and (1.10), λ, τ, μ, and γ are real constants, however, they do not all have the same units. If we assume the field variables g_{ij} and ψ_i are unitless (as is customary[5]), then the local coordinates must have units of length, since the line element $ds^2=g_{ij}\,dx^i\,dx^j$, has units of length squared. Now when using A^{ij} and B^i to determine vector–tensor field theories of gravitation and electromagnetism we plan to set $A^{ij}=8\pi\sqrt{g}\,T^{ij}$ and $B^i=16\pi\sqrt{g}\,J^i$ when sources are present. As a result A^{ij} and B^i must have units of (length)$^{-2}$, since T^{ij} and J^i have these units.[5] Due to this observation and the above remarks it is apparent that λ and γ must be unitless, whereas τ and μ must have units of (length)^2 and (length)$^{-2}$ respectively.[6]

We shall now proceed with a proof of the theorem.

2. THE PROOF THE THEOREM

In this section we shall give a proof of the theorem stated in the Introduction. However, many of the more odious details of the proof will be omitted because of their length.[7]

Throughout this section A^{ij} and B^i will serve to denote a pair of tensorial concomitants satisfying the hypotheses of the theorem.

In order to simplify the form of the ensuing expressions we shall adopt the following notation: If

$$C^{\cdots}_{\cdots}=C^{\cdots}_{\cdots}(g_{ab};g_{ab,c};g_{ab,cd};\psi_a;\psi_{a,b};\psi_{a,bc})$$

is any concomitant, then we define

$$C^{\cdots;ab}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial g_{ab}},\quad C^{\cdots;ab,c}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial g_{ab,c}},\quad C^{\cdots;ab,cd}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial g_{ab,cd}},$$

$$C^{\cdots;a}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial \psi_a},\quad C^{\cdots;a,b}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial \psi_{a,b}},\quad C^{\cdots;a,bc}_{\cdots}=\frac{\partial C^{\cdots}_{\cdots}}{\partial \psi_{a,bc}}.$$

If $C^{\cdots}_{\cdots}$ were a tensorial concomitant, then the quantities $C^{\cdots;ab,cd}_{\cdots}$ and $C^{\cdots;a,bc}_{\cdots}$ would also be tensorial concomitants.

Remark: It should be noted that some of the above derivatives of $C^{\cdots}_{\cdots}$ possess various symmetries. For example, $C^{\cdots;ab,cd}_{\cdots}=C^{\cdots;ba,cd}_{\cdots}=C^{\cdots;ab,dc}_{\cdots}$. These obvious symmetries will be used in the sequel without further mention. In addition it should also be noted that due to assumption (1.6), $A^{ij}=A^{ji}$. □

Lemma 1: The pair of tensorial concomitants A^{ij} and B^i must satisfy the following equations:

$$A^{ij}{}_{|j}+\tfrac12 F^i{}_j B^j=0, \tag{2.1}$$

$$E^{hk}(A^{ij})-A^{hk;ij}=0, \tag{2.2}$$

$$E^{h}(A^{ij})-B^{h;ij}=0, \tag{2.3}$$

$$E^{h}(B^{i})-B^{h;i}=0, \tag{2.4}$$

$$E^{hk}(B^{i})-A^{hk;i}=0. \tag{2.5}$$

Proof: If L is a Lagrange scalar density of the form $L=L(g_{ab};\cdots;g_{ab,i_1\cdots i_\alpha};\psi_a;\cdots;\psi_{a,i_1\cdots i_\beta})$, then the Euler–Lagrange expressions associated with L must satisfy the following identity[8]:

$$E^{ij}(L)_{|j}+\tfrac12 F^i{}_j E^j(L)+\tfrac12\psi^i E^j(L)_{|j}=0. \tag{2.6}$$

Equation (2.1) follows immediately from Eq. (2.6), since we desire $A^{ij}=E^{ij}(L)$ and $B^i=E^i(L)$ for a suitably chosen L, and $B^i{}_{|i}=0$.

If $D^{\cdots}_{\cdots}$ is any quantity of the form

$$D^{\cdots}_{\cdots}=D^{\cdots}_{\cdots}(g_{ab};\cdots;g_{ab,i_1\cdots i_\mu};\psi_a;\cdots;\psi_{a,i_1\cdots i_\nu})$$

which is an ordinary divergence [i.e., there exists a quantity

$$F^{\cdots k}_{\cdots} = F^{\cdots k}_{\cdots}(g_{ab};\ldots;g_{ab,i_1\cdots i_\mu};\psi_a;\ldots;\psi_{a,i_1\cdots i_\nu}),$$

which is such that $D^{\cdots}_{\cdots} = (d/dx^k)F^{\cdots k}_{\cdots}$], then[9] $E^{ij}(D^{\cdots}_{\cdots})$ and $E^i(D^{\cdots}_{\cdots})$ vanish identically.[10]

Now it is apparent that the Euler–Lagrange tensor $E^{ij}(L)$ can be expressed in the form $E^{ij}(L) = \partial L/\partial g_{ij} + (d/dx^a)F^{ija}$. As a result of this fact and the above remarks, $E^{hk}(E^{ij}(L)) = E^{hk}(\partial L/\partial g_{ij})$. It is easily shown that E^{hk} and $\partial/\partial g_{ij}$ commute, consequently

$$E^{hk}(E^{ij}(L)) - \frac{\partial}{\partial g_{ij}} E^{hk}(L) = 0. \tag{2.7}$$

Since we desire $A^{ij} = E^{ij}(L)$ for a suitably chosen L we see that Eq. (2.7) implies Eq. (2.2).

Equations (2.3)–(2.5) are established in a similar manner.[11] □

Now the tensorial concomitants A^{ij} and B^i must satisfy Eqs. (2.2)–(2.5) in all coordinate systems. By examining how the quantities appearing on the left-hand side of these equations transform under a coordinate transformation it can be shown[7] that the partial derivatives of A^{ij} and B^i must satisfy the following equations:

$$A^{ab;cd,ef} = A^{cd;ab,ef}, \tag{2.8}$$

$$2\frac{d}{dx^j}A^{ab;cd,ej} - A^{ab;cd,e} - A^{cd;ab,e} = 0, \tag{2.9}$$

$$A^{ab;c,de} = B^{c;ab,de}, \tag{2.10}$$

$$2\frac{d}{dx^j}A^{ab;c,dj} - A^{ab;c,d} - B^{c;ab,d} = 0, \tag{2.11}$$

$$B^{(a;|b|,cd)} = B^{b;(a,cd)}, \tag{2.12}$$

$$2\frac{d}{dx^j}B^{a;b,cj} + 2\frac{d}{dx^j}B^{c;b,aj} - B^{a;b,c} - B^{c;b,a} - B^{b;a,c} - B^{b;c,a} = 0, \tag{2.13}$$

where round brackets about a collection of indices denotes symmetrization over all of the enclosed indices except for those with vertical bars about them.

Remark: It should be noted that Eq. (2.10) implies that A^{ab} is devoid of terms involving second order derivatives of ψ_a if and only if B^a is independent of terms involving second order derivatives of g_{ab}. □

We shall now proceed to examine some of the implications of the equation $B^h{}_{|h} = 0$, which, due to the fact that B^h is a contravariant vector density, is equivalent to $B^h{}_{,h} = 0$. Upon writing out the latter equation we obtain

$$B^{h;ab}g_{ab,h} + B^{h;ab,c}g_{ab,ch} + B^{h;ab,cd}g_{ab,cdh} + B^{h;a}\psi_{a,h} + B^{h;a,b}\psi_{a,bh} + B^{h;a,bc}\psi_{a,bch} = 0. \tag{2.14}$$

If this equation is now differentiated with respect to $g_{rs,tuv}$ and $\psi_{r,stu}$, we find that

$$B^{(t;|rs|,uv)} = 0 \tag{2.15}$$

and

$$B^{(s;|r|,tu)} = 0. \tag{2.16}$$

Due to Eqs. (2.10), (2.12), (2.15), and (2.16) we see that

$$A^{ab;(c,de)} = 0 \tag{2.17}$$

and

$$B^{r;(s,tu)} = 0. \tag{2.18}$$

Using Eqs. (2.13), (2.16), and (2.18) it is not difficult to show that

$$B^{(a;b,c)} = 0 \tag{2.19}$$

and hence

$$-2\frac{d}{dx^j}B^{j;b,ac} + B^{c;a,b} + B^{a;c,b} = 0. \tag{2.20}$$

To proceed further with our investigation of A^{ij} and B^i it will be necessary to make use of the invariance identities satisfied by these quantities. The invariance identities which we shall require are given below[12]

$$2A^{rs;b(t,uv)}g_{lb} + A^{rs;(t,uv)}\psi_l = 0, \tag{2.21}$$

$$B^{r;(s,tu)}\psi_l + 2B^{r;b(s,tu)}g_{lb} = 0. \tag{2.22}$$

Upon combining Eqs. (2.17), (2.18), (2.21), and (2.22) we find that

$$A^{rs;w(t,uv)} = 0 \tag{2.23}$$

and

$$B^{r;w(s,tu)} = 0. \tag{2.24}$$

Due to Eqs. (2.8) and (2.10) the above equations imply that

$$A^{w(t;|rs|,uv)} = 0 \tag{2.25}$$

and

$$A^{w(s;|r|,tu)} = 0. \tag{2.26}$$

In order to clearly enunciate the implications of the above work we require the following:

Definition[13]: A "quantity" $Q^{i_1i_2\cdots i_{2h-1}i_{2h}\cdots i_{2p-1}i_{2p}}$ $(p > 1)$ is said to have property S if:

(i) it is symmetric in the indices i_{2h-1}, i_{2h} for $h = 1,\ldots,p$;

(ii) it is symmetric under interchange of the pair of indices (i_1i_2) with the pair of indices $(i_{2h-1}i_{2h})$ for $h = 2,\ldots,p$;

(iii) it vanishes upon symmetrizing over any three of the four indices $i_1, i_2, i_{2h-1}, i_{2h}$, for $h = 2,\ldots,p$.[14]

Lovelock[13] has shown that if a quantity has property S then any component of that quantity vanishes whenever three or more of its indices assume the same numerical values. Consequently, in a space of four dimensions, any quantity which has property S in ten or more of its indices is identically zero.

Using Eqs. (2.10), (2.15)–(2.18), and (2.23)–(2.26) it is not difficult to prove[7]

Lemma 2: The partial derivatives of A^{ab} and B^a satisfy the following conditions: $A^{ab;cd,ef}$ has property

S, $A^{ab;c,de;r,st}$ has property S in the indices a,b,d,e,s,t,c,r, $B^{c;ab,de;r,st}$ has property S in the indices a,b,d,e,s,t,c,r,

$$A^{ab;cd,ef;rs,tu}=0, \tag{2.27}$$

$$A^{ab;cd,ef;r,st}=0, \tag{2.28}$$

$$A^{ab;c,de;r,st;u,vw}=0, \tag{2.29}$$

$$B^{a;bc,de;rs,tu}=0,\quad B^{a;bc,de;r,st;u,vw}=0.$$

We shall now employ the above lemma to get a rough idea of the functional form of the tensorial concomitants A^{ab} and B^a.

Due to Eqs. (2.27) and (2.28) we see that

$$A^{hk;rs,tu}=G^{hkrstu}(g_{ab};g_{ab,c};\psi_a;\psi_{a,b}), \tag{2.30}$$

where G^{hkrstu} is a tensorial concomitant of the indicated functions which enjoys property S. Integrating Eq. (2.30) with respect to $g_{rs,tu}$ shows us that

$$A^{hk}=G^{hkrstu}g_{rs,tu}+\tilde{D}^{hk}(g_{ab};g_{ab,c};\psi_a;\psi_{a,b};\psi_{a,bc}). \tag{2.31}$$

Using the symmetries of G^{hkrstu} it can be shown that Eq. (2.31) may be rewritten as follows:

$$A^{hk}=\tfrac{2}{3}G^{hkrstu}R_{rtus}+D^{hk}(g_{ab};g_{ab,c};\psi_a;\psi_{a,b};\psi_{a,bc}). \tag{2.32}$$

Since A^{hk} and $G^{hkrstu}R_{rtus}$ are symmetric tensorial concomitants, D^{hk} must also be a symmetric tensorial concomitant of the specified functional form.

Equations (2.29) and (2.32) tell us that

$$D^{hk;r,st;u,vw}=E^{hkrstuvw}(g_{ab};g_{ab,c};\psi_a;\psi_{a,b}), \tag{2.33}$$

where $E^{hkrstuvw}$ is a tensorial concomitant of the indicated functions and enjoys property S in the indices h,k,s,t,v,w,r,u. Upon integrating Eq. (2.33) with respect to $\psi_{r,st}$ and $\psi_{u,vw}$ it can easily be shown that

$$D^{hk}=\tfrac{1}{2}E^{hkrstuvw}\psi_{r,st}\psi_{u,vw}+\tilde{p}^{hkrst}\psi_{r,st}+\tilde{Q}^{hk}, \tag{2.34}$$

where $\tilde{p}^{hkrst}$ and $\tilde{Q}^{hk}$ are nontensorial quantities constructed from $g_{ab};g_{ab,c};\psi_a$ and $\psi_{a,b}$.

Now it is possible to prove that[7]

$$\begin{aligned}E^{hkrstuvw}=(K/\sqrt{g})\{&\epsilon^{hsvr}\epsilon^{ktwu}+\epsilon^{hsvu}\epsilon^{ktwr}\\&+\epsilon^{hswr}\epsilon^{ktvu}+\epsilon^{hswu}\epsilon^{ktvr}+\epsilon^{ksvr}\epsilon^{htwu}\\&+\epsilon^{ksvu}\epsilon^{htwr}+\epsilon^{kswr}\epsilon^{htvu}+\epsilon^{kswu}\epsilon^{htvr}\},\end{aligned} \tag{2.35}$$

where ϵ^{hsvr} denotes the four-dimensional Levi-Civita symbol, and K is a scalar tensorial concomitant of $g_{ab};g_{ab,c};\psi_a$ and $\psi_{a,b}$.

When Eqs. (2.34) and (2.35) are combined we discover that [see Eq. (1.3)]

$$D^{hk}=(K/\sqrt{g})\,\epsilon^{hsvr}\epsilon^{ktwu}F_{rs,t}F_{uw,v}+\tilde{p}^{hkrst}\psi_{r,st}+\tilde{Q}^{hk}$$

and hence

$$D^{hk}=(K/\sqrt{g})\,\epsilon^{hsvr}\epsilon^{ktwu}F_{rs|t}F_{uw|v}+P^{hkrst}\psi_{r,st}+Q^{hk}, \tag{2.36}$$

where P^{hkrst} and Q^{hk} are concomitants of $g_{ab};g_{ab,c};\psi_a$ and $\psi_{a,b}$. In addition it should be noted that P^{hkrst} can be chosen to be symmetric in s and t.

Upon differentiating Eq. (2.36) with respect to $\psi_{r,st}$, we find that P^{hkrst} must be a tensorial concomitant.

Due to Eqs. (2.32) and (2.36) we have shown that[15]

$$\begin{aligned}A^{hk}=\tfrac{2}{3}G^{hkrstu}R_{rtus}&+K\sqrt{g}\,\delta^{hrsv}_{luwt}\,g^{lk}F_{rs}{}^{|t}F^{uw}{}_{|v}\\&+P^{hkrst}\psi_{r,st}+Q^{hk},\end{aligned} \tag{2.37}$$

where G^{hkrstu}, K, and P^{hkrst} are tensorial concomitants of $g_{ab};g_{ab,c};\psi_a$ and $\psi_{a,b}$, while (for the present) Q^{hk} is simply a symmetric concomitant of these same arguments.

In a similar manner it can be demonstrated that[7]

$$B^h=2K\sqrt{g}\,\delta^{hvrs}_{wukt}F^{wu}{}_{|v}R_{rs}{}^{kt}+\tfrac{2}{3}H^{hjkst}R_{sjkt}+M^h, \tag{2.38}$$

where H^{hjkst} and M^h are tensorial concomitants of the form

$$H^{hjkst}=H^{hjkst}(g_{ab};g_{ab,c};\psi_a;\psi_{a,b})$$

and

$$M^h=M^h(g_{ab};g_{ab,c};\psi_a;\psi_{a,b};\psi_{a,bc}),$$

and K is the same scalar concomitant as the one appearing in Eq. (2.37). In addition, H^{hjkst} has property S in the indices j,k,s,t.

Remark: It must be noted that if A^{hk} and B^h satisfy the assumptions of our theorem then they are necessarily expressible in the form (2.37) and (2.38) for *some* choice of $G^{\cdots}$, K, $P^{\cdots}$, $Q^{\cdot\cdot}$, $H^{\cdots}$, and $M^{\cdot}$. However, that does *not* imply that any pair of tensorial concomitants of the form (2.37) and (2.38) satisfies the assumptions of our theorem. □

We shall now derive a few lemmas which will help us to construct the various concomitants appearing in Eqs. (2.37) and (2.38).

Lemma 3: The tensorial concomitants A^{hk} and B^h are such that

$$A^{hk;r,s}+A^{hk;s,r}=0 \tag{2.39}$$

and

$$B^{h;r,s}+B^{h;s,r}=0. \tag{2.40}$$

Proof: Due to our previous work, equation $B^h{}_{,h}=0$ can be written as follows:

$$B^{h;ab}g_{ab,h}+B^{h;ab,c}g_{ab,ch}+B^{h;a}\psi_{a,h}+B^{h;a,b}\psi_{a,bh}=0. \tag{2.41}$$

Upon differentiating this equation with respect to $\psi_{r,st}$, we obtain

$$\begin{aligned}B^{h;r,st;ab}g_{ab,h}&+B^{h;r,st;ab,c}g_{ab,ch}+B^{h;r,st;a}\psi_{a,h}\\&+B^{h;r,st;a,b}\psi_{a,bh}+\tfrac{1}{2}(B^{s;r,t}+B^{t;r,s})=0,\end{aligned}$$

or equivalently

$$2\frac{d}{dx^h}B^{h;r,st}+B^{s;r,t}+B^{t;r,s}=0.$$

Combining this equation with Eq. (2.20) shows us that

$$B^{a;b,c}+B^{a;c,b}+B^{c;b,a}+B^{c;a,b}=0,$$

and hence we may employ Eq. (2.19) to conclude that

$$B^{b;a,c}+B^{b;c,a}=0$$

as desired.

In an analogous manner[7] we can use Eqs. (2.10), (2.11), (2.15), and (2.41) to establish Eq. (2.39). □

Lemma 4: The tensorial concomitants A^{hk} and B^h are independent of ψ_a, i.e.,

$$A^{hk;a} = 0, \tag{2.42a}$$

and

$$B^{h;a} = 0. \tag{2.42b}$$

Proof: Equation (2.4) tells us that

$$0 = \frac{d^2}{dx^l\, dx^m} B^{i;h,lm} - \frac{d}{dx^l} B^{i;h,l} + B^{i;h} - B^{h;i}.$$

We may use Eq. (2.16) to rewrite the above equation as follows:

$$0 = -2 \frac{d^2}{dx^l\, dx^m} B^{l;h,mi} - \frac{d}{dx^l} B^{i;h,l} + B^{i;h} - B^{h;i}. \tag{2.43}$$

Using Eq. (2.20) to replace $-2(d^2/dx^l\, dx^m) B^{l;h,mi}$ in Eq. (2.43), we find that

$$0 = -\frac{d}{dx^m} B^{m;i,h} - B^{h;i} + B^{i;h}, \tag{2.44}$$

where we have employed Eq. (2.40) to conclude that

$$0 = \frac{d}{dx^m} B^{i;m,h} + \frac{d}{dx^m} B^{i;h,m}.$$

If we now differentiate Eq. (2.41) with respect to $\psi_{i,r}$ we see that

$$0 = \frac{d}{dx^h} B^{h;i,r} + B^{r;i}. \tag{2.45}$$

Upon combining Eqs. (2.44) and (2.45) we discover that $B^{i;h}$ vanishes as desired.

We shall now prove Eq. (2.42a). To begin with, equation (2.5) implies that

$$0 = \frac{d^2}{dx^l\, dx^m} B^{i;hk,lm} - \frac{d}{dx^l} B^{i;hk,l} + B^{i;hk} - A^{hk;i}.$$

Due to Eq. (2.15) we may rewrite the above equation as follows:

$$0 = -2 \frac{d^2}{dx^l\, dx^m} B^{m;hk,li} - \frac{d}{dx^l} B^{i;hk,l} + B^{i;hk} - A^{hk;i}. \tag{2.46}$$

Upon differentiating Eq. (2.41) with respect to $g_{rs,tu}$ and $g_{rs,t}$ we find that

$$2 \frac{d}{dx^h} B^{h;rs,tu} + B^{t;rs,u} + B^{u;rs,t} = 0 \tag{2.47a}$$

and

$$\frac{d}{dx^h} B^{h;rs,t} + B^{t;rs} = 0. \tag{2.47b}$$

Using Eq. (2.47a) we see that Eq. (2.46) can be rewritten as follows:

$$0 = \frac{d}{dx^l} B^{l;hk,i} + B^{i;hk} - A^{hk;i}. \tag{2.48}$$

Equations (2.47b) and (2.48) imply that $A^{hk;i} = 0$. □

We shall presently employ Lemmas 3 and 4 to prove that the tensorial concomitants P^{hkrst} and H^{rhkst} occurring in equations (2.37) and (2.38) must vanish. However, in order to do so we shall require the following technical result.[7]

Lemma 5: In a space of dimension n, any tensorial concomitant $\Phi^{i_1\cdots i_q}$ $(q \geq 0)$ of the form $\Phi^{i_1\cdots i_q} = \Phi^{i_1\cdots i_q}(g_{ab}; g_{ab,c}; \psi_{a,b})$ is independent of $g_{ab,c}$ and such that

$$\Phi^{i_1\cdots i_q;a,b} = -\Phi^{i_1\cdots i_q;b,a}.$$

In particular, if n is even and q is odd then $\Phi^{i_1\cdots i_q}$ vanishes.[16]

We are now in a position to prove

Lemma 6: If the pair of tensorial concomitants A^{hk} and B^h satisfy the assumptions of the theorem, then they must necessarily be expressible as follows:

$$A^{hk} = \tfrac{2}{3} G^{hkrstu} R_{rtus} + K\sqrt{g}\, \delta^{hrsv}_{luwt}\, g^{lk} F_{rs}{}^{|t} F^{uw}{}_{|v} + Q^{hk}, \tag{2.49}$$

$$B^h = 2K\sqrt{g}\, \delta^{hvjs}_{wukt} F^{wu}{}_{|v} R_{js}{}^{kt} + M^h, \tag{2.50}$$

where

(a) G^{hkrstu}, K, and Q^{hk} are tensorial concomitants of g_{ab} and $\psi_{a,b}$;

(b) M^h is a tensorial concomitant of $g_{ab}; g_{ab,c}; \psi_{a,b}$ and $\psi_{a,bc}$;

(c) G^{hkrstu} has property S and Q^{hk} is symmetric.

Proof: Due to Lemma 4 it is an elementary matter to prove that the tensorial concomitants G^{hkrstu}, K, P^{hkrst}, H^{rhkst}, and M^h appearing in Eqs. (2.37) and (2.38) are independent of ψ_a. Thus we may now use Lemma 5 to deduce that

(i) G^{hkrstu} and K are tensorial concomitants of g_{ab} and $\psi_{a,b}$;

(ii) P^{hkrst} and H^{rhkst} vanish.

As a result of (ii), Eqs. (2.37) and (2.38) now assume the form (2.49) and (2.50), respectively.

Since P^{hkrst} vanishes in Eq. (2.37), Q^{hk} must be a tensorial concomitant. Thus we may once again apply Lemma 4 to Eq. (2.37) to deduce that Q^{hk} is independent of ψ_a. Consequently Lemma 5 may be invoked to conclude that Q^{hk} is a concomitant of g_{ab} and $\psi_{a,b}$. □

Using many of our previous results it can be shown[7] that the scalar concomitant K appearing in Eqs. (2.49) and (2.50) must be a *real constant* and that

$$G^{hkrstu} = \frac{1}{\sqrt{g}} \left\{\frac{K}{2} F_{ac} F_b{}^c + m g_{ab}\right\} \times [\epsilon^{hrta}\epsilon^{ksub} + \epsilon^{hrua}\epsilon^{kstb} + \epsilon^{hsta}\epsilon^{krub} + \epsilon^{hsua}\epsilon^{krtb}], \tag{2.51}$$

where m is a real constant.

Upon combining Lemma 6 with Eq. (2.51) and the fact that K is a real constant we easily obtain

Lemma 7: Any pair of tensorial concomitants A^{hk} and B^h satisfying the assumptions of the theorem must be expressible in the following fashion:

$$A^{hk} = \lambda\sqrt{g}\, G^{hk} + \tau\sqrt{g}\, \delta^{habc}_{defj} g^{dk} F_{al} F^{el} R_{bc}{}^{fj} + \tau\sqrt{g}\, \delta^{habc}_{defj}\, g^{dk} F_{ab}{}^{|j} F^{ef}{}_{|c} + Q^{hk}, \tag{2.52}$$

and

$$B^h = 2\tau\sqrt{g}\ \delta^{habc}_{defj} F^{de}{}_{|a} R_{bc}{}^{fj} + M^h, \tag{2.53}$$

where λ and τ are real constants. Furthermore, Q^{hk} $(=Q^{kh})$ and M^h are tensorial concomitants of the following form:

$$Q^{hk} = Q^{hk}(g_{ab};\psi_{a,b})$$

and

$$M^h = M^h(g_{ab}; g_{ab,c};\psi_{a,b};\psi_{a,bc}).$$

It is well known that the term with coefficient λ in Eq. (2.52) is an Euler–Lagrange tensor; in fact

$$\sqrt{g}\ G^{hk} = E^{hk}(-\sqrt{g}\,R). \tag{2.54}$$

The purpose of our next lemma is to show that the terms with coefficient τ in Eqs. (2.52) and (2.53) are also Euler–Lagrange tensors.

Lemma 8: The Euler–Lagrange tensors of the Lagrange scalar density

$$L = -(\tau/2)\sqrt{g}\ \delta^{tvab}_{wulm} F_{tv} F^{wu} R_{ab}{}^{lm} \tag{2.55}$$

are given by

$$E^{hi}(L) = \tau\sqrt{g}\ \delta^{habc}_{defk} g^{di} F_{al} F^{el} R_{bc}{}^{fk} + \tau\sqrt{g}\ \delta^{habc}_{defk} g^{di} F_{ab}{}^{|k} F^{ef}{}_{|c}, \tag{2.56}$$

and

$$E^i(L) = 2\tau\sqrt{g}\ \delta^{iabc}_{defj} F^{de}{}_{|a} R_{bc}{}^{fj}. \tag{2.57}$$

Proof: We begin by computing $E^i(L)$.

Under the present circumstances $E^i(L)$ is given by $E^i(L) = -(d/dx^j)L^{;i,j}$. Since L is a scalar density and $L^{;i,j} = -L^{;j,i}$, our expression for $E^i(L)$ can be rewritten as follows:

$$E^i(L) = -L^{;i,j}{}_{|j}. \tag{2.58}$$

Using Eq. (2.55) we easily find that

$$L^{;i,j} = -2\tau\sqrt{g}\ \delta^{ijab}_{wulm} F^{wu} R_{ab}{}^{lm}. \tag{2.59}$$

Due to Eqs. (2.58), (2.59), and the fact that $\delta^{ijab}_{wulm} R_{ab}{}^{lm}{}_{|j}$ vanishes identically (in view of the Bianchi identity), we see that

$$E^i(L) = 2\tau\sqrt{g}\ \delta^{ijab}_{wulm} F^{wu}{}_{|j} R_{ab}{}^{lm}$$

as claimed.

We shall now proceed to compute $E^{hi}(L)$.

Rund[17] has shown that under the present assumptions on the form of L, $E^{hi}(L)$ can be expressed as follows:

$$E^{hi}(L) = L^{;hi,jk}{}_{|jk} - L^{;rs,hk} R_r{}^i{}_{ks} + \tfrac{1}{3} L^{;rs,ik} R_r{}^h{}_{ks} + \tfrac{1}{2} g^{hi} L - \tfrac{1}{2} L^{;r,h} F_r{}^i. \tag{2.60}$$

Using Eq. (2.55), it is not difficult to show that

$$L^{;hi,jk} = -(\tau/2)\sqrt{g}\,\{\delta^{tvhj}_{wulm} F_{tv} F^{wu} g^{lk} g^{mi} + \delta^{tvhk}_{wulm} F_{tv} F^{wu} g^{lj} g^{mi}\}. \tag{2.61}$$

Upon combining Eqs. (2.59)–(2.61) we find, after a lengthy calculation, that[7]

$$E^{hi}(L) = \tau\sqrt{g}\ \delta^{habc}_{defk} g^{di} F_{al} F^{el} R_{bc}{}^{fk} + \tau\sqrt{g}\ \delta^{habc}_{defk} g^{di} F_{ab}{}^{|k} F^{ef}{}_{|c} - \tfrac{3}{4}\tau\sqrt{g}\ \delta^{habcd}_{efklm} g^{ei} F_{ab} F^{fk} R_{cd}{}^{lm}. \tag{2.62}$$

Since we are working in a space of four dimensions, δ^{habcd}_{efklm} vanishes identically and consequently Eq. (2.62) reduces to Eq. (2.56). □

Our next lemma provides us with a partial converse to Lemma 7.

Lemma 9: Any pair of tensorial concomitants a^{hi} and b^i of the form

$$a^{hi} = \lambda\sqrt{g}\ G^{hi} + \tau\sqrt{g}\ \delta^{habc}_{defj} g^{di} F_{al} F^{el} R_{bc}{}^{fj} + \tau\sqrt{g}\ \delta^{habc}_{defj} g^{di} F_{ab}{}^{|j} F^{ef}{}_{|c}, \tag{2.63}$$

$$b^i = 2\tau\sqrt{g}\ \delta^{iabc}_{defj} F^{de}{}_{|a} R_{bc}{}^{fj}, \tag{2.64}$$

where λ and τ are arbitrary real constants, satisfy the assumptions of the theorem.

Proof: Owing to Eq. (2.54) and Lemma 8 it is apparent that a^{hi} and b^i satisfy all of the assumptions of our theorem except perhaps the second [viz., Eq. (1.7)]. We shall now show that this condition is also met.

Due to Eqs. (2.57) and (2.64) we have

$$b^i = -\frac{d}{dx^j} L^{;i,j}, \tag{2.65}$$

where L is given by Eq. (2.55). Since b^i is a contravariant vector density $b^i{}_{|i} = db^i/dx^i$, and hence we may use Eq. (2.65) to conclude that

$$b^i{}_{|i} = -\frac{d^2}{dx^i dx^j} L^{;i,j}. \tag{2.66}$$

Upon combining Eq. (2.66) with the fact that $L^{;i,j} = -L^{;j,i}$ [see Eq. (2.59)] we find that $b^i{}_{|i}$ vanishes as required. □

Remark: Due to the above lemma it is clear that the tensorial concomitant M^h appearing in Lemma 7 must be divergence-free. □

Our proof of the theorem will follow immediately from Lemmas 7 and 9 once the following result is established.

Lemma 10: If $\mathcal{A}^{hi} = \mathcal{A}^{hi}(g_{ab};\psi_{a,b})$ and $\mathcal{B}^i = \mathcal{B}^i(g_{ab}; g_{ab,c};\psi_{a,b};\psi_{a,bc})$ are tensor densities which satisfy assumptions (i), (ii), and (iii) of the theorem, then

$$\mathcal{A}^{hi} = (\gamma/2)\sqrt{g}\ (F^{ha}F^i{}_a - \tfrac{1}{4} g^{hi} F^{ab} F_{ab}) + \mu\sqrt{g}\ g^{hi},$$

and

$$\mathcal{B}^i = \gamma\sqrt{g}\ F^{ij}{}_{|j},$$

where μ is some real constant. Moreover, a Lagrangian which yields $\mathcal{A}^{hi}$ and $\mathcal{B}^i$ as its Euler–Lagrange expressions is

$$L = -(\gamma/4)\sqrt{g}\ F^{ab} F_{ab} + 2\mu\sqrt{g}.$$

Proof: Due to Eq. (2.11) we know that $\mathcal{A}^{ab;c,d} = -\mathcal{B}^{c;ab,d}$, and hence $\mathcal{B}^{c;r,st;ab,d} = 0$. As a result of this fact the tensorial concomitant $\beta^{i;r,st}$ must be independent of $g_{ab,c}$.

Since we desire $\beta^i = \gamma\sqrt{g}\, F^{ij}{}_{|j}$, when evaluated for a flat metric, we must have $\beta^{i;r,st,u,vw}$ and $\beta^{i;r,st;u,v}$ equal to zero when evaluated for a flat metric. Upon combining this observation with the fact that $\beta^{i;r,st}$ is always independent of $g_{ab,c}$ we may now conclude that $\beta^{i;r,st}$ is a concomitant of only g_{ab}. Consequently, there exists a tensorial concomitant $\Phi^{irst} = \Phi^{irst}(g_{ab})$ which is such that

$$\beta^{i;r,st} = \Phi^{irst}. \tag{2.67}$$

Due to Eq. (2.67) and (2.22) we see that Φ^{irst} has the following symmetries:

$$\Phi^{irst} = \Phi^{irts} \quad \text{and} \quad \Phi^{i(rst)} = 0.$$

Using the results of McKiernan[18] it is now easy to prove that

$$\Phi^{irst} = \alpha\sqrt{g}\,(g^{ir}g^{st} - \tfrac{1}{2}(g^{is}g^{rt} + g^{it}g^{rs})), \tag{2.68}$$

where α is some real constant.

Upon integrating Eq. (2.67) with the aid of Eq. (2.68) we find (through the use of Lemma 5) that

$$\beta^i = \alpha\sqrt{g}\, F^{ij}{}_{|j}$$

and hence $\alpha = \gamma$.

Now it is easily seen that if

$$\mathcal{L} = -(\gamma/4)\sqrt{g}\, F^{ab}F_{ab}, \tag{2.69}$$

then $E^i(\mathcal{L}) = \beta^i$. However, this does *not* imply that the $\mathcal{A}^{hi}$ corresponding to β^i need be $E^{hi}(\mathcal{L})$. What we can say though is that if $L = L(g_{ab};\ldots;g_{ab,i_1\ldots i_\alpha};\psi_a;\ldots;\psi_{a,i_1\ldots i_\beta})$ is a Lagrange scalar density which is such that $E^{hi}(L) = \mathcal{A}^{hi}$ and $E^i(L) = \beta^i$, then $E^i(L - \mathcal{L}) = 0$. Consequently, we may employ Eq. (2.6) to deduce that $E^{hi}(L - \mathcal{L})_{|i} = 0$. Since $E^{hi}(L - \mathcal{L})$ is a concomitant of only g_{ab} and $\psi_{a,b}$ we can now use a result of Lovelock's[19] to deduce that $E^{hi}(L - \mathcal{L}) = \mu\sqrt{g}\, g^{hi}$, where μ is some real constant. Upon combining this fact with Eq. (2.69) and the fact that $E^{hi}(L) = \mathcal{A}^{hi}$, we find that

$$\mathcal{A}^{hi} = (\gamma\sqrt{g}/2)(F^{ha}F^i{}_a - \tfrac{1}{4}g^{hi}F^{ab}F_{ab}) + \mu\sqrt{g}\, g^{hi}.$$

Since $E^{hi}(2\mu\sqrt{g}) = \mu\sqrt{g}\, g^{hi}$, and $E^i(2\mu\sqrt{g}) = 0$, our proof of Lemma 10 is now complete. □

Due to Lemmas 7, 9, and 10 our proof of the theorem is finally finished. □

At this time I would like to point out that owing to Lemmas 7 and 9 we can conclude that any pair of tensorial concomitants A^{hk} and B^h of the form (1.5a) and (1.5b), which satisfy assumptions (i) and (ii) of the theorem must be expressible in the form

$$A^{hk} = \lambda\sqrt{g}\, G^{hk} + \tau\sqrt{g}\,\delta^{habc}_{defj}\, g^{dk}F_{al}\, F^{el}R_{bc}{}^{fj} + \tau\sqrt{g}\,\delta^{habc}_{defj}\, g^{dk}\, F_{ab}{}^{|j}F^{ef}{}_{|c} + Q^{hk}$$

and

$$B^h = 2\tau\sqrt{g}\,\delta^{habc}_{defj}\, F^{de}{}_{|a}R_{bc}{}^{fj} + M^h,$$

for some choice of real constants λ and τ. Furthermore, Q^{hk} and M^h must be tensorial concomitants of the form $Q^{hk} = Q^{hk}(g_{ab};\psi_{a,b})$ and $M^h = M^h(g_{ab};g_{ab,c};\psi_{a,b};\psi_{a,bc})$ which are such that $Q^{hk} = E^{hk}(L)$ and $M^h = E^h(L)$ for some Lagrange scalar density L of the form (1.4). Now I presently believe that the Lagrangian L which yields Q^{hk} and M^h as its Euler–Lagrange expressions must be equivalent to a Lagrange scalar density $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{L}(g_{ab};F_{ab})$, in the sense that $E^{hk}(L) = E^{hk}(\mathcal{L})$ and $E^h(L) = E^h(\mathcal{L})$. However, so far I have not been able to either prove or disprove this conjecture.

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[1]Throughout this paper small Latin indices may assume the values 1 to 4 and obey the summation convention. Indices will be lowered and raised by means of g_{ij} and its (matrix) inverse g^{ij} respectively, and g will denote $|\det g_{ij}|$. A partial derivative with respect to the local coordinate x^i (say) will be indicated by a subscript i preceded by a comma; whereas a covariant derivative with respect to the Christoffel connection built from g_{ab} in the direction $\partial/\partial x^i$ will be denoted by a subscript i preceded by a vertical bar. The curvature tensor $R_a{}^b{}_{cd}$ is defined in accordance with $V^b{}_{|cd} - V^b{}_{|dc} = V^a R_a{}^b{}_{cd}$, where V^b is an arbitrary contravariant vector field. The Ricci tensor, curvature scalar, and Einstein tensor are defined by $R_{ij} = R_i{}^a{}_{ja}$, $R = g^{ij}R_{ij}$, and $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$, respectively. The symbol $\delta^{i_1\cdots i_p}_{j_1\cdots j_p}$ will be used to denote the $p\times p$ generalized Kronecker delta. Lastly, geometrized units will be used, in terms of which $c = G = 1$, and the line element $ds^2 = g_{ij}\,dx^i\,dx^j$, has units of length squared.

[2]See, for example, H. Rund, Abh. Math. Sem. Univ. Hamburg **29**, 243 (1966); or S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space—Time* (Cambridge U.P., London, 1973).

[3]If L is a Lagrange scalar density of the form (1.4), then its Euler—Lagrange tensors are given by

$$E^{ij}(L) = \sum_{\mu=0}^{\alpha}(-1)^\mu \frac{d^\mu}{dx^{i_1}\cdots dx^{i_\mu}}\left(\frac{\partial L}{\partial g_{ij,i_1\cdots i_\mu}}\right),$$

and

$$E^i(L) = \sum_{\mu=0}^{\beta}(-1)^\mu \frac{d^\mu}{dx^{i_1}\cdots dx^{i_\mu}}\left(\frac{\partial L}{\partial \psi_{i,i_1\cdots i_\mu}}\right).$$

$E^{ij}(L)$ is obtained from L through a variation of g_{ij} holding ψ_i (and its derivatives) fixed; while $E^i(L)$ is obtained from L through a variation of ψ_i holding g_{ij} (and its derivatives) fixed.

[4]The field equations of the Einstein—Maxwell field theory with cosmological term included are obtained from the Einstein—Maxwell field equations by adding Λg^{ij} to the left-hand side of the first set of the Einstein—Maxwell field equations, and leaving the second set unaltered, where Λ is the so-called "cosmological constant."

[5]See, for example, Chapter 17 of C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[6]For more information concerning the use of units in determining the field equations of physical field theories see S. Aldersley, "Dimensional Analysis in Relativistic Gravitational Theories," submitted for publication.

[7]The details omitted from this section may be found in G.W. Horndeski, "A Theorem on Second-Order Vector-Tensor Field Theories," (unpublished), preprint available from the Department of Applied Mathematics of the University of Waterloo, Waterloo, Ontario.

[8]A proof of this identity may be found in G.W. Horndeski, "Tensorial Concomitants of Relative Tensors, Affine Connections and their Derivatives of Arbitrary Order," (unpublished), preprint available from the Department of

Applied Mathematics of the University of Waterloo, Waterloo, Ontario.

[9]If $D^{\cdots}_{\cdots}$ is *any* concomitant of g_{ab}, ψ_a, and their derivatives (of some finite order), then we define $E^{ij}(D^{\cdots}_{\cdots})$ and $E^{i}(D^{\cdots}_{\cdots})$ as follows:

$$E^{ij}(D^{\cdots}_{\cdots}) = \sum_{\mu=0}^{\infty} (-1)^{\mu} \frac{d^{\mu}}{dx^{i_1}\cdots dx^{i_\mu}} \left(\frac{\partial D^{\cdots}_{\cdots}}{\partial g_{ij,i_1\cdots i_\mu}} \right),$$

and

$$E^{i}(D^{\cdots}_{\cdots}) = \sum_{\mu=0}^{\infty} (-1)^{\mu} \frac{d^{\mu}}{dx^{i_1}\cdots dx^{i_\mu}} \left(\frac{\partial D^{\cdots}_{\cdots}}{\partial \psi_{i,i_1\cdots i_\mu}} \right).$$

Since $D^{\cdots}_{\cdots}$ is of finite order in the derivatives of g_{ab} and ψ_a, the above infinite series have only a finite number of non-zero terms.

[10]See Theorem 3 in D. Lovelock, J. Aust. Math. Soc. **14**, 482 (1972).

[11]For more information on the operators introduced in Eqs. (2.2)–(2.5) see G.W. Horndeski, Tensor N.S. **28**, 303 (1974).

[12]These results are derived in Ref. 8 above and are also given in G.W. Horndeski, Utilitas Mathematica **9**, 3 (1976).

[13]D. Lovelock, Aequationes Math. **4**, 127 (1970).

[14]In what follows, I shall often say things like "the quantity C^{abcdef} has property S in the indices a, b, c, f, d, e." By that I mean that the quantity Φ^{abcfde} defined by $\Phi^{abcfde} = C^{abcdef}$ has property S.

[15]Since we are dealing with a metric whose signature is Lorentzian $\epsilon^{abcd} g_{ap} g_{bq} g_{cr} g_{ds} = -g\epsilon_{pqrs}$, where (recall that) $g = |\det g_{ij}|$, and hence $-g = \det g_{ij}$.

[16]The first sentence in the statement of Lemma 5 is due to D. Lovelock. See, Appendix 9 of D. Lovelock, "Mathematical Aspects of Variational Principles in the General Theory of Relativity" (unpublished D.Sc. thesis, University of Natal, South Africa, 1973).

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Continuum calculus and Feynman's path integrals

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An operational calculus is set up with the specific aim of resolving the problem of the integral of functionals in a general complex Banach algebra. The functionals that occur frequently in physics are Feynman's path integrals for quantum mechanics which usually appear in the form of an exponential of an integral. Through the establishment of two operations, the r differentiation and p integration, we succeed in constructing a formula, in closed form, for the integrals of Feynman type functionals. Applications to known problems (quantum harmonic oscillator, the electron–phonon system) corroborate conventional results. This formula is found to be at once consistent and more general than the method of projection into cylinder functionals of Frederichs type. It does not require an additive Gaussian measure, and it admits integration with finite limit functions. The methodology developed is general and applicable to other branches of mathematics. It is particularly suited to the study of infinitely divisible distributions in probability theory. We rederive, with facility, Lévy's formulas for continuous sums, and Lévy–Khintchine and Kolmogorov formulas. We also find it applicable to continuum matrix algebra, where the formula for the determinant of matrices of continuous indices is given as a p integral. As to algebraic identities, we give a continuum version of the binary expansion, and retrieve Stirling's formula of factorials by p integration. The *idée-clef* lies in the concept of infinitesimal ratio of a function in the same way that differential calculus deals with infinitesimal differences. Then the functional integral appears to be a natural product of the interaction between the conventional integration and the proposed p integration. It also heralds the possibility of a generalized measure theory for integrals where the basic operation between the measure and the integrand is not bilinear.

I. INTRODUCTION

The Feynman formulation of quantum mechanics mathematically involves an integral of a functional.[1] Specifically, the action of the system is defined in terms of its Lagrangian, $L(\dot{x},x,t)$,

$$S[b,a]=\int_{ta}^{tb} L(\dot{x},x,t), \qquad (1.1)$$

where $x(t)$ is the trajectory, $\dot{x}(t)$ the velocity, t the time, for a point in the phase space. The propagator is then given[2] as the "sum over all paths," $x(t)$, of contributions from $S[b,a]$,

$$K(b,a)=\int_a^b \mathcal{D}x(t)\cdot \exp(i/\hbar)S[b,a]. \qquad (1.2)$$

This is a functional integral, the evaluation of which was usually carried out by expanding (1.2) into an n-fold integral, then taking the limit of the result as $n\to\infty$. Several versions of the expansion have been proposed: the $\mathcal{P}$ projection method,[3] the Fourier series expansion,[4] the central moment expansion,[5] etc., with varying degree of effectiveness. However, most methods will break down upon deleting the Gaussian measure which is subsumed in all cases. The theoretical study of the functional integrals so far also suffers from the fundamental difficulty of establishing a satisfactory measure from conventional theory for the integration in a general Hilbert space.[6]

Therefore, the present investigation purports to put forward an operational calculus, called the continuum calculus, through which the integrals of a definite class of functionals frequently encountered in quantum mechanics as Feynman path integrals can be uniquely characterized and evaluated in closed form; thereby the expansion into cylinder functionals, a useful but cumbersome approach, is entirely circumvented. This characterization is found to be independent of the assumption of Gaussian measures and admissible of finite limits of integration. Thus it is more general than the expansion methods proposed so far.

The continuum calculus consists essentially of two operations, the r differentiation and the p integration. It studies the rules of operation and properties of the operators, $R/Rt(\cdot)$ and $\mathbf{P}\,dt \gg (\cdot)$, to be specified later. The theory is an independent subject of study in itself, and can be shown to have applications in other branches of mathematics as well.

A step-by-step construction for the r differentiation and p integration is carried out in Secs. II and III. They are shown to be closely related to the ordinary differentiation and integration through two important correspondence theorems. Heuristically, the r differentiation studies the behavior of the instantaneous ratio of a function $f(t)$ in the immediate neighborhood of point t in the same way that the ordinary differentiation is concerned with the instantaneous difference of a function. The p integration is then recognized as the "inverse" operation, in a loose sense of the term, of the r differentiation.

In Sec. IV, we apply the p integration method to algebra. A version of the Stirling formula for factorials is recovered. Section V witnesses the application of the p integrals to the probability theory of distributions of sums of random variables.[7] We rederive the formulas of Lévy[8] for "continuous" sums with much ease, and examine from a new perspective the characteristic functions of infinitely divisible distributions.

Through the operator approach we are able to set up in Sec. VI the definition of the integral of a class of

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functionals that admit an exponential integral representation. The p integral plays a large role in this definition. The proof of the consistency of the new definition with conventional results is then taken on as the major task in the subsequent sections (VIII, IX). We are able to demonstrate this for functional integrals studied variously by Fredericks,[3] Feynman and Hibbs,[2] Abé,[9] Montroll,[10] Brush,[11] Cameron and Martin,[12] and Morette,[13] etc., in physics as well as in mathematics. Furthermore, the present method sheds light on other aspects of the functional integral, e.g., the evaluation of the normalization constant for a Feynman path integral, an interesting and intriguing question. Sections VIII and IX take up the cases of quantum harmonic oscillators and the electron—phonon system after a brief survey of the continuum matrix algebra in Sec. VII.

We indicate possible developments in Sec. X. Furthermore, a heuristic discussion is entertained on the generalization of the integral formula to cover a wider class of integral representations of functionals.

II. THE METHOD OF r DIFFERENTIATION

In this section we establish a noval type of differential operator, denoted by R/Rt, on a certain class of functions. Let f be a function (or a form) mapping a complex Banach space B into its base field, C, of complex numbers, $f:B\to C$. Let $N\subset B$ be the kernel of the mapping f, and $\|b\|$ the norm of $b\in B$. We propose the following definition.

2.1. *Definition*: *the r derivative*: Let C^B be the set of functions from B to C. The operator R/Rt takes a function $f\in C^B$ and yields a function $f^*\in C^B$ which satisfies the following condition: For any $\epsilon>0$, and t not in the closure of kernel $\bar{N}$, there exists a δ such that

$$|f(t+b)-f(t)[f^*(t)^{\|b\|}]|<\epsilon \tag{2.1}$$

whenever $\|b\|<\delta$. If such an f^* exists at the point $t\in B-\bar{N}$, we denote it by

$$\frac{Rf}{Rt}(t)=f^*(t) \tag{2.2}$$

and call it the r derivative of f at t. f is called r differentiable at t; the operation is called r differentiation. (Or for brevity, rationative, rationable, and rationation, respectively, for reasons to be specified later).

We note that since t is not in $\bar{N}$, $f(t)\neq 0$. There exists a neighborhood V of t, such that the image $f(V)$ does not contain the origin 0. When f is r differentiable on a set $E\subseteq B-\bar{N}$, f is differentiable at each point $t\in E$. Higher r derivatives can be obtained recursively by repeated applications of definition (2.1).

The r derivative, Rf/Rt, bears a close relationship to the ordinary derivative, df/dt, of a function $f(t)$. In fact a one—to—one correspondence can be established, as will be seen later. We examine now the algebraic properties of the operation R/Rt. First, the function set, C^B, can be made into a Banach algebra[14] A_B by the usual construction. Addition and scalar multiplication are given by $(\alpha f+\beta g)(t)\equiv\alpha f(t)+\beta g(t)$, $\forall\ f,g\in C^B$, $t\in B$, and $\alpha,\beta\in C$. Multiplication is defined as $(fg)(t)\equiv f(t)g(t)$, and $(\alpha fg)(t)\equiv(f(\alpha g))(t)\equiv\alpha f(t)g(t)$. The function $f_0(t)\equiv 0$, is the zero function, and $f_1(t)\equiv 1$ the unit of multiplication. The space C^B can then be completed by incorporating all the ideal elements with a given norm, which is left undefined at this moment. Then we show that R/Rt is distributive with respect to the product of elements of A_B, and homogeneous with respect to exponentiation by a scalar.

2.2 *Lemma*: *distributivity of R/Rt*: The operator R/Rt is distributive with respect to the product of elements of A_B.

Proof: The definition 2.1 can be written in a more transparent form,

$$\frac{R}{Rt}f(t)=\lim_{\|b\|\to 0}\left|\frac{f(t+b)}{f(t)}\right|^{1/\|b\|}, \tag{2.3}$$

since $f(t)\neq 0$. For r differentiable f, and $g\in A_B$, and t not in the union of the closures of the kernels, $\bar{N}_f\cup\bar{N}_g$, of f and g,

$$\begin{aligned}\frac{R}{Rt}(fg)(t)&=\lim_{\|b\|\to 0}\left|\frac{f(t+b)g(t+b)}{f(t)g(t)}\right|^{\|b\|^{-1}}\\&=\lim_{\|b\|\to 0}\left|\frac{f(t+b)}{f(t)}\right|^{\|b\|^{-1}}\lim_{\|b\|\to 0}\left|\frac{g(t+b)}{g(t)}\right|^{\|b\|^{-1}}\\&=\frac{R}{Rt}f(t)\frac{R}{Rt}g(t).\end{aligned} \tag{2.4}$$

2.3 *Lemma*: *the homogeneity of R/Rt with respect to scalar exponentiation*: The exponentiation by a scalar for $f\in A_B$ can be defined conventionally. Then

$$\frac{R}{Rt}(f(t)^a)=\left(\frac{R}{Rt}f(t)\right)^a. \tag{2.5}$$

We omit the proof, which is straightforward. The distributivity can be extended to products of more than two factors by mathematical induction. We examine some examples. If f is a constant function, $f(t)\equiv a$, we have

$$\frac{R}{Rt}f(t)=\lim_{\|b\|\to 0}\left|\frac{a}{a}\right|^{\|b\|^{-1}}=1, \tag{2.6}$$

$f^*(t)=f_1(t)$, the unit of multiplication. If $f(t)=\exp(t)$, for t real,

$$\frac{R}{Rt}f(t)=\lim_{|b|\to 0}\left|\frac{\exp(t+b)}{\exp(t)}\right|^{|b|^{-1}}=e. \tag{2.7}$$

If $f(t)=t^n$, t real,

$$\begin{aligned}\frac{R}{Rt}f(t)&=\lim_{|b|\to 0}\left|\frac{(t+b)^n}{t^n}\right|^{|b|^{-1}}\\&=\lim_{|b|\to 0}|1+(n/t)+O(b^2)|^{|b|^{-1}}=\exp(n/t).\end{aligned} \tag{2.8}$$

In the following we prove the important theorem linking the r derivative with the ordinary derivative, which establishes the existence of Rf/Rt, whenever df/dt exists.

2.4 *Theorem*: *the correspondence theorem of r differentiation*: A function, $f\in A_B$, is r differentiable on $E\subseteq B-\bar{N}$ if and only if f is differentiable on E in the ordinary sense. And the derivative Rf/Rt is given by

$$\frac{R}{Rt}f(t)=\exp\left(\frac{d}{dt}\ln f(t)\right). \tag{2.9}$$

Proof: We prove the second part of the theorem first. Since f is differentiable, we can expand $f(t)$ into Taylor's series, at least to the first order,

$$f(t+b)=f(t)+f'(t)b+O(b^2), \tag{2.10}$$

where $f'(t)=df/dt$. Substitution of (2.10) into the alternative definition (2.3) for Rf/Rt, gives

$$\frac{R}{Rt}f(t)=\lim_{\|b\|\to 0}\left|\frac{f(t+b)}{f(t)}\right|^{\|b\|^{-1}}=\lim_{\|b\|\to 0}\left|1+\frac{f'(t)}{f(t)}b+O(b^2)\right|^{\|b\|^{-1}}$$
$$=\exp\left(\frac{f'(t)}{f(t)}\right)=\exp\left(\frac{d}{dt}\ln f(t)\right). \tag{2.11}$$

Therefore, Rf/Rt exists and is given by (2.11). This proves the sufficiency. The necessity part is proved easily from ordinary analysis[15] by noting that $\ln(\cdot)$ is differentiable, and the differentiability of the composition of differentiable functions. QED

This interesting theorem translates equally between the ordinary derivative, dt/dt, and r derivative, Rf/Rt, of a function. The examples considered previously can be easily demonstrated by this new relationship. If $f(t)=a$, $d\ln(a)/dt=0$; therefore, $Rf/Rt=\exp(0)=1$. If now $f(t)=\exp(t)$, $d\ln(\exp(t))/dt=1$, thus $Rf/Rt=e$. Similarly, for $f(t)=t^n$, $d\ln(t^n)/dt=n/t$, so $Rf/Rt=\exp(n/t)$. Theorem 2.4 introduces into the r differentiation the panoply of methods that are available in the differential calculus.

The differential calculus, as we recognize, is concerned with the study of the infinitesimal difference of a function, therefore the name "differential." The r differentiation, as proposed above, can be viewed as the study of the infinitesimal "ratio" of a function [see Eq. (2.3)]. Thus the calculus developed for this purpose can correspondingly be called the "rational" calculus. For example, the infinitesimal difference of a constant function is zero, the unit of addition ($df/dt=0$), while the infinitesimal ratio of the constant function is unity, the unit of multiplication ($Rf/Rt=1$). And in general, the rational calculus bears toward the multiplication in an analogous fashion as the differential calculus to addition. From Theorem 2.4, the two branches of calculus are closely related. In fact a parallel development of the rational calculus along the lines of differential calculus is entirely possible. We give, for instance, the rational version of Taylor's expansion without proof in the following theorem.

2.5 *Theorem*: *Rational Taylor's expansion*: Let R be the set of real numbers, and $f:R\to C$ be n times r differentiable on $E\subseteq R-\tilde{N}$, then for $t\in E$

$$f(t+b)=f(t)\left[\frac{R}{Rt}f(t)\right]^{b}\left[\frac{R^2}{Rt^2}f(t)\right]^{b^2/2!}\cdots\left[\frac{R^n}{Rt^n}f(t)\right]^{b^n/n!}\cdot R_n, \tag{2.12}$$

where R_n is the residual term given by

$$R_n=\exp\left(\frac{1}{n!}\int_t^{t+b}dx\,\frac{d^nf}{dt^n}(x)(t-x)^n\right). \tag{2.13}$$

In differential calculus, we were also interested in the inverse operation, i.e., to find the primitive of a function $f(t)$ or the function whose derivative is $f(t)$. The search led us to the integral calculus. Similarly, the search for the primitive of a function $f(t)$ in the rational sense, or the function whose r derivative coincides with $f(t)$, will lead us to the study of the p integrals in the next section. And the calculus so developed will be called the "potential" calculus.

III. THE METHOD OF p INTEGRATION

In this section we shall investigate the method of obtaining the rational primitives of a given function, $f(t)$. We shall find ourselves dealing with an integral theory where the integral is not bilinear with respect to its integrand and the underlying measure.[16] Actually, an exponential relationship is found. We define the exponential operation in the Banach algebra as:

3.1 *Definition*: *Exponential mapping* $\langle\exp\langle$[15]: Let A_B be the complex Banach algebra of the function space C^B as before. For $f,g\in A_B$ and $u,v\in B$, the binary operation $\langle\exp\langle$ is defined as: $\langle\exp\langle: C\times B\to C$,

$$f(t)\langle\exp\langle(u+v)\equiv(f(t)\langle\exp\langle u)(f(t)\langle\exp\langle v),\quad\forall\ t\in B, \tag{3.1}$$

$$(f(t)g(t))\langle\exp\langle u\equiv(f(t)\langle\exp\langle u)(g(t)\langle\exp\langle u),\quad\forall\ t\in B. \tag{3.2}$$

For convenience, we write interchangeably,

$$f(t)\langle\exp\langle u=u\rangle\exp\rangle f(t)=u\gg f(t)=f(t)\ll u=f(t)^u. \tag{3.3}$$

We now turn to an examination of the infinite products, as a necessary step to the definition of a p integral. In the integral calculus, we are first concerned with an infinite series:

$$I_d=\sum_{i=1}^{\infty}f_i=f_1+f_2+\cdots+f_n+\cdots, \tag{3.4}$$

and the (Riemann) integral[17] is construed to be the result of the limiting process of (3.4) as each term is weighted by a differential Δt_i,

$$I_c=\lim_{\sup\Delta t_i\to 0}\sum_{i=1}^{\infty}\Delta t_i f_i$$
$$=\lim_{\sup\Delta t_i\to 0}(\Delta t_1f_1+\Delta t_2f_2+\cdots+\Delta t_nf_n+\cdots),$$
$$a\leq t\leq b,\quad f_n=f(t_n). \tag{3.5}$$

In the spirit of the analogy between sums and products, the infinite product,

$$P_d=\prod_{i=1}^{\infty}f_i=f_1f_2\cdots f_n\cdots \tag{3.6}$$

would pass into the continuum case, as the index i becomes the continuous variable t:

$$P_c=\lim_{\sup\Delta t_i\to 0}\prod_{i=1}^{\infty}f_i^{\Delta t_i}$$
$$=\lim_{\sup\Delta t_i\to 0}f_1^{\Delta t_1}f_2^{\Delta t_2}\cdots f_n^{\Delta t_n}\cdots,\quad a\leq t\leq b,\quad f_n=f(t_n) \tag{3.7}$$

now with the weighting factors Δt_i to be in the exponents. The recognition of this fact is of crucial importance in the development of the theory of p integration. Otherwise, the infinite product (3.6) is not necessarily convergent; just as in (3.4), the infinite series is not necessarily convergent. When the individual terms in

(3.5) are weighted we are able to study a much wider class of sums of the function $f(t)$ than in (3.4). Therefore, the weighted product (3.7) will allow us to study a wider class of functions $f(t)$ in the product form. We write (3.7) symbolically as

$$P_c = \lim_{\sup \Delta t_i \to 0} \prod_{i=1}^{\infty} f_i^{\Delta t_i} = \mathbf{P}_a^b dt \gg f(t) = \mathbf{P}_a^b [f(t)]^{dt} \tag{3.8}$$

when such product exists. Equation (3.8) will be the Riemann–Stieltjes version of the p integral. We formalize our definition by the following definition.

3.2 *Definition*: *the Riemann–Stieltjes p integral*: Let f be a function from R (real numbers) to C (complex numbers). Then the p integral of f on an interval (a, b) in R is defined as

$$I_p(f) \equiv \mathbf{P}_a^b [f(t)]^{dt} \equiv \lim_{\substack{n\to\infty \\ \sup \Delta t_i \to 0}} \prod_{i=1}^{n} f(t_i)^{\Delta t_i},$$
$$a = t_0 < t_1 < \cdots < t_n = b, \quad \Delta t_i = t_i - t_{i-1}, \tag{3.9}$$

whenever the limit exists.

If I_p exists, we call f to be p integrable on (a, b), the result a p integral, the operation, p integration (or for brevity, potentiable, potential, and potentiation, respectively). We shall extend the definition to a Lebesque type in the future. Some of the algebraic properties of the p integration are summarized below.

3.3 *Lemma*: *distributivity and homogeneity of p integration*: The p integration, $\mathbf{P}_a^b dt \gg \cdot$, is distributive with respect to products of functions $f, g: R \to C$. It is also homogeneous with respect to exponentiation by a constant a, i.e.,

$$\mathbf{P}_a^b dt \gg (fg)(t) = (\mathbf{P}_a^b dt \gg f(t))(\mathbf{P}_a^b dt \gg g(t)), \tag{3.10}$$

$$\mathbf{P}_a^b dt \gg (f(t)^a) = (\mathbf{P}_a^b dt \gg f(t))^a. \tag{3.11}$$

Definition 3.2 is not a constructive definition. However, this is remedied by the correspondence theorem given below, which correlates the p integral with the ordinary integral of a function $f(t)$.

3.4 *Theorem*: *the correspondence theorem of p integration*: Let f be a function from R to C. Then f is p integrable on a set $E \subseteq R$ if and only if f is Riemann–Stieltjes integrable in the ordinary sense on E. The p integral is given in terms of the ordinary integral by

$$\mathbf{P}_E [f(t)]^{dt} = \exp\left(\int_E dt \ln f(t)\right). \tag{3.12}$$

Proof: We shall be brief on the proof. If we take the logarithm of (3.9), we have

$$\ln I_p(f) = \ln \lim_{\substack{n\to\infty \\ \sup \Delta t_i \to 0}} \prod_{i=1}^{n} f(t_i)^{\Delta t_i} = \lim_{\substack{n\to\infty \\ \sup \Delta t_i \to 0}} \sum_{i=1}^{n} \Delta t_i \ln f(t_i)$$
$$= \int_a^b dt \ln f(t). \tag{3.13}$$

We are now in a position to find the primitive of a function $f(t)$.

3.5 *Theorem*: *the r primitive of a function f(t)*: Let f be a function from R to C. If f is p integrable on the interval (a, x), then $I_p(f)$ on (a, x) is a function of the limits a and x and the r derivative of $I_p(f)$ with respect to x is $f(x)$. The r primitive of $f(x)$, $I_p(f)$, is unique up to a multiplicative constant.

That $I_p(f)$ is a function of the limits of integration is easy to see. The r derivative is found by using the two correspondence Theorem (2.9) and (3.12),

$$\frac{R}{Rx} I_p(f)(x) = \exp \frac{d}{dx} \ln I_p(f)(x) = \exp \frac{d}{dx} \ln \exp \int_a^x dt \ln f(t)$$
$$= \exp \ln f(x) = f(x). \tag{3.14}$$

We look at some examples. If f is a constant function, $f(t) = \alpha$, then

$$I_p(f) = \exp \int_a^b dt \ln \alpha = \exp(b \ln \alpha - a \ln \alpha) = \alpha^b / \alpha^a, \tag{3.15}$$

while the reverse operation

$$\frac{R}{Rt} \alpha^t \alpha^{-a} = \frac{R}{Rt} \alpha^t \frac{R}{Rt} \alpha^{-a} = \left(\exp \frac{d}{dt} \ln \alpha^t\right) \cdot 1$$
$$= \exp \ln \alpha = \alpha. \tag{3.16}$$

Now $f(t) = t^n$,

$$I_p(f) = \exp \int_a^b dt \ln t^n = \exp(n[t \ln t - t]_a^b)$$
$$= (b^b e^{-b})^n (a^a e^{-a})^{-n}. \tag{3.17}$$

The inverse operation gives

$$\frac{R}{Rt} [(t^t e^{-t})^n (a^a e^{-a})^{-n}] = \left(\exp \frac{d}{dt} \ln (t^t e^{-t})^n\right) \cdot 1$$
$$= \exp n \ln t = t^n. \tag{3.18}$$

The generalization of the p integral (3.9) to general measures can be made through the correspondence Theorem 3.4. The step-by-step construction will not be given here. We give the result directly. Let B be a complex separable (Hausdorff) Banach space, (B, S_B, μ) be a measure space with S_B a σ algebra on B, and μ the positive measure (more precisely, real and nonnegative). Then we propose the following definition.

3.6 *Definition*: *the general measure theoretical p integral*: Let f be a measurable form on the complex separable Banach space B, $f: B \to C$, with the given measure space (B, S_B, μ). Then the p integral of f, $I_p^E(f)$, on a set $E \subseteq B$ is given by

$$I_p^E(f) = \mathbf{P}_E\, \mu(dt) \gg f(t) \equiv \exp\left(\int_E \mu(dt) \ln f(t)\right) \tag{3.19}$$

whenever the given integral exists. Note that $\int_E \mu(dt)$ is the Lebesque integral.

The definition can also be generalized to complex measures.

The method of p integration is found to be vitally important in the characterization of the integrals of functionals. In fact, the functional integral represents the natural results of the interaction between the conventional integral and the newly defined p integration. Other fruitful interactions of this new methodology with conventional mathematics can also be brought forth. We shall consider a few in the following sections, such as applications to algebra, probability theory, and to the theory of matrices of continuous indices. The major result will be the characterization of the functional integral in Sec. VI. We can also envisage the

TABLE I. The Binary expansion of $(a+b)^n$.

n	a	b	$\log_{10}(a+b)^n$ exact	$\log_{10}(a+b)^n$ calc*
1000	0.6	0.5	41.39	42.98
10 000	0.6	0.5	413.9	415.9
100 000	0.6	0.5	4139.	4141.2

*From Eq. (4.5).

application of p integration to continuous functional differentiation. This will be a subject for future study.

IV. APPLICATIONS TO ALGEBRA

The ordinary algebraic formulas have their counterparts in the continuum approach here. We examine two cases here, the factorial of an integer n and the binary expansion of $(a+b)^n$.

Factorial: The factorial $n!$ of an integer n is defined as $n(n-1)\cdots(3)(2)(1)$ or, in product form,

$$n! = \prod_{i=1}^{n} i. \tag{4.1}$$

As i becomes a continuous parameter t, (4.1) becomes the p integral of the function $f(t)=t$,

$$\mathbf{P}_0^x[t]^{dt} = \exp\int_0^x dt \ln t = x^x e^{-x}. \tag{4.2}$$

This is the Stirling formula of factorials for large x that is extensively used in statistical mechanics.[18] The interpretation for this approximation is that for large n the difference between successive indices, $\Delta i = 1$, becomes insignificant in comparison with the magnitude n and formula (4.1) passes into the p integral,

$$n! = \prod_{i=1}^{n} i^{\Delta i} \rightarrow \lim_{\Delta t_i \to 0} \prod_{t_i=0}^{n} t_i^{\Delta t_i} = \mathbf{P}_0^n[t]^{dt}. \tag{4.3}$$

Binary Expansion: The expansion of the discrete formula $(a+b)^n$ is given by

$$(a+b)^n = \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!} a^{n-m} b^m. \tag{4.4}$$

When n becomes large the sum becomes an integral and the factorials become those given by (4.2) (with $m \rightarrow t$, $n \rightarrow x$),

$$\int_0^x dt \left(\frac{ax}{x-t}\right)^{x-t} \left(\frac{bx}{t}\right)^t. \tag{4.5}$$

Table I gives the calculation for some cases when n approaches the order of 10^5. Formula (4.5) follows (4.4) closely, as a step size $\Delta t = 5$ in the numerical integration is used.

V. APPLICATIONS TO PROBABILITY THEORY

The study of the probability distribution of the sum of independent identically distributed random variables is important to the limit theorems in probability theory. Let $x_1, x_2, \ldots, x_n$ be a sequence of independent random variables. We are interested in the probability distribution of the sum Z_n,

$$Z_n = x_1 + x_2 + \cdots + x_j + \cdots + x_n, \tag{5.1}$$

for any natural number n. We are also interested in the continuum case when the index j takes on continuous values t. By analogy we study the random variable Z_t, for which the differences

$$Z_{t_2} - Z_{t_1}, \ldots, Z_{t_j} - Z_{t_{j-1}}, \ldots, Z_{t_n} - Z_{t_{n-1}} \tag{5.2}$$

are independently distributed for $t_1 < t_2 < \cdots < t_n$. Such investigation is important in the study of stationary Markov processes[19,20] and Brownian motions.[21] The probability distribution of such random variables is called the infinitely divisible distribution. We first examine the case when the differences in (5.2) are identically distributed according to a same distribution law and derive the formulas of Lévy by our p integral approach.

This distribution of the sum of two independent random variables is known to be the convolution of the individual distribution functions. Therefore, it is more convenient to study the case in the Fourier space, i.e., with respect to their characteristic functions. For the sum (5.1), the characteristic function of Z_n is

$$f_{Z_n}(k) = \prod_{j=1}^{n} f_j(k) = f_1(k) f_2(k) \cdots f_j(k) \cdots f_n(k), \tag{5.3}$$

where $f_j(k) = \int dx \exp(ikx)\, dP_j(x)$ is the characteristic function of x_j, $P_j(x)$ being the probability distribution of x_j. As the index j becomes continuous, the number of factors in the product (5.3) becomes infinite, we find that the p integral notion readily applies. Thus for the random variable $Z_{t'}$ of (5.2), we have

$$f_{Z_{t'}}(k) = \mathbf{P}_0^{t'}[f_t(k)]^{dt}. \tag{5.4}$$

When the factors in (5.2) are identically distributed according to the normal law, with characteristic function

$$f_t(k) = \exp[i\mu k - (\sigma^2/2)k^2], \tag{5.5}$$

where μ and σ^2 are the mean and variance, respectively, of the normal distribution, (5.4) becomes

$$f_{Z_{t'}}(k) = \exp\int_0^{t'} dt \ln f_t(k) = \exp\{t'[i\mu k - (\sigma^2/2)k^2]\}, \tag{5.6}$$

which is a formula given by Lévy.[8] For Poisson distribution

$$P[x = nh] = (\lambda^n/n!)e^{-n}, \quad f_t(k) = \exp\lambda(\exp(ihk) - 1), \tag{5.7}$$

where h is the span. We have

$$f_{Z_{t'}}(k) = \exp\int_0^{t'} dt \ln f_t(k) = \exp\lambda t'(\exp(ihk) - 1) \tag{5.8}$$

which is again verified by Lévy's result.[8]

We can also apply the present approach to study the infinitely divisible distributions of random variables whose summands (5.2) are not identically distributed. For example, if $t=0$, $f_0(k)$ is distributed according to the normal law, and at $t=\infty$, $f_\infty(k)$ is distributed according to the Poisson law, and suppose that a homotopy $H[f;s]$ exists[22] such that $H[f;1] = f_0(k)$ and $H[f;0] = f_\infty$, $s = e^{-t}$ and $H[f;s]$ is continuous with respect to s, the parameter, then the distribution of the sum, $Z_{t'}$, for $t'=1$, is given by

$$f_{Z_1}(k) = \mathbf{P}_0^1[H[f;s]]^{ds}. \tag{5.9}$$

If the homotopy is given by

$$H[f;s]=H[f;e^{-t}]=\exp\left[i\mu k+\left(\exp(ikt)-1-\frac{ikt}{1+t^2}\right)\frac{1+t^2}{t^2}\bar{G}(t)\right], \quad (5.10)$$

i.e., as $t\to 0$, we have $-\frac{1}{2}t^2k^2$ in the parentheses with $\bar{G}(t)$ of the order $O(t^2)$, we have the normal distribution. As $t\to\infty$, we essentially have $(\exp ikt-1)\bar{G}$ in the exponet, which is the Poisson form. Then the distribution of $Z_{t'}$ is given by

$$f_{Z_1}(k)=\exp\int_0^1 ds\ln H[f;s]$$
$$=\exp\left(i\mu k+\int_0^\infty dG(t)\left[\exp(ikt)-1-\frac{ikt}{1+t^2}\right]\frac{1+t^2}{t^2}\right),$$
$$dG(t)=e^t\bar{G}(t)\,dt, \quad (5.11)$$

which is the Lévy–Khintchine formula[7] for infinitely divisible distributions with modified interpretation (i.e., nonidentically distributed summands). Since different homotopies can be constructed, if we choose

$$H[f;s]=H[f;e^{-t}]=\exp[i\mu k+\textstyle\int_0^\infty dK(t)(\exp(ikt)-1-ikt)t^2], \quad (5.12)$$

where $K(t)$ is of bounded variation with dK/dt of the order, $O(t^2)$, as $t\to 0$, we have the so-called Kolmogorov formula[7] for the infinitely divisible distribution. Note the difference in the interpretation of the distribution laws for the summand (5.2) in our case.

Since the sum of a finite number of independent infinitely divisible random variables is itself infinitely divisible, so is the (weak) limit of such a sequence of random variables. The theorem of canonical representation of infinitely divisible distributions can be now interpreted, in light of the development here, as that for each infinitely divisible random variable its characteristic function corresponds to the homotopy with a certain value for s, constructed from the discrete Poisson distribution either according to the Lévy–Khintchine formula or the Kolmogorov formula. That any infinitely divisible distribution is a finite superposition of Poisson distribution is already a known fact.[7] the extension to a homotopy appears natural.

VI. THE INTEGRAL OF FUNCTIONALS

We start our study of the functional integrals by considering the n-fold integral of a function of n variables,

$$\int_{a_1}^{b_1}dy_1\int_{a_2}^{b_2}dy_2\cdots\int_{a_n}^{b_n}dy_nf(y_1,y_2,\ldots,y_n)=\left(\prod_{i=1}^n\int_{a_i}^{b_i}dy_i\right)\cdot f(y_i). \quad (6.1)$$

As the index i takes on continuous values t, the function $f(y_i)$ becomes a functional $f[y(t)]$, and the product of the n-fold integral signs also involves infinitely many factors. (6.1) can then be written symbolically as

$$\lim_{n\to\infty}\left(\prod_{i=1}^n\int_{a_i}^{b_i}dy_i\right)\cdot f(y_i)\to(\mathbf{P}\,dt\gg[\textstyle\int_{a(t)}^{b(t)}dy(t)])\cdot(f[y]) \quad (6.2)$$

by adopting p integral concepts. This formula is found to be a correct symbolism for the integral of a functional. We examine an example studied by Frederichs.[3]

Let $\Phi[y]$ be a functional of the function $y(t)$ given by

$$\Phi[y]=\pi^{-1/2}(\exp\textstyle\int_0^1 dr\,b(r)y(r))(\exp[-\int_0^1 ds\,y(s)^2]). \quad (6.3)$$

Then the integral of (6.3) can be written symbolically as

$$(\mathbf{P}_0^1\,dt\gg[\textstyle\int dy(t)])\cdot(\Phi[y]). \quad (6.4)$$

However, by the correspondence theorem (3.4) we can write $\mathbf{P}\,dt\gg[\int dy(t)]\cdot(\cdot)$ in the operator form,

$$\mathbf{P}\,dt\gg[\textstyle\int dy(t)]\cdot(\cdot)\equiv(\exp\int dt\ln[\int dy(t)])\cdot(\cdot). \quad (6.5)$$

We shall show that the operator form of functional integration is equivalent to the p projection method of Frederichs in the case of (6.4). If we expand (6.3) into cylinder functionals, we have

$$\Phi[y]=\lim_{\substack{\Delta t_i\to 0\\ n\to\infty}}\pi^{-1/2}\exp[(b(t_1)y(t_1)-y(t_1)^2)\Delta t_1]\cdots\times\exp[(b(t_n)y(t_n)-y(t_n)^2)\Delta t_n]. \quad (6.6)$$

When we assign an exponential weighting factor Δt_i to the integral signs of (6.1), (6.4) becomes

$$\pi^{-1/2}\prod_{i=1}^n[\textstyle\int_{-\infty}^\infty dy_i\exp(-y(t_i)^2+b(t_i)y(t_i))]^{\Delta t_i}$$
$$=\pi^{-1/2}\prod_{i=1}^n[\textstyle\int_{-\infty}^\infty dy_i\exp(-(y_i+b_i/2)^2+b_i^2/4)]^{\Delta t_i}$$
$$=\pi^{-1/2}\prod_{i=1}^n[\exp(b_i^2/4)\textstyle\int_{-\infty}^\infty\exp(-z_i^2)]^{\Delta t_i}$$
$$=\pi^{-1/2}\prod_{i=1}^n[\exp(b_i^2/4)\pi^{1/2}]^{\Delta t_i} \quad (6.7)$$

in the limit $\sup\Delta t_i\to 0$ and $n\to\infty$. But that is exactly the definition of a p integral [see (3.9)], i.e.,

$$\pi^{-1/2}\mathbf{P}_0^1\left[\pi^{1/2}\exp\frac{b(t)^2}{4}\right]^{dt}=\pi^{-1/2}\exp\int_0^1 dt\ln\left[\pi^{1/2}\exp\frac{b(t)^2}{4}\right]=\exp\int_0^1 dt\frac{b(t)^2}{4}, \quad (6.8)$$

the same as the result of Frederichs.[3] The p projection method can be most conveniently applied to functionals which are "separable," i.e., which admit an n-product form of (6.6) where each factor is a function of only one y_i. A more general approach will be the operator approach of (6.5) to be examined next.

We therefore prepare $\Phi[y]$ of (6.3) into a form that is ready to interact with the operator form (6.5) of the functional integration,

$$\Phi[y]=\pi^{-1/2}\exp\textstyle\int_0^1 dt\ln\exp[b(t)y(t)-y(t)^2]. \quad (6.9)$$

(6.4) then becomes

$$(\mathbf{P}_0^1\,dt\gg[\textstyle\int dy(t)])\cdot(\Phi[y])$$
$$=\pi^{-1/2}\exp\textstyle\int_0^1 dt\ln\int dy(t)\cdot\exp[b(t)y(t)-y(t)^2]$$
$$=\pi^{-1/2}\exp\textstyle\int_0^1 dt\ln\exp[b(t)^2/4]\int_{-\infty}^{+\infty}\exp[-(y-b/2)^2]$$
$$=\pi^{-1/2}\exp\textstyle\int_0^1 dt\ln\exp[b(t)^2/4]\int_{-\infty}^{\infty}dz\exp(-z^2)$$
$$=\pi^{-1/2}\exp\textstyle\int_0^1 dt\ln(\exp[b(t)^2/4]\pi^{1/2})$$

$$= \pi^{-1/2} \exp \int_0^1 dt (b(t)^2/4 + \ln \pi^{1/2})$$
$$= \exp \int_0^1 dt\, b(t)^2/4 \qquad (6.10)$$

which is the result given by Frederichs. We note that the second factor in (6.3) is actually a Gaussian measure. We remark also that in the derivation (6.10) the inner integral with respect to the path function $y(t)$ is carried out at constant t, consistent with the common practice in functional integration. We immediately generalize to the following definition of the integral of functionals, omitting the intervening developments.

Let A_B be the complex separable (Hausdorff) Banach algebra of the function set C^B as before. Let Φ be a function (functional) from A_B to C. Then the functional integral of Φ is given by the following definition.

6.1 *Definition: the integral of functionals*: Let $y \in A_B$ be a function from a complex, separable (Hausdorff) Banach space, B, to C (complex numbers). Let (B, S_B, μ) be a measure space on B, and (A_B, S_A, m) a measure space on A_B (i.e., S_B and S_A are the σ algebras on B and A_B, respectively, and μ and m are the corresponding real and nonnegative measures).[23] Then if the functional $\Phi : A_B \rightarrow C$ admits an integral representation of the form

$$\Phi[y] = \exp \int_E \mu(dt) f(y(t)), \qquad (6.11)$$

where f is a complex valued function, $f: C \rightarrow C$, $E \subseteq B$, we define an $x_t(y)$ function by

$$x_t(y) \equiv \exp f(y(t)) \qquad (6.12)$$

and the functional integral, $I_f^F(\Phi)$ of $\Phi[y]$ on the set $F \subseteq A_B$ is defined as

$$I_f^F(\Phi) = \mathbf{P}_E\, \mu(dt) \gg [\textstyle\int_F m(dy(t))] \cdot \Phi[y]$$
$$\equiv \exp \int_E \mu(dt) \ln[\textstyle\int_F m(dy(t)) \cdot x_t(y)] \qquad (6.13)$$

whenever the successive integrals exist.

Expression (6.13) simplifies when $B = R$ (real numbers), and $A_B = R^R$ as we take S_B and S_A to be the induced (in the sense of Wiener[24]) Borel sets. Furthermore, if $y(t)$ varies from $a(t)$ to $b(t)$, (6.13) reduces to

$$I_f^{(a,b)}(\Phi) = \exp \int_{t_1}^{t_2} dt \ln[\textstyle\int_{a(t)}^{b(t)} dy(t) \exp f(y(t))], \qquad (6.14)$$

where $f: R \rightarrow C$.

We note that Definition 6.1 gives the integral of a functional of the type (6.11) in closed form, in comparison with the expansion methods. It also allows calculations with finite limits from $a(t)$ to $b(t)$. The Wiener integral is a special case of (6.13) when the Gaussian measure is selected for m. For most applications in physics and probability theory the integral form (6.11) is quite sufficient. We shall examine some known sample cases in quantum physics where the Feynman path integral is to be evaluated. For certain more general functionals, the functional integral can also be defined in closed form. We shall take this up in the discussion section.

VII. CONTINUUM MATRIX ALGEBRA

Owing to later requirements, we develop here, without proof, the continuum counterpart of the matrix theory. Only elementary aspects will be touched upon, some of which are apparent generalizations of the discrete theory.

(1) The continuum matrix, $C(s,t)$, is a function from $E \times F \rightarrow R$, where $E, F \subseteq R$, the real numbers. $C(s,t)$ is called a square matrix if $E = F$.

(2) The transpose, $C^T(s,t)$, is obtained from $C(s,t)$ by $C^T(s,t) = C(t,s)\ \forall\ s \in E$ and $t \in F$. If $C(s,t)$ is square and invariant under transposition, $C(s,t)$ is called symmetrical.

(3) $C(s,t)$ is called an orthogonal matrix iff $\int dx\, C^T(s,x) C(x,t) = \delta(s,t)$, where $\delta(s,t)$ is the Dirac delta function.

(4) The eigenvalue of a square matrix $C(s,t)$ is the function $\lambda(r)$ such that the condition $\int dx\, v_r(x) C(x,t) = \lambda(r) v_r(t)$ for some nonzero function $v_r(t)$.

(5) If $C(s,t)$ is a square matrix, and is symmetrical in s and t, then there exists an orthogonal matrix $A(s,t)$ such that $C(s,t)$ can be diagonalized, $\int dx\, dy\, A^T(s,x) C(x,y) A(y,t) = \Lambda(s,t)\delta(s,t)$, where $\Lambda(s,t)$ is diagonal, and $\Lambda(rr) = \lambda(r)$, the eigenvalues of C.

(6) The product of two matrices A and B is given by $(AB)(s,t) = \int dx\, A(s,x) B(x,t)$.

(7) The determinant of a square matrix, $C(s,t)$, with eigenvalues $\lambda(r)$ is defined to be the potential (p integral) of its eigenvalues, i.e.,

$$\det(C(s,t)) = \mathbf{P}_E [\lambda(r)]^{dr}.$$

(8) For A, B, square matrices, $\det(AB) = \det(A)\det(B)$.

(9) $\det(\delta(s,t)) = 1$.

The principal result so far is contained in (7), whereby the definition of the determinant of a continuum matrix is made possible through the potential calculus. The discrete concept of a determinant as the multilinear alternating (exterior) product is, admittedly, difficult to be generalized to the continuum case.

VIII. THE QUANTUM HARMONIC OSCILLATOR

Feynman's path integral method for the evaluation of the propagator of a quantum system is a ready example for the application of the method of functional integration developed here. To demonstrate the utility of (6.14), we examine the known case of quantum harmonic oscillators.[2,10] The Lagrangian for this system is given by

$$L(\dot{x}, x, t) = (m/2)(\dot{x}^2 - \omega^2 x^2), \qquad (8.1)$$

where $x = x(t)$ is the path with fixed ends, $x(0) = x_0$, $x(\tau) = x_\tau$. The propagator is then given by the functional integral

$$K(\tau, 0) = \mathbf{P}_0^\tau dt \gg \left[\int x(t)\right] \cdot \exp\left[\frac{i}{\hbar} \int_0^\tau dt \frac{m}{2} (\dot{x}^2 - \omega^2 x^2)\right]. \qquad (8.2)$$

Feynman[2] showed that the kernel can be expanded around the classical path $\bar{x}(t)$, i.e., $x(t) = \bar{x}(t) + y(t)$. After the extraction of the classical contribution, a quadratic functional in $y(t)$ is obtained,

$$K(\tau, 0) = \exp\frac{i}{\hbar} S_{\mathrm{C1}}(\tau, 0)\, \mathbf{P}_0^{\tau} dt \gg \left[\int dy(t)\right] \cdot \exp\left[\frac{i}{\hbar}\int_0^{\tau} dt \frac{m}{2}(\dot{y}^2 - \omega^2 y^2)\right], \tag{8. 3}$$

with $y(0) = y(\tau) = 0$. S_{C1} is the classical result,[2]

$$S_{\mathrm{C1}}(\tau, 0) = \frac{m\omega}{2\sin\omega\tau}[(x_0^2 + x_\tau^2)\cos\omega\tau - 2x_0 x_\tau]. \tag{8. 4}$$

The usual practice[2,10] is to discretize the functional in $y(t)$,

$$\int_0^1 ds \frac{\tau m}{2\hbar}\left[\frac{1}{\tau^2}\left(\frac{dy}{ds}\right)^2 - \omega^2 y^2\right] \to \sum_{j=1}^{n-1} \epsilon\left[\frac{m}{2\hbar\tau\epsilon^2}(y_j - y_{j-1})^2 - \frac{\tau m}{2\hbar}\omega^2 y_j^2\right] = \sum_j [(z_j - z_{j-1})^2 - \tau^2\epsilon^2\omega^2 z_j^2], \tag{8. 5}$$

where $t = s\tau$, $z_j = (m/2\hbar\tau\epsilon)^{1/2} y_j$. Note that the integral is now dimensionless. The summation on j is from 1 to $n-1$ since the initial and final positions of x are fixed. This contributes a factor of $(i\pi)^{-1}$ to the normalization constant N. (8. 5) is a quadratic form, and can be written as

$$(Ni\pi)^{-1} \exp i \sum_j \sum_k z_j A_{jk} z_k. \tag{8. 6}$$

In the continuum case, (8. 6) has the form

$$\Phi[y] = \Phi[z] = (Ni\pi)^{-1} \exp[-i^{-1} \textstyle\int_0^1 ds \int_0^1 dt\, z(s) A(s, t) z(t)]. \tag{8. 7}$$

The matrix $A(s, t)$ is as given by Montroll.[10] If $A(s, t)$ is symmetrical, it can be diagonalized by an orthogonal matrix $C(s, t)$,

$$\int dr\, dr'\, C^T(s, r) A(r, r') C(r', t) = \Lambda(s, s)\delta(s, t). \tag{8. 8}$$

Upon defining a new variable, $u(t) = \int ds\, z(s) C^T(s, t)$, we have

$$\Phi[y] = \Phi[u] = (Ni\pi)^{-1} \exp[-i^{-1} \textstyle\int_0^1 dt\, u(t)\Lambda(t, t) u(t)] \tag{8. 9}$$

which is of the form (6. 11). Therefore the integral is given, according to (6. 14), by

$$\begin{aligned}
\left[\exp\frac{-i}{\hbar} S_{\mathrm{C1}}(\tau, 0)\right] K(\tau, 0) &= (Ni\pi)^{-1}\, \mathbf{P}\left[\int dy(t)\right]^{dt} \cdot \Phi[y] \\
&= (Ni\pi)^{-1}\, \mathbf{P}\left[\int dz(t)\right]^{dt} |J|\, \Phi[z] \\
&= (Ni\pi)^{-1}\, \mathbf{P} \quad \int du(t)^{\;dt} |J| \cdot |C| \cdot \Phi[u] \\
&= (Ni\pi)^{-1} |J| \cdot |C| \cdot \exp \int_0^1 dt \ln\left[\int_{-\infty}^{+\infty} du \exp(-i^{-1}\Lambda u^2)\right] \\
&= (Ni\pi)^{-1} |J| \cdot \exp \int_0^1 dt \ln[i\pi \det(\Lambda)^{-1}]^{1/2} \\
&= (Ni\pi)^{-1} |J|\, (i\pi)^{1/2} (\det\Lambda)^{-1/2},
\end{aligned} \tag{8. 10}$$

where J is the Jacobian of the transformation from y to z, i. e., $J = \delta(s, t)(m/2\hbar\tau\epsilon)^{-1/2}$; by (7. 7), $|J| = \det(\delta(s, t)(m/2\hbar\tau\epsilon)^{-1/2}) = (m/2\hbar\tau\epsilon)^{-1/2}$. $|C|$ arises from the transformation z to u, and is equal to ± 1 for orthogonal matrices, we took it to be $+1$. The normalization constant is taken such that,[2]

$$K(b, a) = \int dx_c K(b, c) K(c, a). \tag{8. 11}$$

It has the dimension of length squared. Therefore, $N = |J|^2$. The determinant $\det(\Lambda) = \det(A)$, and is given by Montroll[10] to be $(\tau\epsilon\omega)^{-1}\sin(\omega\tau)$. We have then,

$$K(\tau, 0) = \left(\frac{m\omega}{i\hbar\sin\omega\tau}\right)^{1/2} \exp\frac{i}{\hbar} S_{\mathrm{C1}}(\tau, 0). \tag{8. 12}$$

This is the complete expression for a quantum harmonic oscillator with fixed end points. For $x(0) = 0$, and free $x(\tau)$, the same methodology can be shown to apply and give

$$K = \sec^{1/2}(\tau\lambda^{1/2}), \quad \tau^2\lambda < \pi/2, \tag{8. 13}$$

λ being the spring constant.[10]

IX. THE ELECTRON-PHONON SYSTEM

The electron–phonon system considered by Abé[9] consists of a Hamiltonian,

$$H(p, q) = p^2/2m + \gamma p + V(q) \tag{9. 1}$$

with

$$V(q) = (m\omega^2/2) q^2 + m\omega\alpha q, \tag{9. 2}$$

where the symbols have the usual meanings. The density matrix is then given by the functional integral

$$\rho(q_1, q_0) = \mathbf{P}\left[\int dq(t)\right.^{dt} \cdot \exp\left[-\int_0^{\beta} dt \frac{m}{2\hbar^2} \times \left(\frac{dq}{dt} + i\hbar\gamma(t)^2 - V(q(t)\right)\right. . \tag{9. 3}$$

As before, Abé extracted the classical path contribution out of (9. 3) by setting $q = \bar{q} + y$, $\bar{q}$ being the classical path, and discretized the integrand

$$\rho(q_1, q_0) = [\exp(-f^*)]\, \mathbf{P}\left[\int dy(t)\right.^{dt} \cdot \lim_{\epsilon \to 0} \exp\left[-\frac{1}{2}\left(\sum_j \sum_k y_j \frac{\partial^2 f}{\partial q_j \partial q_k} y_k\right)\right], \tag{9. 4}$$

where $\exp(-f^*)$ is the classical contribution,

$$\exp(-f^*) = \exp -\left(\frac{m\omega}{2\hbar\sinh(\hbar\omega\beta)}((q_0^2 + q_1^2)\cosh(\hbar\omega\beta) - 2q_0 q_1 + 2Aq_1 + 2Bq_0 + 2C)\right). \tag{9. 5}$$

A, B, and C are known integrals[9] of the elements of the Hamiltonian (9. 1); and the quadratic form in (9. 4) has elements

$$\begin{array}{ll}
\dfrac{2m}{\beta\epsilon\hbar^2} + \beta\epsilon\dfrac{\partial^2 V}{\partial q_k^2}, & j = k, \\[2ex]
-\dfrac{m}{\beta\epsilon\hbar^2}, & j = k \pm 1, \\[2ex]
0, & \text{otherwise.}
\end{array} \tag{9. 6}$$

As before, we set $z_j = (m/\beta\epsilon\hbar^2)^{1/2} y_j$, and diagonalize the quadratic form by a C matrix and transformation of variables from z_j to $u_j = \sum_k C_{jk}^T z_k$,

$$\rho(q_1q_0) = (\exp(-f^*))(2\pi N)^{-1}\, \mathbf{P}\left[\int du(t)\right]^{dt} \times |J|\,|C| \lim_{\epsilon\to 0} \exp(-\tfrac{1}{2}\sum_j u_j\Lambda_{jj}u_j). \quad (9.7)$$

The continuum case is then

$$\begin{aligned}\rho(q_1q_0) &= (\exp(-f^*))(2\pi N)^{-1}|J|\, \mathbf{P}\left[\int du(t)\right]^{dt} \cdot \exp\int_0^1 dt - \tfrac{1}{2}u(t)\Lambda(tt)u(t)\\ &= (\exp(-f^*))(2\pi N)^{-1}|J| \cdot \exp\int_0^1 dt \times \ln\int_{-\infty}^{+\infty} du \exp(-\tfrac{1}{2}u\Lambda u)\\ &= (\exp(-f^*))(2\pi N)^{-1}|J| \exp\int_0^1 dt \ln\sqrt{2\pi/\det\Lambda}\\ &= (\exp(-f^*))N^{-1}|J|(2\pi\det\Lambda)^{-1/2}, \end{aligned} \quad (9.8)$$

where N is the normalization constant, J the Jacobian of transformation, $|J| = (m/\beta\epsilon\hbar^2)^{-1/2}$, $|C| = 1$. N is determined as before, $N = |J|^2$, $\det(\Lambda)$ is given by Abé as $\sinh(\hbar\omega\beta)/(\beta\epsilon\hbar\omega)$. We have then,

$$\rho(q_1q_0) = \left(\frac{m\omega}{h\sin(\hbar\omega\beta)}\right)^{1/2}\exp(-f^*), \quad (9.9)$$

where $\exp(-f^*)$ is given by (9.5). This result checks with that of Abé.[9]

We note that in the derivation, the integral was rendered dimensionless due to the presence of the exponential and logarithm functions, also the velocity (or momentum) terms were discretized and the resulting expression absorbed into the quadratic form, which was in turn found to obey some differential equation that could be solved to give the value of the determinant $\det(\Lambda)$. The structure among these various elements was correctly given by the functional integral formula (6.14), and conventional results were obtained.

X. DISCUSSION

Through the development of the methodology of the continuum calculus, i.e., the "rational" and "potential" calculi, we were able to characterize the integral for functionals in Sec. VI. The formula given is in closed form and amenable to conventional mathematical manipulations. Applications to algebra, probability theory, and Feynman's path integrals in quantum physics yielded valid results. We remark that in the presentation, the adherence to generality and rigor was at times sacrificed for the sake of giving a concrete demonstration, in a short time span and space, of the applicability of the present approach. The fecundity of the theory was also not fully explored. A case in point is the cross-fertilization of the potentiation operation and the ordinary differentiation, which will be called the homogeneous continuous differentiation and the heterogeneous continuous differentiation. The result will constitute the "inverse" operation of functional integration. We hope to be able to present a study on this in the future. The detailed nature of the p integral and the functional integral also deserves to be scrutinized from the measure theoretical point of view. To facilitate the calculation of the functional integrals, numerical methods should be developed. Another urgent task will be the characterization of the integral of functionals of different representation. For instance, what is the functional integral of the following functional:

$$\Phi[y] = \int dt f(y(t)), \quad f: R \to C. \quad (10.1)$$

We can give only a tentative solution here by invoking a "differentiable" homotopy to replace $y(t)$ in the inner integral, and a Riccati type of construction for $x_t(y)$ (see Definition 6.1). Instead of (6.12), $x_t(y)$ is now a functional,

$$x_t[y] = \exp[f(y(t))/\int_0^t ds f(y(s))]. \quad (10.2)$$

The term $\int_0^t ds f(y(s))$ in the denominator acts like some kind of a "memory functional." If for the given lower limit $a(t)$ and upper limit $b(t)$, a homotopy $H[y;\lambda]$ exists such that $H[y;0](t) = a(t)$ and $H[y;1](t) = b(t)$, and $\partial H/\partial\lambda$ exists for $\lambda \in (0,1)$, then (6.14) is replaced by

$$I_f^{(a,b)}(\Phi) \equiv \exp\int dt \ln\int_0^1 d\lambda\left(\frac{\partial H}{\partial\lambda}\right)x_t[H] \quad (10.3)$$

where

$$x_t[H] = \exp\{f(H[y;\lambda](t))/\int_0^t ds f(H[y;\lambda](s))\}. \quad (10.4)$$

The class of all homotopies $H[y;\lambda]$ that assign the same value α to the functional integral (10.3) will be called an invariant class $C_\alpha[\Phi]$ relative to $\Phi[y]$ of (10.1). Clearly, all the invariant classes $C_\alpha[\Phi]$ form an equivalent decomposition of the set K of all admissible homotopies of $y(t)$, and definition (10.3) would have meaning only when α is uniquely determined for all admissible homotopies. In other words, a general theorem on the class of admissible homotopies is needed. We present a case study for an improper integral of a functional of type (10.1) in the Appendix. Further investigation is required.

Finally, with the ascertainment of the integral of functionals, it is hoped that new results can be derived from its use, on witnessing the current interests[25] in path integrals in various branches of physics.

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APPENDIX: THE HOMOTOPIC CONSTRUCTION OF FUNCTIONAL INTEGRALS

In this Appendix, we shall determine one possible invariant class of admissible homotopies that are associated with a (diagonalized) quadratic functional

$$\Phi[y] = \int_0^1 dt\, B(t,t)y(t)^2, \quad (A1)$$

where $B(t,t)$ is a given fixed matrix. Friederichs[3] considered the functional integral of a similar (non-diagonalized) version of (A1), with a Gaussian measure and for the limit $-\infty \le y(t) \le +\infty$.

We first apply (10.2),

$$x_t[y] = \exp[B(t,t)y(t)^2/\int_0^t ds\, B(s,s)y(s)^2]. \quad (A2)$$

Then (6.13) gives for the Gaussian measure

$$I_f(\Phi) = \exp\int_0^1 dt \ln\int_{-\infty}^{+\infty} dy(t)[\exp(-y(t)^2)]x_t[y]. \quad (A3)$$

Now if we construct a homotopy with the lower limit $a(t) = 0$,

$$H[y;\lambda](t)=a(t)+\frac{\lambda}{1-\lambda}[b(t)-a(t)]=\frac{\lambda}{1-\lambda}b(t), \quad \text{(A4)}$$

we can define a functional integral, $J_f(\Phi)$,

$$J_f(\Phi)\equiv\exp\int_0^1 dt\ln\int_0^1 d\lambda\, b(1-\lambda)^{-2} \times[\exp(-b^2(\lambda/(1-\lambda))^2)]\cdot x_t[H]. \quad \text{(A5)}$$

However, we make a transformation of variables, $\gamma=\lambda/(1-\lambda)$, and let $Q(t)=\int_0^t ds\,B(s,s)\,b(s)^2$,

$$J_f(\Phi)=\exp\int_0^1 dt\ln\int_0^\infty d\gamma\, b\exp(-b^2\gamma^2)\exp(Bb^2\gamma^2/Q\gamma^2)$$
$$=\exp\int_0^1 dt\ln b(t)\exp[B(t)b(t)^2/Q(t)]\tfrac{1}{2}(\pi/b^2(t)^2)^{1/2}$$
$$=\exp\int_0^1 dt\ln\tfrac{1}{2}(\pi)^{1/2}\exp[B(t)b(t)^2/Q(t)]. \quad \text{(A6)}$$

The functional integral $I_f(\Phi)$ is simply related to $J_f(\Phi)$ by a factor of 2 in the argument of the logarithm function,

$$I_f(\Phi)=\exp\int_0^1 dt\ln\pi^{1/2}\exp[B(tt)b(t)^2/Q(t)]$$
$$=(\pi)^{1/2}\exp\int_0^1 dt\,[B(tt)b(t)^2/\int_0^t ds\,B(ss)b(s)^2]. \quad \text{(A7)}$$

If we choose $b(t)$ to be a constant function, $b(t)=C$,

$$I_f(\Phi)=(\pi)^{1/2}\exp\int_0^1 dt\,B(tt)/\int_0^t ds\,B(ss)$$
$$=(\pi)^{1/2}\int_0^1 dtB(t,t). \quad \text{(A8)}$$

Aside from a normalization factor $(\pi)^{1/2}$, the result is identical to that of Frederichs.[3] Therefore, for improper integrals $[-\infty\le y(t)\le\infty]$ of the functional (A1), the invariant class of admissible homotopies is the one equivalent to the form (A4) with homogeneous lower limit, $a(t)=0$, and constant upper limit, $b(t)=C$.

We note that as to functionals of the exponential representation (6.11), no uniqueness question arises, since no homotopy is involved. For the representation (10.1), a purely formal and invariant definition of its functional integral independent of the homotopy construction could also be given via (10.3) by restoring $y(t)$ in place of $H[y;\lambda]$ everywhere in (10.3). However, then practical ways of evaluation must be found to calculate the invariant $I_f(\Phi)$, overcoming the difficulty due to the presence of the "memory" functional $\int_0^t ds\, f(y(s))$. One possible solution may lie in a Fourier series expansion of $y(t)$. The homotopy route can then be viewed as a constructive definition of $I_f(\Phi)$, but at the cost of the incurrence of the uniqueness question and the restriction to homotopic limits of integration.

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On the irreducible representations of the Lie algebra chain $G_2 \supset A_2$*

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In the first part of this article we solve the "state labelling problem" for the irreducible finite dimensional representations of the $G_2 \supset A_2$ chain, using a method applicable to other algebras-subalgebras chains. In the second part we define, for these representations, operators analogous to those introduced by Nagel and Moshinsky for the $A_n \supset A_{n-1}$ chain and explicitly construct the representations belonging to the equivalent classes $\circ\overset{\Lambda}{\Rrightarrow}\bullet^{0}$ and $\circ\overset{0}{\Rrightarrow}\bullet^{\Lambda}$.

INTRODUCTION AND SUMMARY

Some irreducible representations of the Lie algebra G_2 (or of its two real forms) appear in the well-known $B_3 \supset G_2 \supset A_1$ chain used by Racah in atomic spectroscopy[1] and occasionally in elementary particle physics through the $G_2 \supset A_2$ chain[2]; This latter model, conveniently generalized in the wider context of the other exceptional Lie algebras F_4, E_6, E_7, E_8 is the subject of much current interest.[3]

General results on the representations of the $G_2 \supset A_1$ chain (where A_1 is a principal three-dimensional algebra[4]) are very meagre. The "basis labeling problem" is not yet solved; a general method for this exists[5] but a complete answer requires the displaying of an explicit construction of the representations of G_2. The representations of the chain $G_2 \supset A_2$ afford such a possibility. Some years ago, an algorithm was published,[6] which permits the construction of the representations of G_2 by reduction to this algebra of the irreducible representations of the particular Gel'fand–Tseitlin $A_1 \supset A_6$ chain. This method however is effective and well adapted only for numerical computations.

When restricting a representation of G_2 to its subalgebra A_2, the carrier space of this representation contains a vector subspace of crucial importance: the vector subspace of the maximal vectors. The whole problem of the construction of the representations of the $G_2 \supset A_2$ chain can be reduced to the construction of a basis for this subspace. (This remark is valid for any representation of an algebra-sub-algebra chain.)

As a first step, we must find a convenient labeling for such a basis. Applying a very natural method (described in Appendix B) we solve this problem easily. The result is an interesting one: *A basis of maximal vectors of the representation of A_2 subduced by a representation of the class $\overset{\Lambda_1}{\circ}\Rrightarrow\overset{\Lambda_2}{\bullet}$ of G_2 can be put in a one-to-one correspondance with a Gel'fand–Tseitlin basis of a representation belonging to the class $\overset{\Lambda_1}{\circ}\!-\!\overset{\Lambda_2}{\circ}$ of A_2.*

As a second step, we find a set of operators which, acting repeatedly, generate this subspace starting from a given maximal vector of the G_2 representation. These operators are just the equivalent for $G_2 \supset A_2$ of the operators introduced by Nagel and Moshinsky for the representations of the chain $A_n \supset A_{n-1}$.[7] These operators permit an effective construction of a basis of maximal vectors and, thanks to the Asherova–Smirnov operator,[8] a construction of a basis of the whole carrier space of the representation. Being expressible in terms of the G_2 generators, they also permit the computation of the matrix elements of these latter.

As a last step we carry out this program explicitly for the representations belonging to the classes $\overset{\Lambda}{\circ}\Rrightarrow\overset{0}{\bullet}$ and $\overset{0}{\circ}\Rrightarrow\overset{\Lambda}{\bullet}$.

1. NOTATIONS AND CONVENTIONS FOR THE CHAIN $G_2 \supset A_2$

The base field of G_2 is here $\mathbb{C}$ (or $\mathbb{R}$). Let $\mathcal{H}$ denote a Cartan subalgebra of G_2. As usual we identify it with its dual by means of the restriction to $\mathcal{H}$ of the Killing form defined on G_2. We choose for the root system

$$\Sigma = \{\pm \alpha_1, \pm \alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2), \pm(\alpha_1+3\alpha_1), \pm(2\alpha_1+3\alpha_2)\},$$

the following normalization:

$$(\alpha_1,\alpha_1)=2, \quad (\alpha_1,\alpha_2)=-1, \quad (\alpha_2,\alpha_2)=\tfrac{2}{3}. \tag{1.1}$$

For each root $\rho \in \Sigma$, we know there exists an element e_ρ and an element $e_{-\rho}$ of G_2, such that for each $h \in \mathcal{H}$ we have

$$[h, e_\rho] = (h,\rho)e_\rho, \quad [e_\rho, e_{-\rho}] = h_\rho \quad (h_\rho \equiv \rho)$$

and (for all $\rho, \mu \in \Sigma$)

$$[e_\rho, e_\mu] = \begin{cases} N_{\rho,\mu} e_{\rho+\mu} & \text{if } \rho+\mu \in \Sigma, \\ 0 & \text{if } \rho+\mu \notin \Sigma. \end{cases}$$

The structure constants $N_{\rho,\mu}$ of course satisfy

$$N_{\rho,\mu} = -N_{\mu,\rho},$$

but it is always possible to impose the further two conditions

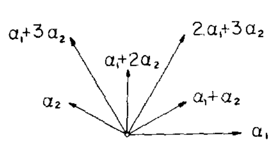

FIG. 1. The positive roots of G_2.

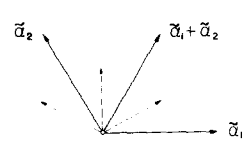

FIG. 2. The positive roots of $\widetilde{A}_2 \hookrightarrow G_2$.

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$$N_{\rho,\mu} = -N_{-\rho,-\mu} \quad \text{and} \quad N_{\rho,\mu} = -N_{\rho,-\mu-\rho}.$$

With respect to the normalization (1.1) we can set

$$N_{\alpha_1,\alpha_2} = 1, \quad N_{\alpha_1+\alpha_2,\alpha_2} = -2\sqrt{3}/3, \quad N_{\alpha_1+2\alpha_2,\alpha_2} = 1,$$
$$N_{\alpha_1+2\alpha_2,\alpha_1+\alpha_2} = 1, \quad N_{\alpha_1+3\alpha_2,\alpha_1} = -1.$$

From these, all the other structure constants are immediately found with the help of the above two conditions.

The regular subalgebra[4] $\widetilde{A}_2$ of G_2 is completely defined (up to an equivalence) by the root subsystem

$$\widetilde{\Sigma} = \{\pm\widetilde{\alpha}_1, \pm\widetilde{\alpha}_2, \pm(\widetilde{\alpha}_1+\widetilde{\alpha}_2) : \widetilde{\alpha}_1 = \alpha, \ \widetilde{\alpha}_2 = \alpha_1 + 3\alpha_2\} \subset \Sigma. \tag{1.2}$$

The Cartan subalgebra $\widetilde{\mathcal{H}}$ of $\widetilde{A}_2$ generated by $\widetilde{\Sigma}$ is contained in $\mathcal{H}$; more exactly $\widetilde{\mathcal{H}} = \mathcal{H}$ and, if $\widetilde{W}$ denotes the Weyl group of this subalgebra, it is a subgroup of the Weyl group W of G_2. Actually,

$$W = \widetilde{W}1 \dotplus \widetilde{W}S_{\alpha_1} \tag{1.3}$$

(where $S_\rho : M \to S_\rho M = M - [2(M,\rho)/(\rho,\rho)]\rho$, $\rho \in \Sigma$, $M \in \mathcal{H}$).

2. STRUCTURE OF THE IRREDUCIBLE REPRESENTATIONS OF $G_2 \supset A_2$

Both equivalence classes of irreducible finite dimensional representations of G_2 and A_2 are characterized by any two nonnegative integers Λ_1 and Λ_2; those of G_2 will be denoted by $\overset{\Lambda_1\ \Lambda_2}{\circ\!\equiv\!\bullet}$, those of A_2 by $\overset{\Lambda_1\ \Lambda_2}{\circ\!-\!\circ}$.

If $D \in \overset{\Lambda_1\ \Lambda_2}{\circ\!\equiv\!\bullet}$, we know there exists a maximal weight $\Lambda \in \mathcal{H}$ such that

$$\Lambda_1 = 2(\Lambda, \alpha_1)/(\alpha_1, \alpha_1), \quad \Lambda_2 = 2(\Lambda, \alpha_2)/(\alpha_2, \alpha_2),$$

and each weight P of D is of the form

$$P = \Lambda - k_1\alpha_1 - k_2\alpha_2,$$

where k_1 and k_2 are two nonnegative integers. We write $\Delta(D)$ for the set of weights of D. If V_P denotes the weight space of P, V admits a direct decomposition

$$V = \bigoplus_{P \in \Delta(D)} V_P$$

and for each $h \in \mathcal{H}$ and each $\rho \in \Sigma$ we have ($v_P \in V_P$)

$$D(h)v_P = (h, P)v_P,$$

$$D(e_\rho)v_P \begin{cases} \in V_{P+\rho} & \text{if } P+\rho \in \Delta(D), \\ = 0 & \text{if } P+\rho \notin \Delta(D). \end{cases}$$

To simplify our notation we put

$$E_\rho = D(e_\rho), \quad H = D(h).$$

The subalgebra $\widetilde{A}_2$ of G_2 being simple, the representation $\widetilde{D}$ obtained by restriction of D to $\widetilde{A}_2 \subset G_2$ is completely reducible into a sum of irreducible ones. Since $\widetilde{\mathcal{H}} \subseteq \mathcal{H}$, for each $\widetilde{h} \in \widetilde{\mathcal{H}}$, we have $\widetilde{H}v_P = (\widetilde{h}, P)v_P$ and therefore $\Delta(\widetilde{D}) = \Delta(D)$.

Let Z denote the subspace of V of the maximal vectors of the representation $\widetilde{D}$, i.e.,

$$Z = \{v \in V \mid E_{\alpha_1}v = 0, \ E_{\alpha_1+3\alpha_2}v = 0\}.$$

[By virtue of (1.2), $D(e_{\widetilde{\alpha}_1}) = E_{\alpha_1}$, $D(e_{\widetilde{\alpha}_2}) = E_{\alpha_1+3\alpha_2}$.] It is generated by weight vectors v_M,

$$Hv_M = (h, M)v_M$$

such that

$$M_1 = \frac{2(M,\alpha_1)}{(\alpha_1,\alpha_1)} \quad \text{and} \quad M_2 = \frac{2(M, \alpha_1+3\alpha_2)}{(\alpha_1+3\alpha_2, \alpha_1+3\alpha_2)}$$

belong to $\mathbf{N}$; so we have

$$Z = \bigoplus_{\{M\}} Z_M,$$

where $\dim Z_M$ equals the number of representations belonging to $\overset{M_1\ M_2}{\circ\!-\!\circ}$ contained in $\widetilde{D}$.

A labeling for a basis of weight vectors of Z is given by the "path-label" method (cf. Appendix B): To each weight vector $v \in Z$, there corresponds a unique triple $\{p, q, r\}$ of nonnegative integers defined by

$$E^r_{\alpha_1+\alpha_2}v \neq 0, \quad E^{r+1}_{\alpha_1+\alpha_2}v = 0,$$
$$E^q_{\alpha_1+2\alpha_2}E^r_{\alpha_1+\alpha_2}v \neq 0, \quad E^{q+1}_{\alpha_1+2\alpha_2}E^r_{\alpha_1+\alpha_2}v = 0,$$
$$E^p_{\alpha_2}E^q_{\alpha_1+2\alpha_2}E^r_{\alpha_1+\alpha_2}v \neq 0, \quad E^{p+1}_{\alpha_2}E^q_{\alpha_1+2\alpha_2}E^r_{\alpha_1+\alpha_2}v = 0.$$

The weight M to which v belongs is then given by

$$M = \Lambda - p\alpha_2 - q(\alpha_1 + 2\alpha_2) - r(\alpha_1 + \alpha_2);$$

consequently v is a maximal vector of a representation belonging to $\overset{M_1\ M_2}{\circ\!-\!\circ}$ with

$$M_1 = \Lambda_1 + p - r, \quad M_2 = \Lambda_1 + \Lambda_2 - p - q. \tag{2.1}$$

The domains of definition of the numbers p, q, and r are given by the following theorem.

Theorem: If $D \in \overset{\Lambda_1\ \Lambda_2}{\circ\!\equiv\!\bullet}$, $\widetilde{D}(M_1, M_2) \in \overset{M_1\ M_2}{\circ\!-\!\circ}$, and if $\widetilde{D}$ denotes the restriction of D to $A_2 \subset G_2$, then

$$\widetilde{D} = \dotplus_{\{p,q,r\}} \widetilde{D}(\Lambda_1 + p - r, \ \Lambda_1 + \Lambda_2 - p - q)$$

with

$$0 \leq p \leq \Lambda_2, \quad 0 \leq q \leq \Lambda_1, \quad 0 \leq r \leq \Lambda_1 + p - q$$

(all permissible integers p, q, r).

It is straightforward to verify directly that

$$\dim(D) = \sum_{\{p,q,r\}} \dim[\widetilde{D}(\Lambda_1 + p - r, \Lambda_1 + \Lambda_2 - p - q)],$$

but this equality of course does not constitute a proof of the theorem. However, by virtue of the uniqueness of the decomposition of the character of a representation into the sum of the characters of its irreducible components, the theorem will be proved as soon as we have established the equality

$$\widetilde{\chi} = \sum_{p=0}^{\Lambda_2} \sum_{q=0}^{\Lambda_1} \sum_{r=0}^{\Lambda_1+p-q} \widetilde{\chi}_{\Lambda - p\alpha_2 - q(\alpha_1+2\alpha_2) - r(\alpha_1+\alpha_2)}$$

in which $\widetilde{\chi}$ is the character of $\widetilde{D}$, i.e., the restriction to $\widetilde{\mathcal{H}}$ of the character χ_Λ of D and $\widetilde{\chi}_{\Lambda-p\alpha_2-q(\alpha_1+2\alpha_2)-r(\alpha_1+\alpha_2)}$ is the character of $\widetilde{D}(\Lambda_1 + p - r, \Lambda_1 + \Lambda_2 - p - q)$.

For all results on characters used here we refer the reader to the Jacobson[9]; although, due to our special choice of base field, the formal exponentials can here be thought of as complex valued functions $e(M) : \mathcal{H} \to C$ defined by $e(M) : h \to \exp\{(M, h)\}$ with any M in $\mathcal{H}$.

Let $\mathcal{A} = \sum_{S \in W} (\det S)S$ and $\widetilde{\mathcal{A}} = \sum_{\widetilde{S} \in \widetilde{W}} (\det \widetilde{S})\widetilde{S}$ be the alternation operators constructed on W and $\widetilde{W}$. Further, let δ and $\widetilde{\delta}$ be the half-sum of the positive roots of G_2 and of $\widetilde{A}_2 \subset G_2$: $\delta = \widetilde{\delta} + \alpha_1 + 2\alpha_2$. We need the following lemma.

Lemma:

$$\mathcal{A}e(\delta)=(\tilde{\mathcal{A}}e(\tilde{\delta}))\cdot F,$$

where

$$F=-[1-e(-\alpha_2)][1-e(-\alpha_1-\alpha_2)][1-e(\alpha_1+2\alpha_2)]$$

and

$$\tilde{S}F=F \text{ for all } \tilde{S}\in\tilde{W}.$$

Proof: Since $\mathcal{A}e(\delta)=e(\delta)\Pi_{\alpha>0}[1-e(-\alpha)]$ (Ref. 9, p. 250), a direct computation gives the result. The invariance of F with respect to $\tilde{W}$ follows at once from the fact that $\{\alpha_1+2\alpha_2,-\alpha_2,-\alpha_1-\alpha_2\}$ is the weight set of the representation belonging to $\overset{0}{\circ}\!-\!\overset{1}{\circ}$ contained in the restriction to $\tilde{A}_2\subset G_2$ of the adjoint representation of G_2. ∎

Proof of the theorem: Starting from the Weyl formula (Ref. 9, p. 255),

$$\chi_\Lambda\cdot\mathcal{A}e(\delta)=\mathcal{A}e(\Lambda+\delta),$$

it suffices to show that

$$\sum_{p=0}^{\Lambda_2}\sum_{q=0}^{\Lambda_1}\sum_{r=0}^{\Lambda_1+p-q}\tilde{\chi}_{\Lambda-p\alpha_2-q(\alpha_1+2\alpha_2)-r(\alpha_1+\alpha_2)}\mathcal{A}e(\delta)=\mathcal{A}e(\Lambda+\delta).$$

Using the above lemma and the Weyl formula again, this time for the character $\tilde{\chi}_{\Lambda-p\alpha_2-q(\alpha_1+2\alpha_2)-r(\alpha_1+\alpha_2)}$, this equality becomes

$$\sum_{p=0}^{\Lambda_2}\sum_{q=0}^{\Lambda_1}\sum_{r=0}^{\Lambda_1+p-q}\tilde{\mathcal{A}}[e(\Lambda-p\alpha_2-q(\alpha_1+2\alpha_2)-r(\alpha_1+\alpha_2)+\tilde{\delta})\cdot F]$$
$$=\mathcal{A}e(\Lambda+\delta).$$

Let B denote the left-hand side; since

$$\Lambda+\tilde{\delta}=\Lambda+\delta-\alpha_1-2\alpha_2$$
$$=(\Lambda_1+1)(\alpha_1+\alpha_2)+(\Lambda_1+\Lambda_2+1)(\alpha_1+2\alpha_2),$$
$$B=\sum_{p=0}^{\Lambda_2}\sum_{q=0}^{\Lambda_1}\sum_{r=0}^{\Lambda_1+p-q}\tilde{\mathcal{A}}[e(-p\alpha_2+(\Lambda_1+\Lambda_2-p+1)(\alpha_1+2\alpha_2)$$
$$+(\Lambda_1-r+1)(\alpha_1+\alpha_2))\cdot F].$$

The sums are easy to evaluate: They are geometric progressions of ratio $e(-\alpha_1-\alpha_2)$, $e(-\alpha_2)$, and $e(\alpha_1+2\alpha_2)$. The denominators in the formulas cancel with the elements of F. Taking into account the fact that if $\tilde{S}_\rho M=M$ for some $S_\rho\in\tilde{W}$ and $M\in\tilde{\mathcal{H}}$, then $\tilde{\mathcal{A}}e(M)=0$, we obtain

$$B=\tilde{\mathcal{A}}[e((\Lambda_1+1)(\alpha_1+\alpha_2)+(\Lambda_1+\Lambda_2+2)(\alpha_1+2\alpha_2))$$
$$-e((\Lambda_1+1)(\alpha_1+2\alpha_2)+(\Lambda_1+\Lambda_2+2)(\alpha_1+\alpha_2))],$$

but

$$S_{\alpha_2}(\alpha_1+\alpha_2)=\alpha_1+2\alpha_2$$

and

$$(\Lambda_1+1)(\alpha_1+\alpha_2)+(\Lambda_1+\Lambda_2+2)(\alpha_1+2\alpha_2)=\Lambda+\delta.$$

Therefore,

$$B=\tilde{\mathcal{A}}[e(\Lambda+\delta)-e(S_{\alpha_2}(\Lambda+\delta))],$$

i.e., by virtue of (1.3) ($\det S_{\alpha_2}=-1$)

$$B=\mathcal{A}e(\Lambda+\delta).$$ ∎

This theorem furnishes a very remarkable labeling for the basis vectors of a representation $D\in\overset{\Lambda_1}{\circ}\!\equiv\!\overset{\Lambda_2}{\bullet}$ by means of two Gel'fand–Tseitlin patterns.[10]

Setting

$$a=\Lambda_1+\Lambda_2-q,\quad b=\Lambda_2-p,\quad c=a-r,$$

we obtain, form the theorem, the inequalities

$$\Lambda_2\leqslant a\leqslant\Lambda_1+\Lambda_2,\quad 0\leqslant b\leqslant\Lambda_2,\quad b\leqslant c\leqslant a$$

and, from (2.1),

$$M_1+M_2=\Lambda_1+c,\quad M_2=a+b-\Lambda_2,$$

which gives the label

$$\begin{bmatrix}\Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0\\ & a & & b & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix}. \tag{2.2}$$

The pattern on the right gives a label for the basis vectors of the representation $\tilde{D}(M_1,M_2)$ associated with the pattern on the left. According to the usual rules of correspondance, the vector associated with such a label belongs to the weight

$$P=\Lambda-(\Lambda_1+\Lambda_2-a)(\alpha_1+2\alpha_2)-(\Lambda_2-b)\alpha_2$$
$$-(a-c)(\alpha_1+\alpha_2)-(\Lambda_1+c-d)(2\alpha_1+3\alpha_2)$$
$$-(a+b-\Lambda_2-e)(\alpha_1+3\alpha_2)-(d-f)\alpha_1.$$

It is quite surprising that the representations $\tilde{D}(M_1,M_2)$ can be labeled by the pattern

$$\begin{bmatrix}\Lambda_1+\Lambda_2 & & \Lambda_2 & & 0\\ & a & & b & \\ & & c & & \end{bmatrix},$$

which is the pattern of a representation of A_2 belonging to $\overset{\Lambda_1}{\circ}\!-\!\overset{\Lambda_2}{\circ}$! This labeling should be compared with the one given some years ago by Lam and Sharp.[11]

3. CONSTRUCTION OF THE REPRESENTATIONS $\tilde{D}$

We look for "unitary" representations of G_2. By this we mean that, with respect to a Hermitian scalar product $(\,|\,)$: $V\times V\to C$ defined on the carrier space V of the representation, we impose the conditions

$$E_\rho^\dagger=E_{-\rho}\text{ for all }\rho\in\Sigma,\quad H^\dagger=H\text{ for all }h\in\mathcal{H}.$$

It can be shown that such representations always exist (in using our special choice of structure constants and the existence of a compact real form of G_2).

Starting from any orthonormal basis of Z consisting of weight vectors of D, we know how to construct an orthonormal basis of V. In fact if we write, according to the above labeling, these basis vectors of Z under the form

$$\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & \Lambda_1+c & & a+b-\Lambda_2 & \\ & & c & & & & & \Lambda_1+c & & \end{bmatrix},$$

an orthonormal basis vector of V is given by

$$\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix} = S\begin{bmatrix} a+b, c \\ d, e, f \end{bmatrix} \psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & \Lambda_1+c & & a+b-\Lambda_2 & \\ & & c & & & & & \Lambda_1+c & & \end{bmatrix},$$

where $S\binom{a+b,\,c}{d,e,f}$ denotes the operator of Asherova and Smirnov[8] which is written here as

$$S\begin{bmatrix} a+b, c \\ d, e, f \end{bmatrix} = C \sum_{i=0}^{\Lambda_1+c-d} \begin{bmatrix} i \\ \Lambda_1+c-d \end{bmatrix} \frac{(d-e+1)!}{(d-e+i+1)!} E_{-\alpha_1}^{d-f+i} E_{-\alpha_1-3\alpha_2}^{a+b-\Lambda_2-e+i} E_{-2\alpha_1-3\alpha_2}^{\Lambda_1+c-d-i}$$

with

$$C = \left(\frac{(d-a-b)!\,(d+1)!\,(\Lambda_1-e+1)!\,e!\,(f-e)!}{(a+b-\Lambda_2-e)!\,(\Lambda_1+c-d)!\,(\Lambda_1+\Lambda_2-a-b+c)!\,(a+b-\Lambda_2)!\,(d-e)!\,(d-e+1)!\,(d-f)!} \right)^{1/2}. \tag{3.1}$$

The action of the generators of $\tilde{A}_2 \subset G_2$ on this basis is well known: It is given by the Gel'fand–Tseitlin formulas.[10] If we write for simplicity

$$\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix} \equiv \psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix},$$

these formulas take here the following form

$$\begin{aligned} &H_{\alpha_1}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = [2f-(d+e)]\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix}, \quad H_{\alpha_1+3\alpha_2}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = [2(d+e)-f-(\Lambda_1+a+b+c-\Lambda_2)]\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix}, \\ &E_{\alpha_1}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = [(d-f)(f-e+1)]^{1/2}\psi\begin{bmatrix} d & & e \\ & f+1 & \end{bmatrix}, \quad E_{-\alpha_1}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = [(d-f+1)(f-e)]^{1/2}\psi\begin{bmatrix} d & & e \\ & f-1 & \end{bmatrix}, \\ &E_{\alpha_1+3\alpha_2}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = C_1\psi\begin{bmatrix} d+1 & & e \\ & f & \end{bmatrix} + C_2\psi\begin{bmatrix} d & & e+1 \\ & f & \end{bmatrix}, \quad E_{-\alpha_1-3\alpha_2}\psi\begin{bmatrix} d & & e \\ & f & \end{bmatrix} = K_1\psi\begin{bmatrix} d-1 & & e \\ & f & \end{bmatrix} + K_2\psi\begin{bmatrix} d & & e-1 \\ & f & \end{bmatrix}, \\ &C_1 = \left[\frac{(\Lambda_1+c-d)(\Lambda_2-a-b+d+1)(d+2)(d-f+1)}{(d-e+1)(d-e+2)}\right]^{1/2} \quad K_1 = \left[\frac{(\Lambda_1+c-d+1)(\Lambda_2-a-b+d)(d+1)(d-f)}{(d-e)(d-e+1)}\right]^{1/2}, \\ &C_2 = \left[\frac{(\Lambda_1+c-e+1)(a+b-\Lambda_2-e)(e+1)(f-e)}{(d-e)(d-e+1)}\right]^{1/2} \quad K_2 = \left[\frac{(\Lambda_1+c-e+2)(a+b-\Lambda_2-e+1)e(f-e+1)}{(d-e+1)(d-e+2)}\right]^{1/2}. \end{aligned} \tag{3.2}$$

4. NAGEL-MOSHINSKY TYPE OPERATORS FOR THE REPRESENTATIONS OF $G_2 \supset A_2$

If, as we have just seen, the representatives of A_2 in $\mathrm{End}(V)$ have matrix elements independent of the chosen basis for Z, it is of course not the case for those of G_2. Actually the whole problem of the construction of the representations D of $G_2 \supset A_2$ consists precisely of constructing a basis of Z and of deducing from it the action of the generators of $G_2 \backslash A_2$ on the latter. However, we will show that this problem can be simplified somewhat by defining on Z some operators (the equivalent ones for $G_2 \supset A_2$ of those introduced by Nagel and Moshinsky for the irreducible representations of $A_n \supset A_{n-1}$[7]).

Let $\Pi: V \to Z$ be the projection operator associated with Z ($\Pi^2 = \Pi$, $\Pi^\dagger = \Pi$). If $T: V \to V$ is a linear operator, we will denote by $\overline{T}$ the restriction to Z of the operator $\Pi T \Pi$. Since $\Pi^\dagger = \Pi$, we have $(\Pi T \Pi)^\dagger = \Pi T^\dagger \Pi$ and therefor $\overline{T^\dagger} = \overline{T}^\dagger$. (It should be clear that $\overline{T}^\dagger$ is the Hermitian conjugate of $\overline{T}: Z \to Z$ with respect to the restriction to $Z \times Z$ of the Hermitian scalar product.)

By definition of Z,

$$E_{\tilde{\rho}}: Z \to \{0\}, \quad (E_{-\tilde{\rho}}Z) \cap Z = \{0\},$$

when $\tilde{\rho}$ is a positive root of $\tilde{A}_2 \subset G_2$; therefore $\overline{E}_{\pm\alpha_1} = 0$, $\overline{E}_{\pm(\alpha_1+3\alpha_2)} = 0$, and $\overline{E}_{\pm(2\alpha_1+3\alpha_2)} = 0$. On the other hand, $\overline{H} = H$ (more exactly $\overline{H}$ is equal to the restriction of H to Z) for all $h \in \mathcal{H}$, which implies

$$[\overline{H}, \overline{E}_\rho] = (h, \rho)\overline{E}_\rho \quad (\rho \in \Sigma).$$

For the remaining generators of G_2 we obtain (all operators restricted to Z)

$$\begin{aligned} \overline{E}_{\alpha_1+2\alpha_2} &= E_{\alpha_1+2\alpha_2}, \quad \overline{E}_{\alpha_1+\alpha_2} = E_{\alpha_1+\alpha_2}, \\ \overline{E}_{\alpha_2} &= E_{\alpha_2} - (H_{\alpha_1}+2)^{-1}E_{-\alpha_1}E_{\alpha_1+\alpha_2}, \\ \overline{E}_{-\alpha_2} &= E_{-\alpha_2} - (H_{\alpha_1+3\alpha_2}+2)^{-1}E_{-\alpha_1-3\alpha_2}E_{\alpha_1+2\alpha_2}, \\ \overline{E}_{-\alpha_1-\alpha_2} &= E_{-\alpha_1-\alpha_2} + (H_{\alpha_1}+2)^{-1}E_{-\alpha_1}E_{-\alpha_2} - (H_{\alpha_1}+2)^{-1} \\ &\quad \times (H_{2\alpha_1+3\alpha_2}+3)^{-1}E_{-\alpha_1}E_{-\alpha_1-3\alpha_2}E_{\alpha_1+2\alpha_2} \\ &\quad - (H_{2\alpha_1+3\alpha_2}+3)^{-1}E_{-2\alpha_1-3\alpha_2}E_{\alpha_1+2\alpha_2}, \\ \overline{E}_{-\alpha_1-2\alpha_2} &= E_{-\alpha_1-2\alpha_2} + (H_{\alpha_1+3\alpha_2}+2)^{-1}E_{-\alpha_1-3\alpha_2}E_{\alpha_2} \\ &\quad - (H_{\alpha_1+3\alpha_2}+2)^{-1}(H_{2\alpha_1+3\alpha_2}+3)^{-1}E_{-\alpha_1-3\alpha_2} \\ &\quad \times E_{-\alpha_1}E_{\alpha_1+\alpha_2} + (H_{2\alpha_1+3\alpha_2}+3)^{-1}E_{-\alpha_1-3\alpha_2}E_{\alpha_1+\alpha_2}. \end{aligned} \tag{4.1}$$

The first two equalities are obvious since $E_{\alpha_1+2\alpha_2}$ and $E_{\alpha_1+\alpha_2}$ commute with E_{α_1} and $E_{\alpha_1+3\alpha_2}$ (and therefore $E_{\alpha_1+2\alpha_2}: Z \to Z$, $E_{\alpha_1+\alpha_2}: Z \to Z$). All the others are found in the same manner. Verify, for example, the third one.

Let $v_M \in Z$ be a weight vector belonging to the weight M. By the above formula,

$$\bar{E}_{\alpha_2} v_M = E_{\alpha_2} v_M - \frac{1}{M_1 + 1} E_{-\alpha_1} E_{\alpha_1+\alpha_2} v_M,$$

where $M_1 = (M, \alpha_1)$.

We must have

$$E_{\alpha_1} \bar{E}_{\alpha_2} v_M = 0 \quad \text{and} \quad E_{\alpha_1+3\alpha_2} \bar{E}_{\alpha_2} v_M = 0.$$

The second condition is trivially verified taking into account the commutation relations in G_2. For the first one, we have

$$\begin{aligned} 0 &= E_{\alpha_1} E_{\alpha_2} v_M - \frac{1}{M_1+1} E_{\alpha_1} E_{-\alpha_1} E_{\alpha_1+\alpha_2} v_M \\ &= E_{\alpha_1+\alpha_2} v_M - \frac{1}{M_1+1} H_{\alpha_1} E_{\alpha_1+\alpha_2} v_M \\ &= E_{\alpha_1+\alpha_2} v_M - \frac{1}{M_1+1} (M_1+1) E_{\alpha_1+\alpha_2} v_M = 0. \end{aligned}$$

It is clear that the inverse operators appearing in the formulas (4.1) are well defined since for each weight vector $v_M \in Z$ we have

$$H_{\alpha_1} v_M = (M, \alpha_1) v_M \quad \text{and} \quad H_{\alpha_1+3\alpha_2} v_M = (M, \alpha_1 + 3\alpha_2) v_M$$

with $(M, \alpha_1) \geq 0$, $(M, \alpha_1 + 3\alpha_2) \geq 0$. (Recall that M is a maximal weight for some representation of $\tilde{A}_2$.)

The commutation relations (or rather the generalized commutation relations) of these operators are given in Appendix A. All these formulas are obtained directly from the expressions (4.1) for the operators $\bar{E}_\rho$. Some simplifications occur if we take into account that $\bar{E}_\rho^\dagger = \bar{E}_{-\rho}$ and $[\bar{E}_\rho, \bar{E}_\mu]^\dagger = [\bar{E}_{-\mu}, \bar{E}_{-\rho}]$.

Although the weight set attached to Z can be put, as we saw it, in correspondance with the weight set (including multiplicities) of an irreducible representation of A_2, it is clear that the eight operators $\bar{H}_{\alpha_2}$, $\bar{H}_{\alpha_1+\alpha_2}$, $\bar{E}_{\pm\alpha_2}$, $\bar{E}_{\pm(\alpha_1+\alpha_2)}$, and $\bar{E}_{\pm(\alpha_1+2\alpha_2)}$ do not generate [in End(Z)] a Lie algebra of type A_2 in spite of a "slight resemblance" in their commutation relations (a similar situation appears in Ref. 7 with the $\underbrace{A_1 \times A_1 \times \cdots \times A_1}_{n}$ Lie algebra).

Let

$$\left\{ \phi\begin{pmatrix} ab \\ c \end{pmatrix} \equiv \psi \left[\begin{array}{cccccccccc} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & \Lambda_1+c & & a+b-\Lambda_2 & \\ & & c & & & & & \Lambda_1+c & & \end{array}\right] \right\}$$

be an orthonormal basis of weight vectors of Z. Then

$$\bar{H}_{\alpha_1} \phi\begin{pmatrix} ab \\ c \end{pmatrix} = (\Lambda_1 + \Lambda_2 - a - b + c) \phi\begin{pmatrix} ab \\ c \end{pmatrix},$$

$$\bar{H}_{\alpha_1+2\alpha_2} \phi\begin{pmatrix} ab \\ c \end{pmatrix} = (a + b - \Lambda_2) \phi\begin{pmatrix} ab \\ c \end{pmatrix},$$

and since

$$[\bar{H}, \bar{E}_\rho] = (h, \rho) \bar{E}_\rho,$$

we obtain

$$\begin{aligned} \bar{E}_{\pm\alpha_2} \phi\begin{pmatrix} ab \\ c \end{pmatrix} &= \sum_{i \in I} A_i^{(\pm\alpha_2)}(a, b, c) \phi\begin{pmatrix} a \pm 1 + i & & b - i \\ & i & \end{pmatrix}, \\ \bar{E}_{\pm(\alpha_1+\alpha_2)} \phi\begin{pmatrix} ab \\ c \end{pmatrix} &= \sum_{j \in J} A_j^{(\pm(\alpha_1+\alpha_2))}(a, b, c) \\ &\quad \times \phi\begin{pmatrix} a + i & & b - i \\ & c \pm 1 & \end{pmatrix}, \\ \bar{E}_{\pm(\alpha_1+2\alpha_2)} \phi\begin{pmatrix} ab \\ c \end{pmatrix} &= \sum_{k \in K} A_k^{(\pm(\alpha_1+2\alpha_2))}(a, b, c) \\ &\quad \times \phi\begin{pmatrix} a \pm 1 + i & & b - i \\ & c \pm 1 & \end{pmatrix}, \end{aligned} \tag{4.2}$$

where the indices i, j, and k run in principle over all values permissible by the Gel'fand–Tseitlin patterns. It is impossible to say more before having an explicit basis $\{\phi(^{ab}_{c})\}$ for Z.

If $\phi(^{\Lambda_1+\Lambda_2}{}_{\Lambda_1+\Lambda_2}{}^{\Lambda_2})$ is a maximal (unitary) vector of D it is not very difficult to verify that the set of vectors of Z,

$$\left\{ \bar{E}^{a-c}_{-\alpha_1-\alpha_2} \bar{E}^{\Lambda_1+\Lambda_2-a}_{-\alpha_1-2\alpha_2} \bar{E}^{\Lambda_2-b}_{-\alpha_2} \phi\begin{pmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 \\ & \Lambda_1+\Lambda_2 & \end{pmatrix} \right\}, \tag{4.3}$$

where the numbers a, b, and c are any integers compatible with the pattern

$$\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 \\ & a & & b & \\ & & c & & \end{bmatrix},$$

are linearly independent and consequently form a basis for Z. It is then easy, using the generalized commutation relations of Appendix A, to determine the action of the generators $\bar{E}_\mu$ on them and with some additional effort to normalize them. In general however, with the exception of the representations belonging to the classes $\overset{\Lambda_1}{\circ}\!\equiv\!\overset{0}{\bullet}$ and $\overset{0}{\circ}\!\equiv\!\overset{\Lambda_2}{\bullet}$ *this basis is not on orthonormal one.* Vectors of this type, with the same $a+b$ and c belong to the same weight of D and some orthogonalization technique is needed in order to extract from them an equipotent set of orthogonal vectors. Such a procedure involves of course a large amount of arbitrariness; a more systematic way consists of constructing an orthonormal basis compatible with the "path-labels," i.e., constructing a basis such that

$$\bar{E}^{a-c}_{\alpha_1+\alpha_2} \phi\begin{pmatrix} ab \\ c \end{pmatrix} \neq 0, \quad \bar{E}^{a-c+1}_{\alpha_1+\alpha_2} \phi\begin{pmatrix} ab \\ c \end{pmatrix} = 0.$$

[From this follows $\bar{E}_{\alpha_1+2\alpha_2} \phi(^{ab}_{a}) \sim \phi(^{a+1}{}_{a+1}{}^{b})$ and $\bar{E}_{-\alpha_2} \phi(^{ab}_{a}) \sim \phi(^{a}{}_{a}{}^{b-1})$.]

An algorithm can be developed consisting of the systematic calculation of the basis vectors defined by "path-labels," starting from $\phi(^{\Lambda_1+\Lambda_2}{}_{\Lambda_1+\Lambda_2}{}^{\Lambda_2})$. Although a little easier to handle than the algorithms mentioned in the Introduction, it leads to very messy computations. For any given representation however this algorithm can be carried out by computer. We limit ourself here to the tractable cases $\overset{\Lambda_1}{\circ}\!\equiv\!\overset{0}{\bullet}$ and $\overset{0}{\circ}\!\equiv\!\overset{\Lambda_2}{\bullet}$ for which the basis vectors are of the form (4.3).

5. THE REPRESENTATIONS $\overset{\Lambda_1\ 0}{\circ\!\!\equiv\!\!\bullet}$ AND $\overset{0\ \Lambda_2}{\circ\!\!\equiv\!\!\bullet}$

A. $\overset{\Lambda_1\ 0}{\circ\!\!\equiv\!\!\bullet}$

An orthonormal basis of Z

$$\left\{\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} \equiv \psi\begin{bmatrix} \Lambda_1 & & 0 & & 0 & \Lambda_1+c & & a & & 0\\ & a & & 0 & & & \Lambda_1+c & & a & \\ & & c & & & & & \Lambda_1+c & & \end{bmatrix}\right\}$$

is given by

$$\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} = \left(\prod_{i=1}^{a-c}\frac{i(2\Lambda_1-i+3)(2\Lambda_1+a-i+4)(a-i+1)}{3(\Lambda_1-a-i+3)(\Lambda_1-i+2)}\prod_{j=1}^{\Lambda_1-a}\frac{j(3\Lambda_1-j+4)(\Lambda_1-j+1)}{3(\Lambda_1-j+2)}\right)^{-1/2}\bar{E}^{a-c}_{-\alpha_1-\alpha_2}\bar{E}^{\Lambda_1-a}_{-\alpha_1-2\alpha_2}\phi\begin{bmatrix} \Lambda_1 & & 0\\ & \Lambda_1 & \end{bmatrix}.$$

The orthogonality is obvious since any two basis vectors belong to different weights. The normalization factor is obtained from the two equalities

$$\bar{E}_{\alpha_1+2\alpha_2}\bar{E}^{l}_{-\alpha_1-2\alpha_2}\phi\begin{bmatrix} \Lambda_1 & & 0\\ & \Lambda_1 & \end{bmatrix} = \frac{l(3\Lambda_1-l+4)(\Lambda_1-l+1)}{3(\Lambda_1-l+2)}\bar{E}^{l-1}_{-\alpha_1-2\alpha_2}\phi\begin{bmatrix} \Lambda_1 & & 0\\ & \Lambda_1 & \end{bmatrix},$$

$$\bar{E}_{\alpha_1+\alpha_2}\bar{E}^{l}_{-\alpha_1-\alpha_2}\phi\begin{bmatrix} \Lambda_1 & & 0\\ & \Lambda_1 & \end{bmatrix} = \frac{l(3\Lambda_1-l+3)(2\Lambda_1+a-l+4)(a-l+1)}{3(\Lambda_1+a-l+3)(\Lambda_1-l+2)}\bar{E}^{l-1}_{-\alpha_1-\alpha_2}\begin{bmatrix} \Lambda_1 & & 0\\ & \Lambda_1 & \end{bmatrix},$$

which follow from the generalized commutation relations of Appendix A. [Recall $\bar{E}_{\alpha_2}\phi\left(\begin{smallmatrix}\Lambda_1 & & 0\\ & \Lambda_1 & \end{smallmatrix}\right) = \bar{E}_{\alpha_1+2\alpha_2}\phi\left(\begin{smallmatrix}\Lambda_1 & & 0\\ & \Lambda_1 & \end{smallmatrix}\right) = \bar{E}_{\alpha_1+\alpha_2}\phi\left(\begin{smallmatrix}\Lambda_1 & & 0\\ & \Lambda_1 & \end{smallmatrix}\right) = 0$ and $\bar{E}_{\alpha_1+\alpha_2}\phi\left(\begin{smallmatrix}a & & 0\\ & a & \end{smallmatrix}\right) = 0$.]

Thanks to these same relations, we deduce the action of the operators $\bar{E}_\rho$ on these basis vectors,

$$\begin{aligned}
\bar{E}_{\alpha_1+\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= \left[\frac{(a-c)(c+1)(2\Lambda_1-a+c+3)(2\Lambda_1+c+4)}{3(\Lambda_1+c+3)(\Lambda_1-a+c+2)}\right]^{1/2}\phi\begin{bmatrix} a & & 0\\ & c+1 & \end{bmatrix},\\
\bar{E}_{-\alpha_1-\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= \left[\frac{(a-c+1)c(2\Lambda_1-a+c+2)(2\Lambda_1+c+3)}{3(\Lambda_1+c+2)(\Lambda_1-a+c+1)}\right]^{1/2}\phi\begin{bmatrix} a & & 0\\ & c-1 & \end{bmatrix},\\
\bar{E}_{\alpha_1+2\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_1-a)(c+1)(\Lambda_1+a+3)(2\Lambda_1+c+4)}{3(\Lambda_1+c+3)(a+2)}\right]^{1/2}\phi\begin{bmatrix} a+1 & & 0\\ & c+1 & \end{bmatrix},\\
\bar{E}_{-\alpha_1-2\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_1-a+1)c(\Lambda_1+a+2)(2\Lambda_1+c+3)}{3(\Lambda_1+c+2)(a+2)}\right]^{1/2}\phi\begin{bmatrix} a-1 & & 0\\ & c-1 & \end{bmatrix},\\
\bar{E}_{\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= -\left[\frac{(\Lambda_1-a)(a-c+1)(\Lambda_1+a+3)(2\Lambda_1-a+c+2)}{3(\Lambda_1-a+c+1)(a+2)}\right]^{1/2}\phi\begin{bmatrix} a+1 & & 0\\ & c & \end{bmatrix},\\
\bar{E}_{-\alpha_2}\phi\begin{bmatrix} a & & 0\\ & c & \end{bmatrix} &= -\left[\frac{(\Lambda_1-a+1)(a-c)(\Lambda_1+a+2)(2\Lambda_1-a+c+3)}{3(\Lambda_1-a+c+2)(a+1)}\right]^{1/2}\phi\begin{bmatrix} a-1 & & 0\\ & c & \end{bmatrix}.
\end{aligned}\tag{5.1}$$

B. $\overset{0\ \Lambda_2}{\circ\!\!\equiv\!\!\bullet}$

An orthonormal basis of Z,

$$\left\{\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} \equiv \psi\begin{bmatrix} \Lambda_2 & & \Lambda_2 & & 0 & c & & b & & 0\\ & \Lambda_2 & & b & & & c & & b & \\ & & c & & & & & c & & \end{bmatrix}\right\},$$

is given by

$$\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} = \left(\prod_{i=1}^{\Lambda_2-c}\frac{i(2\Lambda_2-i+5)(\Lambda_2-i-b+1)}{3(\Lambda_2-i+3)}\prod_{j=1}^{\Lambda_2-b}\frac{j(\Lambda_2-j+1)}{3}\right)^{-1/2}\bar{E}^{\Lambda_2-c}_{-\alpha_1-\alpha_2}\bar{E}^{\Lambda_2-b}_{-\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & \Lambda_2\\ & \Lambda_2 & \end{bmatrix}$$

and the action of the generators $\bar{E}_\rho$ on these vectors is furnished by the formulas

$$\begin{aligned}
\bar{E}_{\alpha_1+\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_2-c)(c-b+1)(\Lambda_2+c+5)}{3(c+3)}\right]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c+1 & \end{bmatrix},\\
\bar{E}_{-\alpha_1-\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_2-c+1)(c-b)(\Lambda_2+c+4)}{3(c+2)}\right]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c-1 & \end{bmatrix},\\
\bar{E}_{\alpha_1+2\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_2-c)(b+1)(\Lambda_2+c+5)}{3(c+3)}\right]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b+1\\ & c+1 & \end{bmatrix},\\
\bar{E}_{-\alpha_1-2\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} &= \left[\frac{(\Lambda_2-c+1)b(\Lambda_2+c+4)}{3(c+2)}\right]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b-1\\ & c-1 & \end{bmatrix},\\
\bar{E}_{\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} &= [\tfrac{1}{3}(c-b)(b+1)]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b+1\\ & c & \end{bmatrix}, \quad \bar{E}_{-\alpha_2}\phi\begin{bmatrix} \Lambda_2 & & b\\ & c & \end{bmatrix} = [\tfrac{1}{3}(c-b+1)b]^{1/2}\phi\begin{bmatrix} \Lambda_2 & & b-1\\ & c & \end{bmatrix}.
\end{aligned}\tag{5.2}$$

6. MATRIX ELEMENTS OF THE GENERATORS E_ρ

Since, by (3.2), the matrix elements of the generators of the subalgebra A_2 are known, it is easy to ensure oneself that the matrix elements of all generators of G_2 will be calculable (in making use of the commutation relations in G_2) as soon as those of two other generators, for example $E_{\alpha_1+2\alpha_2}$ and $E_{-\alpha_1-2\alpha_2}$, are known. Actually, thanks to the condition $E_\mu^\dagger = E_{-\mu}$, it is sufficient to know only one of the two.

We show here that the matrix elements of $E_{\alpha_1+2\alpha_2}$ are expressed very simply with the help of the coefficients $A_i^{(\pm\mu)}(a,b,c)$ appearing in formulas (4.3).

After some computations, using the Asherova–Smirnov operator, the commutation relations of G_2 and the definitions (4.1) of the $\bar{E}_\mu$, it can be established that

$$E_{\alpha_1+2\alpha_2} S\begin{bmatrix} a+b, & & e \\ d, & e, & f \end{bmatrix} = \left(\frac{(d+2)(e+1)}{(a+b-\Lambda_2+1)(\Lambda_1+c+2)}\right)^{1/2} S\begin{bmatrix} a+b+1, & & c+1 \\ d+1, & e+1, & f+1 \end{bmatrix} \bar{E}_{\alpha_1+2\alpha_2}$$
$$-\left(\frac{(a+b-\Lambda_2-e)(d-a-b+\Lambda_2+1)}{(\Lambda_1+\Lambda_2-a-b+c+1)(a+b-\Lambda_2)}\right)^{1/2} S\begin{bmatrix} a+b-1, & & c \\ d, & e, & f \end{bmatrix} \bar{E}_{-\alpha_2}$$
$$-\left(\frac{(\Lambda_1+c-d)(\Lambda_1+c-e+1)}{(\Lambda_1+\Lambda_2-a-b+c)(\Lambda_1+c+1)}\right)^{1/2} S\begin{bmatrix} a+b, & & c+1 \\ d, & e, & f \end{bmatrix} \bar{E}_{-\alpha_1-\alpha_2}.$$

Acting with the two sides of this expression on the vector

$$\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix}$$

we obtain [taking into account the formulas (4.3)]:

$$E_{\alpha_1+2\alpha_2}\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2 & & 0 \\ & a & & b & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix}$$
$$= \sum_{i\in I}\left(\frac{(d+2)(e+1)}{(a+b-\Lambda_2+1)(\Lambda_1+c+2)}\right)^{1/2} A_i^{(\alpha_1+2\alpha_2)}(a,b,c)\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c+1 & & a+b-\Lambda_2+1 & & 0 \\ & a+1+i & & b-i & & & d+1 & & e+1 & \\ & & c+1 & & & & & f+1 & & \end{bmatrix}$$
$$- \sum_{j\in J}\left(\frac{(a+b-\Lambda_2-e)(d-a-b+\Lambda_2+1)}{(\Lambda_1+\Lambda_2-a-b+c+1)(a+b-\Lambda_2)}\right)^{1/2} A_j^{(-\alpha_2)}(a,b,c)\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c & & a+b-\Lambda_2-1 & & 0 \\ & a-1+j & & b-j & & & d & & e & \\ & & c & & & & & f & & \end{bmatrix}$$
$$- \sum_{k\in K}\left(\frac{(\Lambda_1+c-d)(\Lambda_1+c-e+1)}{(\Lambda_1+\Lambda_2-a-b+c)(\Lambda_1+c+1)}\right)^{1/2} A_k^{(-\alpha_1-\alpha_2)}(a,b,c)\psi\begin{bmatrix} \Lambda_1+\Lambda_2 & & \Lambda_2 & & 0 & \Lambda_1+c+1 & & a+b-\Lambda_2 & & 0 \\ & a+k & & b-k & & & d & & e & \\ & & c-1 & & & & & f & & \end{bmatrix}$$

The representations belonging to the classes $\overset{\Lambda_1}{\bigcirc}\!\equiv\!\overset{0}{\bigcirc}$ and $\overset{0}{\bigcirc}\!\equiv\!\overset{\Lambda_2}{\bigcirc}$ are thus completely constructed since the coefficients $A_i^{(\mu)}(a,b,c)$ are given by the formulas (5.1) and (5.2).

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APPENDIX A: GENERALIZED COMMUTATION RELATIONS

w_M is any weight vector of Z:

$$M_1 = (M, \alpha_1), \quad M_2 = (M, \alpha_1 + 3\alpha_2),$$

$$\bar{E}_{\alpha_1+2\alpha_2}\bar{E}^l_{-\alpha_2}w_M = \frac{M_2+2}{M_2-l+2}\,\bar{E}^l_{-\alpha_2}\bar{E}_{\alpha_1+2\alpha_2}w_M - \frac{1}{\sqrt{3}}\,\frac{l(2M_2-l+3)}{M_2-l+2}\,\bar{E}^{l-1}_{-\alpha_2}\bar{E}_{\alpha_1+\alpha_2}w_M,$$

$$\bar{E}_{-\alpha_1-\alpha_2}\bar{E}^l_{-\alpha_2}w_M = \frac{M_1+1}{M_1+l+1}\,\bar{E}^l_{-\alpha_2}\bar{E}_{-\alpha_1-\alpha_2}w_M + \frac{1}{\sqrt{3}}\,\frac{l(2M_1+l+1)}{M_1+l+1}\,\bar{E}^{l-1}_{-\alpha_2}\bar{E}_{-\alpha_1-2\alpha_2}w_M,$$

$$\bar{E}_{\alpha_2}\bar{E}^l_{-\alpha_1-2\alpha_2}w_M = \frac{M_2+2}{M_2-l+2}\,\bar{E}^l_{-\alpha_1-2\alpha_2}\bar{E}_{\alpha_2}w_M, - \frac{1}{\sqrt{3}}\,\frac{l(2M_2-l+3)}{M_2-l+2}\,\bar{E}^{l-1}_{-\alpha_1-2\alpha_2}\bar{E}_{-\alpha_1-\alpha_2}w_M,$$

$$\bar{E}_{\alpha_1+\alpha_2}\bar{E}^l_{-\alpha_1-2\alpha_2}w_M = \frac{M_1+M_2+3}{M_1+M_2-l+3}\,\bar{E}^l_{-\alpha_1-2\alpha_2}\bar{E}_{\alpha_1+\alpha_2}w_M + \frac{1}{\sqrt{3}}\,\frac{l(2M_1+2M_2-l+5)}{M_1+M_2-l+3}\,\bar{E}^{l-1}_{-\alpha_1-2\alpha_2}\bar{E}_{-\alpha_2}w_M,$$

$$\bar{E}_{-\alpha_2}\bar{E}^l_{-\alpha_1-\alpha_2}w_M = \frac{M_1+2}{M_1-l+2}\,\bar{E}^l_{-\alpha_1-\alpha_2}\bar{E}_{-\alpha_2}w_M - \frac{1}{\sqrt{3}}\,\frac{l(2M_1-l+3)}{M_1-l+2}\,\bar{E}^{l-1}_{-\alpha_1-\alpha_2}\bar{E}_{-\alpha_1-2\alpha_2}w_M,$$

$$\bar{E}_{\alpha_1+2\alpha_2}\bar{E}^l_{-\alpha_1-\alpha_2}w_M = \frac{M_1+M_2+3}{M_1+M_2-l+3}\,\bar{E}^l_{-\alpha_1-\alpha_2}\bar{E}_{\alpha_1+2\alpha_2}w_M + \frac{1}{\sqrt{3}}\,\frac{l(2M_1+2M_2-l+5)}{M_1+M_2-l+3}\,\bar{E}^{l-1}_{-\alpha_1-\alpha_2}\bar{E}_{\alpha_2}w_M,$$

$$\bar{E}_{\alpha_2}\bar{E}^l_{-\alpha_2}w_M = \bar{E}^l_{-\alpha_2}\bar{E}_{\alpha_2}w_M + \frac{l}{3}(M_2-M_1-l+1)\bar{E}^{l-1}_{-\alpha_2}w_M + \frac{l(M_1+2)}{(M_1+1)(M_1+l+1)}\,\bar{E}^{l-1}_{-\alpha_2}\bar{E}_{-\alpha_1-\alpha_2}\bar{E}_{\alpha_1+\alpha_2}w_M,$$

$$- \frac{l}{M_2 - l + 2} \bar{E}^{l-1}_{-\alpha_2} \bar{E}_{-\alpha_1 - 2\alpha_2} \bar{E}_{\alpha_1 + 2\alpha_2} w_M$$

$$+ \frac{1}{\sqrt{3}} \frac{l(l-1)(M_1 + M_2 + 3)}{(M_1 + l + 1)(M_2 - l + 2)}$$

$$\times \bar{E}^{l-2}_{-\alpha_2} \bar{E}_{-\alpha_1 - 2\alpha_2} \bar{E}_{-\alpha_1 - \alpha_2} w_M,$$

$$\bar{E}_{\alpha_1 + \alpha_2} \bar{E}^{l}_{-\alpha_1 - \alpha_2} w_M$$

$$= \bar{E}^{l}_{-\alpha_1 - \alpha_2} \bar{E}_{\alpha_1 + \alpha_2} w_M + \frac{l}{3} (2M_1 + M_2 - l + 1) \bar{E}^{l-1}_{-\alpha_1 - \alpha_2} w_M$$

$$- \frac{l}{M_1 - l + 2} \bar{E}^{l-1}_{-\alpha_1 - \alpha_2} \bar{E}_{\alpha_2} \bar{E}_{-\alpha_2} w_M - \frac{l(M_2 + 2)}{(M_2 + 1)(M_1 + M_2 - l + 3)}$$

$$\times \bar{E}^{l-1}_{-\alpha_1 - \alpha_2} \bar{E}_{-\alpha_1 - 2\alpha_2} \bar{E}_{\alpha_1 + 2\alpha_2} w_M + \frac{1}{\sqrt{3}} \frac{l(l-1)}{M_1 - l + 2}$$

$$\times \bar{E}^{l-2}_{-\alpha_1 - \alpha_2} \bar{E}_{\alpha_2} \bar{E}_{-\alpha_1 - 2\alpha_2} w_M - \frac{1}{\sqrt{3}} \frac{(M_2 + 2) l (l-1)}{(M_2 + 1)(M_1 + M_2 - l + 3)}$$

$$\times \bar{E}^{l-2}_{-\alpha_1 - \alpha_2} \bar{E}_{-\alpha_1 - 2\alpha_2} \bar{E}_{\alpha_2} w_M,$$

$$\bar{E}_{\alpha_1 + 2\alpha_2} \bar{E}^{l}_{-\alpha_1 - 2\alpha_2} w_M$$

$$= \bar{E}^{l}_{-\alpha_1 - 2\alpha_2} \bar{E}_{\alpha_1 + 2\alpha_2} w_M + \frac{l}{3} \left(M_1 + 2M_2 - l + 1 - \frac{2(l-1)}{M_1 + M_2 - l + 3} \right)$$

$$\times E^{l-1}_{-\alpha_1 - 2\alpha_2} w_M - \frac{l}{M_2 - l + 2} \bar{E}^{l-1}_{-\alpha_1 - 2\alpha_2} \bar{E}_{-\alpha_2} \bar{E}_{\alpha_2} w_M$$

$$- \frac{l(M_1 + 2)}{(M_1 + 1)(M_1 + M_2 - l + 3)} \bar{E}^{l-1}_{-\alpha_1 - 2\alpha_2} \bar{E}_{-\alpha_1 - \alpha_2} \bar{E}_{\alpha_1 + \alpha_2} w_M$$

$$+ \frac{1}{\sqrt{3}} \frac{(M_1 + 1) l (l-1)}{(M_1 + M_2 - l + 3)(M_2 - l + 2)}$$

$$\times \bar{E}^{l-2}_{-\alpha_1 - 2\alpha_2} \bar{E}_{-\alpha_2} \bar{E}_{-\alpha_1 - \alpha_2} w_M.$$

APPENDIX B: THE "PATH-LABELS"

Let $\tilde{A} \subset A$ be a complex semisimple subalgebra of a semisimple Lie algebra. If $\tilde{A} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{L}}_+ \oplus \tilde{\mathcal{L}}_-$ is a Cartan decomposition of $\tilde{A}$, we know[4] there always exists a Cartan decomposition $A = \mathcal{H} \oplus \mathcal{L}_+ \oplus \mathcal{L}_-$ with $\tilde{\mathcal{H}} \subseteq \mathcal{H}$, $\tilde{\mathcal{L}}_\pm \subset \mathcal{L}_\pm$. The Lie algebra A admits a direct sum decomposition (actually an orthogonal one with respect to the Killing form on A),

$$A = B \oplus \tilde{A},$$

with $[\tilde{A}, \tilde{A}] \subseteq \tilde{A}$, $[B, \tilde{A}] \subseteq B$.

In the subalgebra $\mathcal{L}_+ = B_+ \oplus \tilde{\mathcal{L}}_+$ ($B_+ = \mathcal{L}_+ \cap B$) there always exists a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ ($x_i \in B_+$, $y_j \in \tilde{\mathcal{L}}_+$) and an order relation ($\vdash$) on the set of subspaces $\{Cx_1, \ldots, Cx_n, Cy_1, \ldots, Cy_m\}$ such that $0 \vdash Cy_j \vdash Cx_i$ and $[Cx_i, Cx_j] \vdash \max\{Cx_i, Cx_j\}$.

For each irreducible and finite dimensional representation $D: A \to \mathrm{End}(V)$, we define the subspace

$$Z = \{v \in V : D(x)v = 0 \ \forall \ x \in \tilde{\mathcal{L}}_+\}.$$

For all $v \in Z$, $v \neq 0$, we associate the integers a_k defined by $[D(x)^0 = I]$

$$D(x_i)^{a_i} D(x_{i+1})^{a_{i+1}} \cdots D(x_n)^{a_n} v \neq 0,$$
$$D(x_i)^{a_i + 1} D(x_{i+1})^{a_{i+1}} \cdots D(x_n)^{a_n} v = 0. \qquad \text{(B1)}$$

This correspondance

$$v \mapsto (a_1, \ldots, a_n)$$

is obviously a function

$$Z - \{0\} \to N \times \cdots \times N.$$

Let S denote the range of this function.

The following proposition is a direct consequence of the definition of the numbers a_k and of the order relation $\vdash$.

Proposition 1: $\forall \ 0 \leq c \leq a_i$, $\forall \ c + 1 \leq j \leq n$,

1. $D(x_j) D(x_i)^c D(x_{i+1})^{a_{i+1}} \cdots D(x_n)^{a_n} v = 0$,
2. $D(x_i)^c D(x_{i+1})^{a_{i+1}} \cdots D(x_n)^{a_n} v \in Z$.

On $N \times \cdots \times N$ we consider the following order relation: $(a_1, \ldots, a_n) \geq (b_1, \ldots, b_n)$ if the first nonzero $a_i - b_i$ starting from $i = 1$ is nonnegative. From this, there exists an order preserving function $g: S \to \{1, 2, \ldots, \nu - 1, \nu\}$. [If $(a_1, \ldots, a_n) \geq (b_1, \ldots, b_n)$, then $g(a_1, \ldots, a_n) \geq g(b_1, \ldots, b_n)$.]

Now define the following subspaces of Z:

$$Z_{(i)} = \{v \in Z, D(x_1)^{a_1} \cdots D(x_n)^{a_n} v \in Z_{(1)}\},$$

where $(a_1, \ldots, a_n) \in S$, $i = g(a_1, \ldots, a_n)$ and

$$Z_{(1)} = \{v \in Z, \ D(x_i) v = 0, \ 1 \leq i \leq n\}.$$

($Z_{(1)}$ is the one-dimensional subspace of the maximal vectors of the representation D and $Z_{(\nu)} = Z$.)

It follows from the order relation on S that if $i \geq i'$, then $Z_{(i')} \subseteq Z_{(i)} \cdots$, Z admits therefore a filtration $Z_{(1)} \subseteq Z_{(2)} \subseteq \cdots \subseteq Z_{(\nu)}$; putting $Z_{(0)} = \{0\} \subset Z_{(1)}$, we have

Proposition 2: $\dim(Z_{(i)}/Z_{(i-1)}) = 1$, $1 \leq i \leq \nu$.

Proof: By definition of $(a_1, \ldots, a_n)$ there always exist vectors $v \in Z_{(i)}$ and $v \notin Z_{(i-1)}$ ($i \neq 0$). If no two such linearly independent vectors exist, the proposition is proved. Suppose then that v_1 and v_2 are linearly independent and $v_1, v_2 \in Z_{(i)}$ but $v_1, v_2 \notin Z_{(i-1)}$. We have

$$D(x_1)^{a_1} \cdots D(x_n)^{a_n} v_1 \neq 0 \quad (\in Z_{(1)}),$$

$$D(x_1)^{a_1} \cdots D(x_n)^{a_n} v_2 \neq 0 \quad (\in Z_{(1)}).$$

Since $\dim Z_{(1)} = 1$, there must exist a number $\lambda \neq 0$ such that

$$D(x_1)^{a_1} \cdots D(x_n)^{a_n} v_1 = \lambda D(x_1)^{a_1} \cdots D(x_n)^{a_n} v_2.$$

Hence

$$D(x_1)^{a_1} \cdots D(x_n)^{a_n} (v_1 - \lambda v_2) = 0,$$

which means $v_1 - \lambda v_2 \in Z_{(i-1)}$ and consequently $Z_{(i)} = Cv_1 \oplus Z_{(i-1)}$.

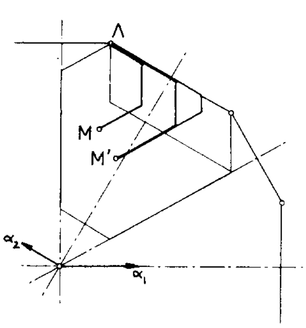

FIG. 3. Typical "path-labels."

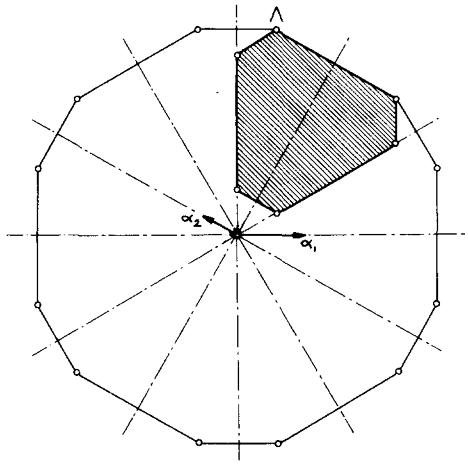

FIG. 4. The "A_2 like hexagonal weights diagram" of the maximal weights of $\tilde{D}$.

Corollary:

1. $\nu = \dim Z$,

2. Z is freely generated by S and consequently the elements $(a_1, \ldots, a_n)$ constitute labels for a basis of Z.

Definition (B1) suggests the terminology used: The vector v is sent onto $Z_{(1)}$ by successive applications of the operators $D(x_i)$ $(1 \leq i \leq n)$, performing a particular "path" in Z. In cases when $\tilde{A}$ is a regular subalgebra, such a path can be visualized in the weight diagram of A (Fig. 3).

This method, conjugated if necessary with the character one (as in our theorem), gives a complete answer to the labeling problem in many given cases and furnishes an effective algorithm allowing a constructing of a basis on the space of the representations.

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Nonlinear response of equilibrium strongly coupled Fermi fluids. I. Formal development

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This is Paper I of a series of three papers in which a self-consistent propagator resummation of self-energy effects in a strongly coupled Fermi fluid, in the presence of an external magnetic field, is performed. In the present paper, an exact expression is obtained for the grand potential in the presence of an external magnetic field which has a constant and a spatially varying part. The grand partition function and grand potential are written in terms of antisymmetrized cluster expansions. The cluster functions are then expanded in terms of a binary expansion. And, finally, an expression for the grand potential is obtained in terms of the reaction matrix for two-body scattering and in a form suitable for the subsequent propagator resummation.

I. INTRODUCTION

In the study of real fluids, both classical and quantum, one is confronted with an interaction potential between particles, of the Lennard–Jones type, in which there is a very strong repulsive core and a short range attractive region which may admit bound states. Conventional perturbation theory is not able to cope with either of these possibilities.

As is well known, the first successful treatment of such systems was due to work by Ursell[1] and Mayer[2,3] on the cluster expansion of thermodynamic properties of classical systems. They were able to define the clusters in terms of an effective interaction which remains well behaved even for an infinite hard core. The method was extended to quantum systems with Boltzmann statistics by Kahn and Uhlenbeck,[4] and to quantum systems with Bose–Einstein and Fermi–Dirac statistics by Lee and Yang.[5] Lee and Yang were able to perform a partial resummation of the quantum cluster expansion and to discuss in some detail the properties of dilute hard sphere Bose gases. The work begun by Lee and Yang was continued by Mohling,[6] who performed a self-consistent resummation of self-energy effects in the quantum cluster expansions through introduction of his Λ transformation. Tuttle[7] has since shown that the Λ transformation is equivalent to a generalized Hartree–Fock transformation.

In general, when one performs a quantum perturbation or binary expansion of equilibrium or nonequilibrium quantities, the expansions are found to contain self-energy structures. In diagrammatic language, these are parts of a diagram which can be removed by cutting two lines of the same momentum and energy. They appear if momentum and energy is conserved during an interaction or sequence of interactions. One is usually interested in two limiting regions: First, the thermodynamic limit (particle number and volume become infinite in such a way that density remains constant) because this allows one to neglect spurious terms in (N^{-1}), where N is the particle number; and, second, the zero temperature limit for equilibrium quantities; or the long time limit for nonequilibrium quantities.

In the thermodynamic limit each term (diagram) in the expansion of a given equilibrium or nonequilibrium quantity remains well behaved, even if it contains a self-energy structure. However, each term (diagram) which contains self-energy structures will have a "secular" (polynomial) dependence on inverse temperature (for an equilibrium quantity) or on time (for a nonequilibrium quantity). These secular terms must be resummed at low temperatures or after a long time, or the expansions can become unbounded.

The Hartree–Fock type of resummation is just one of several methods which exist for resumming self-energy effects in quantum expansions. It has been applied to both equilibrium[8] and nonequilibrium[9,10] perturbation expansions. Mohling's work is the first in which it has been applied systematically to a binary expansion. However, the Hartree–Fock method suffers from regularization problems.[11] In the Hartree–Fock procedure, one removes the secular terms by breaking apart diagrams with self-energy structures and explicitly removing that part of a diagram which has the secular dependence, and placing it in an exponential. As a result, one is left with two pieces, neither of which is well behaved in the thermodynamic limit, because of uncompensated energy denominators which can go to zero.

There is, however, another method for removing self-energy structures which does not suffer from the above difficulties. This is the so-called propagator or Dyson renormalization of quantum field theory.[12] Propagator renormalization is accomplished by adding together all terms with same basic topological structure, but different numbers and types of self-energy structure. This leads to a single composite term with resummed propagators. One never has to break apart a diagram, and therefore is always working with well behaved quantities.

Part of the purpose of this and two subsequent papers is to explore the possibility of performing a propagator resummation on the quantum binary expansion for equilibrium systems. As we shall see, this is not completely straightforward, because the binary expansion lacks much of the symmetry that the perturbation expansion has. As a result the usual method of resummation (the Matsubara method) does not apply. However, as we shall show in a subsequent paper, it *is* possible to

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introduce a self-consistent propagator resummation of self-energy effects for the quantum binary expansion of an equilibrium system.

In view of the recent discovery of superfluid phases[13,14] in liquid He^3, such a study is certainly of interest. The hard core of the He^3 atom plays a very important role in determining which relative angular momentum states will allow the formation of bound pairs in the liquid. Therefore, one would like to have a theory which can treat the hard core, and the attractive region of the He^3 potential, in a consistent manner. At present, one of the major tools for studying the phenomenology of liquid He^3 is perturbation theory, but one must assume that the potential used is some sort of effective potential, and that objects described by the perturbation expansion are quasiparticles. Microscopic information about the scattering of quasiparticles is obtained from studying the T matrix or reaction matrix for scattering of He^3 atoms.

In this and subsequent papers, we will perhaps be able to obtain a clearer understanding as to why perturbation theories work as well as they do. Indeed we will find that the most coherent part of the binary expansion corresponds to precisely those terms we need to describe the phenomenology of liquid He^3 (spin and density fluctuations) and that all quantities that appear in the expansion can be directly calculated in terms of the He^3 potential and are well behaved.

However, as the title indicates we are not solely interested in obtaining a mathematically well behaved microscopic theory of liquid He^3. We are also interested in studying the way in which an external applied field affects the thermodynamic properties of a hard sphere quantum fluid. Normally, when studying the response of such systems, one assumes that the external applied field is small and, therefore, that one can neglect all nonlinear terms in the external field relative to the linear term. However, as well shall see, one must be careful, especially when going past the linear regime. In equilibrium systems the nonlinear terms can be accompanied by secular dependence on the inverse temperature. Therefore, in the low temperature limit the usual argument that nonlinear terms can be neglected relative to the linear term must be modified.

In the present paper, we will be concerned primarily with the derivation of an exact expression for the grand potential of a strongly coupled Fermi fluid in the presence of an external magnetic field. Our expression will be written in terms of the reaction matrix for two body scattering in the fluid, and will be written in a form suitable for our discussion, in subsequent papers, of the magnetic response and the resummation of self-energy effects.

We begin our discussion in Sec. II by deriving a cluster expansion for the grand partition function of a Fermi fluid in the presence of an external magnetic field, and we define the resulting cluster operators in terms of a binary expansion. In Sec. III, we write the grand potential in terms of symmetrized cluster functions and introduce a diagrammatic expansion in terms of "dynamical" cluster functions. In Sec. IV, we write the dynamical cluster functions in terms of a binary expansion; and finally in Sec. V, we write the grand potential in terms of a binary expansion.

The binary expansion differs from perturbation expansions in that each binary operator depends on two temperatures rather than just one temperature as would be the case for a perturbation expansion. Furthermore, certain repeated interactions between pairs of particles are forbidden. In Sec. VI, we write an explicit expression for matrix elements of the binary operator in terms of reaction matrices. We then find that matrix elements of the binary operator break into two parts, one part which looks very much like matrix elements appearing in perturbation expansions but is defined in terms of a reaction matrix, and another part, also defined in terms of reaction matrices but with no counterpart in perturbation theory. In Sec. VII, we write a diagrammatic expansion for the grand potential explicitly in terms of reaction matrices and discuss some interesting features of this expansion. Finally in Sec. VIII, we make some concluding remarks.

II. CLUSTER EXPANSION OF THE GRAND PARTITION FUNCTION

Let us consider a system of spin $\frac{1}{2}$ fermions, with magnetic moment μ, which interact with one another via a spherically symmetric potential $V(|\mathbf{r}_{ij}|)$ ($\mathbf{r}_{ij}$ is the relative position between particles i and j). We shall assume that $V(|\mathbf{r}_{ij}|)$ is short ranged, with a large repulsive core for small values of $|\mathbf{r}_{ij}|$ and a weak attractive region for larger values of $|\mathbf{r}_{ij}|$.

We shall apply an external magnetic field to this system, of the form

$$\mathcal{H}(\mathbf{r}) = (\mathcal{H}_0 + \mathcal{H}_r \cos(\mathbf{k}_0 \cdot \mathbf{r}))\hat{z}. \qquad \text{(II. 1)}$$

The external field is directed in the z direction, but consists of two parts; one part, $\mathcal{H}_0$ is constant and the other part, $\mathcal{H}_r \cos(\mathbf{k}_0 \cdot \mathbf{r})$, oscillates in space. We shall always assume that $\mathbf{k}_0 \neq 0$.

The grand partition function for this system is

$$\mathcal{Z}(\beta, g, \mathcal{H}_0, \mathcal{H}_r) = \sum_{N=0}^{\infty} \mathrm{Tr}_N \exp[-\beta(H^{N'} + \Delta H^N), \qquad \text{(II. 2)}$$

where

$$H^{N'} \equiv H_0^N - gN - M_0^N \mathcal{H}_0 + V^N \qquad \text{(II. 3)}$$

and

$$\Delta H^N = -\int d\mathbf{r} M^N(\mathbf{r}) \mathcal{H}_r \cos(\mathbf{k}_0 \cdot \mathbf{r}). \qquad \text{(II. 4)}$$

In Eq. (II. 2), Tr_N denotes trace with respect to a complete set of N-body states; and $\beta = (k_B T)^{-1}$, where k_B is Boltzmann's constant and T is the absolute temperature. In Eq. (II. 3), H_0^N is the kinetic energy of the system

$$H_0^N = \sum_{i=1}^{N} \frac{k_i^2}{2m}. \qquad \text{(II. 5)}$$

We have set Planck's constant equal to 1, g is the chemical potential, and V^N is the interaction potential energy

$$V^N = \sum_{\mu=1}^{N(N-1)/2} V_\mu. \qquad \text{(II. 6)}$$

[μ denotes a particular pair of particles and $V_\mu \equiv V(|\mathbf{r}_\mu|)$.] M_0^N is the space independent magnetization operator in the z direction which we shall write

$$M_0^N = \mu \sum_{i=1}^{N} s_i. \tag{II.7}$$

In Eq. (II.7), $s_i = \pm \frac{1}{2}$ denotes eigenvalues of the z component of spin for the ith particle. The spin of the ith particle is $\mathbf{s}_i = \frac{1}{2}\boldsymbol{\sigma}_i$, where $\boldsymbol{\sigma}_i$ is the Pauli spin matrix. In Eq. (II.4), $M^N(\mathbf{r})$ is the space dependent magnetization operator

$$M^N(\mathbf{r}) = \mu \sum_{i=1}^{N} s_i(|\mathbf{r}\rangle\langle\mathbf{r}|)_i, \tag{II.8}$$

where $(|\mathbf{r}\rangle\langle\mathbf{r}|)_i$ operates on the state of particle i only. We may now write down the following expansion for the operator $\exp[-\beta(H^{N'}+\Delta H^N)]$ which appears in the grand partition function

$$\begin{aligned}\exp[-\beta(H^{N'}+\Delta H^N)&\\ =\exp(-\beta H_0^{N'})W^N(\beta,0)&\\ +\sum_{n=1}^{\infty}(-1)^n\int_0^\beta d\lambda_1\int_0^{\lambda_1}d\lambda_2\cdots\int_0^{\lambda_{n-1}}d\lambda_n\exp(-\beta H_0^{N'})&\\ \times W^N(\beta,\lambda_1)\Delta H^N(\lambda_1)W^N(\lambda_1,\lambda_2)\times\cdots\times\Delta H^N(\lambda_n)W^N(\lambda_n,0),&\end{aligned} \tag{II.9}$$

where

$$\Delta H^N(\lambda) = \exp(\lambda H_0^{N'})\Delta H^N \exp(-\lambda H_0^{N'}) \tag{II.10}$$

and

$$H_0^{N'} = H_0^N - gN - M_0^N \mathcal{H}_0. \tag{II.11}$$

The operator $W^N(\lambda,\lambda_0)$ which appears in Eq. (II.9) is defined

$$W^N(\lambda,\lambda_0) = \exp(\lambda H_0^{N'})\exp[-(\lambda-\lambda_0)H^N]\exp(-\lambda_0 H_0^{N'}) \tag{II.12}$$

and may be expanded in the following binary expansion:

$$W^N(\lambda,\lambda_0) = 1 + \sum_{\mu=1}^{N(N-1)/2}\int_{\lambda_0}^{\lambda} d\lambda_1 R_\mu(\lambda,\lambda_1)M_\mu^N(\lambda_1,\lambda_0), \tag{II.13}$$

where

$$M_\mu^N(\lambda_1,\lambda_0) = 1 + \sum_{\nu\neq\mu=1}^{N(N-1)/2}\int_{\lambda_0}^{\lambda_1} d\lambda_2 R_\nu(\lambda_1,\lambda_2)M_\nu^N(\lambda_2,\lambda_0). \tag{II.14}$$

The binary operator, $R_\mu(\lambda_1,\lambda_2)$, which appears in Eqs. (II.13) and (II.14) is defined

$$R_\mu(\lambda_1,\lambda_2) = -\frac{\partial}{\partial\lambda_2}W^{(\mu)}(\lambda_1\lambda_2) = -W^{(\mu)}(\lambda_1,\lambda_2)V_\mu(\lambda_2). \tag{II.15}$$

The definition of $W^\mu(\lambda_1,\lambda_2)$ is the same as that of $W^N(\lambda_1,\lambda_2)$ except that all N-body operators are replaced by operators for the pair of particles μ. For example, if μ denotes the pair of particles 1 and 2, $\mu=(1,2)$,

$$\begin{aligned}W^{(1,2)}(\lambda_1\lambda_2) = \exp(\lambda_1 H_0^{(1,2)'})\exp[-(\lambda_1-\lambda_2)H^{(1,2)}]&\\ \times\exp(-\lambda_2 H_0^{(1,2)'}),&\end{aligned} \tag{II.16}$$

where

$$H_0^{(1,2)'} = \frac{k_1^2}{2m} + \frac{k_2^2}{2m} - 2g - \mu(s_1+s_2)\mathcal{H}_0 \tag{II.17}$$

and

$$H^{(1,2)} = H_0^{(1,2)'} + V(|\mathbf{r}_{12}|). \tag{II.18}$$

The quantity $V_\mu(\lambda)$ is defined

$$V_\mu(\lambda) = \exp(\lambda H_0^{(\mu)'})V_\mu\exp(-\lambda H_0^{(\mu)'}). \tag{II.19}$$

The binary operator $R_\mu(\lambda_1,\lambda_2)$ satisfies a very important relation. If we make use of the transitivity of $W_\mu(\lambda_1,\lambda_2)$, integrate Eq. (II.15) with respect to λ_2, and note that $W_\mu(\lambda,\lambda)=1$, we obtain the following identities:

$$\begin{aligned}R_\mu(\lambda_3,\lambda_1) &= -W_\mu(\lambda_3,\lambda_1)V_\mu(\lambda_1)\\ &= -W_\mu(\lambda_3,\lambda_2)W_\mu(\lambda_2,\lambda_1)V(\lambda_1)\\ &= W_\mu(\lambda_3,\lambda_2)R_\mu(\lambda_2,\lambda_1)\\ &= [1+\int_{\lambda_2}^{\lambda_3}R_\mu\, ds\, R_\mu(\lambda_3,s)]R(\lambda_2,\lambda_1).\end{aligned} \tag{II.20}$$

If we now substitute Eqs. (II.13) and (II.14) into Eq. (II.9), and use the definition of ΔH^N given in Eqs. (II.4) and (II.8), we obtain an expansion for the N-body operator $\exp[-\beta(H^{N'}+\Delta H^N)]$ in terms of the binary operator $R_\mu(\lambda_i,\lambda_j)$ and the single-body operator

$$\Delta H_i(\lambda) = \exp(\lambda H_0^{(i)'})\Delta H_i\exp(-\lambda H_0^{(i)'}), \tag{II.21}$$

where

$$\Delta H_i = -\int d\mathbf{r}\,\mu s_i(|\mathbf{r}\rangle\langle\mathbf{r}|)_i\mathcal{H}_r\cos(\mathbf{k}_0\cdot\mathbf{r}) \tag{II.22}$$

and

$$H_0^{(i)'} = k_i^2/2m - g - \mu s_i\mathcal{H}_0. \tag{II.23}$$

The temperature integrations in the expansion may be reordered so that all integrations range from 0 to the temperature immediately to the left.

We can write a compact expression for all terms in the expansion for $\exp[-\beta(H^{N'}+\Delta H^N)]$ by introducing the concept of a ladder diagram (the ladder diagram is a generalized version of the X diagrams used by Lee and Yang and by Mohling). It is convenient to remove first the factors $\exp(-\beta H_0^{N'})$ that appear in Eq. (II.9). We then obtain the following result:

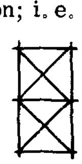
$$\begin{aligned}&\exp(\beta H_0^{N'})\exp[-\beta(H^{N'}+\Delta H^N)\\ &\quad=\sum_{Q=0}^{\infty}(\text{all different } Q\text{th order } N\text{-particle ladder diagrams}).\end{aligned} \tag{II.24}$$

To construct a *Qth order N-particle ladder diagram*, first draw N vertical lines and label them from left to right from 1 to N. At the top, draw a horizontal line labeled β, and below it draw Q horizontal lines labeled from λ_1 to λ_Q from top to bottom. X diagrams are completed by inserting X's in boxes and circles on lines according to the following rules:

(II. i) One and only one cross can occur between any two horizontal lines. Crosses must connect neighboring horizontal lines.

(II. ii) Only one circle can occur on any horizontal line. Circles are placed at the crossing of vertical and horizontal lines.

(II. iii) Two crosses cannot have two points in common; i.e., the structure



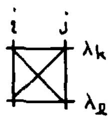
is forbidden.

(II. iv) The bottom points of a cross may not rest on the same horizontal line as a circle.

(II. v) If N_c is the number of circles, then a Qth *order N-particle ladder diagram* contains $N_x = Q - N_c$ crosses ($N_x \geq 0$, $N_c \geq 0$).

An algebraic expression may be associated with a *Qth order N-particle ladder diagram* according to the following rules:

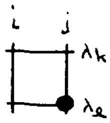
(II. vi) With each cross associate the factor



$$= R_{ij}(\lambda_k, \lambda_l),$$

where i and j are the particle labels of the vertical lines and λ_k and λ_l are the temperature labels of the horizontal lines.

(II. vii) With each circle, associate the factor



$$= -\Delta H_j(\lambda_l),$$

where i, j, λ_k, and λ_l have the same meaning as in rule (II. vi).

(II. vii) Order the algebraic expressions for the various crosses and circles from left to right in the same order that the crosses and circles appear when reading a diagram from top to bottom.

(II. ix) Integrate over each temperature from 0 to the next higher temperature (the temperature of the horizontal line immediately above).

Some examples of *four-particle ladder diagrams* are given in Fig. 1. Algebraic expressions for the various diagrams in Fig. 1 are:

$$\text{Fig. } 1(\text{a}) = 1, \tag{II. 25}$$

$$\text{Fig. } 1(\text{b}) = -\int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 \times R_{12}(\beta, \lambda_1)\Delta H_2(\lambda_2) R_{34}(\lambda_2, \lambda_3), \tag{II. 26}$$

$$\text{Fig. } 1(\text{c}) = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 \int_0^{\lambda_3} d\lambda_4 \times \Delta H_2(\lambda_1)\Delta H_2(\lambda_2)\Delta H_4(\lambda_3)\Delta H_2(\lambda_4), \tag{II. 27}$$

$$\text{Fig. } 1(\text{d}) = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 \int_0^{\lambda_3} d\lambda_4 \times R_{12}(\beta, \lambda_1) R_{23}(\lambda_1, \lambda_2) R_{12}(\lambda_2, \lambda_3) R_{34}(\lambda_3, \lambda_4). \tag{II. 28}$$

We notice that among the ladder diagrams defined above, there are both connected and unconnected Qth order *N-particle ladder diagrams*. A Qth order *connected N-particle ladder diagram* is one for which all N vertical lines remain interconnected by X's when the $(Q+1)$ horizontal lines are removed. In Fig. 1, Fig. 1(d) is the only example of a connected ladder diagram. By using Eq. (II. 20) all unconnected ladder diagrams may be summed to give products of connected ladder diagrams (cf. Appendix A for an example). The operator $\exp(\beta H_0^N)\exp[-\beta(H^{N'} + \Delta H^N)]$ may then be written in terms of cluster operators which are in turn defined in terms of *connected ladder diagrams*. We thus obtain the following result:

$$\exp(\beta H_0^{1'})\exp[-\beta(H^{1'} + \Delta H^1)] = U^1_{(1)}(\beta, 0), \tag{II. 29}$$

$$\exp(\beta H_0^{2'})\exp[-\beta(H^{2'} + \Delta H^2)] = U^1_{(1)}(\beta, 0)U^1_{(2)}(\beta, 0) + U^2_{(1,2)}(\beta, 0), \tag{II. 30}$$

$$\begin{aligned}\exp(\beta H_0^{3'})&\exp[-\beta(H^{3'} + \Delta H^3)] \\ &= U^1_{(1)}(\beta, 0)U^1_{(2)}(\beta, 0)U^1_{(3)}(\beta, 0) + U^1_{(1)}(\beta, 0)U^2_{(2,3)}(\beta, 0) \\ &\quad + U^1_{(2)}(\beta, 0)U^2_{(1,3)}(\beta, 0) + U^1_{(2)}(\beta, 0)U^2_{(1,2)}(\beta, 0) \\ &\quad + U^3_{(1,2,3)}(\beta, 0).\end{aligned} \tag{II. 31}$$

In Eqs. (II. 29)–(II. 31), $U^\ell_{(j_1,\ldots,j_l)}(\beta, 0)$ is an ℓ-particle cluster operator depending on the coordinates of the particular particles $j_1, \ldots, j_\ell$. We write the cluster expansion of $\exp(\beta, H_0^{N'})\exp[-\beta(H^{N'} + \Delta H^N)]$ schematically as

$$\begin{aligned}\exp(\beta H_0^{N'})&\exp[-\beta(H^{N'} + \Delta H^N)] \\ &= N! \sum_{\substack{\{\gamma_\ell\} \\ \sum \ell\gamma_\ell = N}} \left[1 \Big/ \prod_{\ell=1}^{N} (\ell!)^{\gamma_\ell}\gamma_\ell!\right][U^\ell(\beta, 0)]^{\gamma_\ell},\end{aligned} \tag{II. 32}$$

where γ_ℓ is the number of clusters containing ℓ-particles and the summation is performed over all possible combinations of ℓ and γ_ℓ which satisfy the condition $\sum \ell\gamma_\ell = N$.

The cluster operators all commute with one another and are defined in terms of Qth order *connected ladder diagrams* as follows:

$$U^\ell(\beta, 0) = \sum_{Q=\ell-1}^{\infty} (\text{all different } Q\text{th order connected } \ell\text{-particle ladder diagrams}). \tag{II. 33}$$

The *connected ℓ-particle ladder diagrams* are defined and evaluated according to Rules (II. i)–(II. ix). It is important to note that $U^\ell(\beta, 0)$ depends on an infinite number of ladder diagrams

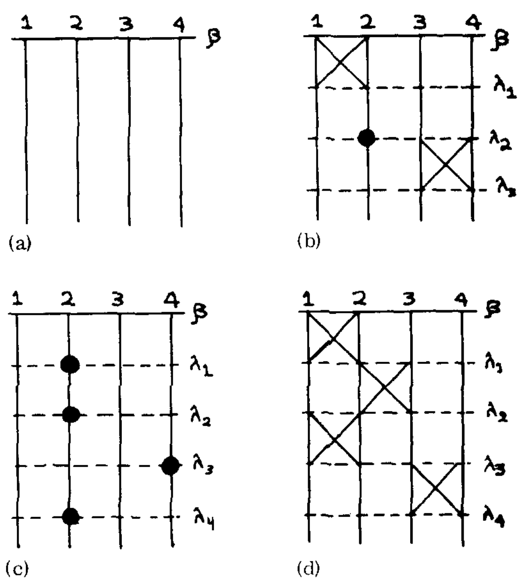


FIG. 1. Four-particle ladder diagrams.

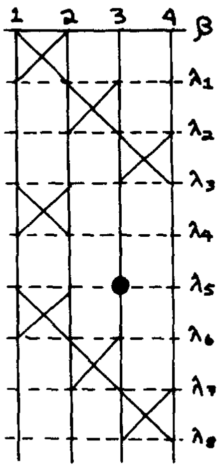
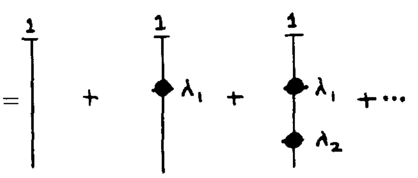


FIG. 2. Connected four-particle ladder diagram with unconnected sequence of crosses and circles between λ_2 and λ_6.

$U_1^1(\beta, 0) =$ 

(II. 34)

From the discussion in Appendix A, it is clear that by introducing the cluster operators, we have greatly simplified the expression for the grand partition function, because we have been able to sum infinite numbers of *unconnected ladder diagrams* into simple products of *connected ladder diagrams*. The *connected ladder diagrams* themselves can be simplified in a similar manner. Let us consider, for example, the *connected four-particle ladder diagram* in Fig. 2. The region between λ_2 and λ_6 corresponds to an unconnected internal sequence of crosses. It can be shown that all diagrams which have the same topological structure between β and λ_2 and between λ_6 and λ_8, but which have disjoint chains of crosses between λ_2 and λ_6, can be summed into a single contracted diagram. The mathematical procedure is similar to that involved in going from unconnected diagrams to products of cluster operators. As a result, we obtain the following definition of the cluster operators in terms of *contracted connected ladder diagrams*:

$$U^{\ell}(\beta, 0) = \sum \text{(all different connected contracted } \ell\text{-particle ladder diagrams).} \tag{II. 35}$$

Contracted ladder diagrams are constructed in the same way as *ladder diagrams* except that rule (II. i) is replaced by

(II. i′) One and only one cross can occur between any two horizontal lines. The upper end of each cross must either rest on the line $\lambda = \beta$ or must connect to a higher cross or circle.

An example of a *connected contracted five-particle diagram* is given in Fig. 3. The algebraic expression corresponding to the diagram in Fig. 3 is

$$\text{Fig. } 3 = \int_0^{\beta} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 \int_0^{\lambda_3} d\lambda_4 \int_0^{\lambda_4} d\lambda_5 \times R_{12}(\beta, \lambda_1) R_{45}(\beta, \lambda_2) \Delta H_4(\lambda_3) R_{23}(\lambda_2 \lambda_4) R_{34}(\lambda_4, \lambda_5). \tag{II. 36}$$

III. GRAND POTENTIAL IN TERMS OF CLUSTER ZERO DIAGRAMS

We now want to write an expression for the grand potential, $\Gamma(\beta, g, H_0, H_r)$. Once we do this we can, in principle, find all interesting thermodynamic properties of the system. We first must explicitly evaluate the trace in Eq. (II. 2). This will lead to an antisymmetrized cluster expansion for the grand partition function which we can then resume to obtain an antisymmetrized cluster expansion for the grand potential.

Since we are dealing with a system of identical fermions we must evaluate the trace with respect to a complete set of antisymmetrized states. We shall let the kets, $|k_1, \ldots, k_N\rangle^{(s)}$, denote a complete set of antisymmetrized momentum and spin eigenstates of $H_0^{N'}$ which can be written

$$|k_1, \ldots, k_N\rangle^{(s)} = \sum_p (\epsilon)^p |k_1, \ldots, k_N\rangle, \tag{III. 1}$$

where $\sum_p$ is the sum over all permutations of the quantities k_j, and $\epsilon = -1$ for Fermi–Dirac statistics (we keep the factor ϵ because it allows us to keep track of terms which result from exchange effects). Each particle in the system is assigned a definite position in a ket, the quantity k_j which occurs in the jth position gives the momentum and spin of the particle; i.e., $k_j = (\mathbf{k}_j, s_j)$. The ket $|k_1, \ldots, k_N\rangle^{(s)}$ is not normalized but must be multiplied by a factor $(N!)^{1/2}$.

The grand potential is related to the grand partition function according to the expression

$$Z(\beta, g, H_0, H_r) = \exp[-\beta \Gamma(\beta, g, H_0, H_r)]. \tag{III. 2}$$

In terms of antisymmetrized kets, the grand partition function can be written

$$\begin{aligned} Z(\beta, g, H_0, H_r) &= \sum_{N=0}^{\infty} \left(\frac{1}{N!}\right)^2 \sum_{k_1 \cdots k_N} {}^{(s)}\langle k_1 \cdots k_N | \exp[-\beta(H^{N'} + \Delta H^N) \\ &\quad \times |k_1 \cdots k_N\rangle^{(s)} \\ &= \sum_{N=0}^{\infty} \left(\frac{1}{N!}\right) \sum_{k_1 \cdots k_N} \exp\left(-\beta \sum_{j=1}^{N} \omega_j'\right) \langle k_1 \cdots k_N | \\ &\quad \times \exp(\beta H_0^{N'}) \exp[-\beta(H^{N'} + \Delta H^N) |k_1 \cdots k_N\rangle^{(s)}, \end{aligned} \tag{III. 3}$$

where

$$\omega_j = k_j^2/2m - g - \mu s_j H_0. \tag{III. 4}$$

In the second term of Eq. (III. 3), one factor $(N!)^{-1}$ comes from the normalization of the antisymmetrized

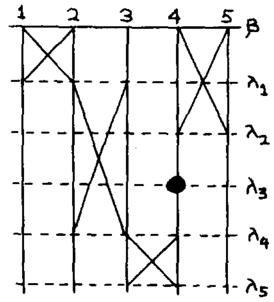


FIG. 3. Connected contracted five-particle ladder diagram.

kets. The other comes from the fact that the summation overcounts momentum states and must be corrected by a factor $(N!)^{-1}$.

The matrix elements appearing in Eq. (III. 3) can be written in terms of antisymmetrized cluster functions as follows:

$$\langle k_1 | \exp(\beta H_0^{1'}) \exp[-\beta(H^{1'} + \Delta H^1)] | k_1 \rangle = U_1\binom{k_1}{k_1}, \tag{III. 5}$$

$$\langle k_1 k_2 | \exp(\beta H_0^{2'}) \exp[-\beta(H^{2'} + \Delta H^2)] | k_1 k_2 \rangle^{(s)}$$
$$= U_1\binom{k_1}{k_1} U_1\binom{k_2}{k_2} + U_2^{(s)}\binom{k_1 k_2}{k_1 k_2}, \tag{III. 6}$$

$$\langle k_1 k_2 k_3 | \exp(\beta H_0^{3'}) \exp[-\beta(H^{3'} + \Delta H^3)] | k_1 k_2 k_3 \rangle^{(s)}$$
$$= U_1\binom{k_1}{k_1} U_1\binom{k_2}{k_2} U_1\binom{k_3}{k_3} + U_1\binom{k_1}{k_1} U_2^{(s)}\binom{k_2 k_3}{k_2 k_3}$$
$$+ U_1\binom{k_2}{k_2} U_2^{(s)}\binom{k_1 k_3}{k_1 k_3} + U_1\binom{k_3}{k_3} U_2^{(s)}\binom{k_1 k_2}{k_1 k_2}$$
$$+ U_3^{(s)}\binom{k_1 k_2 k_3}{k_1 k_2 k_3}, \tag{III. 7}$$

etc., where

$$U_1\binom{k_1}{k_2} = \delta_{k_1, k_2} + T_1\binom{k_1}{k_2}, \tag{III. 8}$$

$$U_2^{(s)}\binom{k_1 k_2}{k_1 k_2} = \epsilon U_1\binom{k_1}{k_2} U_1\binom{k_2}{k_1} + T_2\binom{k_1 k_2}{k_1 k_2}, \tag{III. 9}$$

$$U_3^{(s)}\binom{k_1 k_2 k_3}{k_1 k_2 k_3} = U_1\binom{k_1}{k_2} U_1\binom{k_2}{k_1} U\binom{k_3}{k_2} + U\binom{k_1}{k_2} U\binom{k_2}{k_3} U\binom{k_3}{k_1}$$
$$+ \epsilon U_1\binom{k_1}{k_2} T_2\binom{k_2 k_3}{k_1 k_3} + \epsilon U_1\binom{k_1}{k_3} T_2\binom{k_2 k_3}{k_2 k_1}$$
$$+ \epsilon U_1\binom{k_2}{k_1} T_2\binom{k_1 k_3}{k_2 k_3} + \epsilon U_1\binom{k_2}{k_3} T_2\binom{k_1 k_3}{k_1 k_2}$$
$$+ \epsilon U_1\binom{k_3}{k_1} T_2\binom{k_1 k_2}{k_3 k_2} + \epsilon U_1\binom{k_3}{k_2} T_2\binom{k_1 k_2}{k_1 k_3}$$
$$+ T_3\binom{k_1 k_2 k_3}{k_1 k_2 k_3}, \tag{III. 10}$$

etc., and where

$$T\binom{k_1 \cdots k_\ell}{k_1' \cdots k_\ell'} = \epsilon^\ell \langle k_1 \cdots k_\ell | U^\ell(\beta, 0) | k_1' \cdots k_\ell' \rangle^{(s)}. \tag{III. 11}$$

Equations (III. 5)—(III. 11) are quite easy to reproduce simply by taking antisymmetrized matrix elements of Eqs. (II. 29)—(II. 31). The cluster functions $U_\ell^{(s)}\binom{k_1 \cdots k_\ell}{k_1 \cdots k_\ell}$ contain the effects of both statistical and "dynamical" clustering. By introducing the functions $T_\ell\binom{k_1 \cdots k_\ell}{k_1 \cdots k_\ell}$ we have explicitly separated out "dynamical" clustering effects. When the interaction between particles is turned off, $T_\ell\binom{k_1 \cdots k_\ell}{k_1 \cdots k_\ell}$ is identically zero. However, we notice that even when the interaction between particles is zero, there still may be statistical clustering between particles.

Now that we have introduced an antisymmetrized cluster expansion for the grand partition function, we may very easily write an expression for $\Gamma(g, \beta, H_0, H_r)$ in terms of antisymmetrized cluster functions. The procedure is identical to that used for classical systems.[3] We obtain

$$\Gamma(g, \beta, H_0, H_r)$$
$$= -\frac{1}{\beta} \sum_{N=1}^{\infty} \sum_{k_1 \cdots k_N} \frac{1}{N!} \exp\left(-\beta \sum_{j=1}^{N} \omega_j'\right) U_N^{(s)}\binom{k_1 \cdots k_N}{k_1 \cdots k_N}. \tag{III. 12}$$

It is useful at this point to write down a diagrammatic expansion for $\Gamma(g, \beta, H_0, H_r)$ in terms of the "dynamical" cluster functions $T_\ell\binom{k_1 \cdots k_\ell}{k_1 \cdots k_\ell}$. We therefore introduce *cluster* 0 *diagrams* (these are also used by Mohling but called *primary* 0 *diagrams*).

In terms of *cluster* 0 *diagrams*, the grand potential is defined

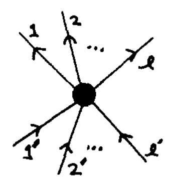
$$\Gamma(g, \beta, H_0, H_r)$$
$$= (-1/\beta) \sum \text{(all different cluster 0 diagrams).} \tag{III. 13}$$

A *cluster* 0 *diagram* is a collection of T_ℓ vertices entirely interconnected by directed lines. T_ℓ vertices have ℓ lines entering and ℓ lines leaving. *Cluster* 0 *diagrams* may contain no broken lines (external lines), and two *cluster* 0 *diagrams* differ if they have different topological structure.

An algebraic expression may be assigned to a *cluster* 0 *diagram* according to the following rules:

(III. i) Label each line from 1 to n, where n is the number of lines, and assign to the jth line a spin and momentum $k_j = (\mathbf{k}_j, s_j)$.

(III. ii) To each T_ℓ vertex, assign a factor



$$= T_\ell\binom{k_1 \cdots k_\ell}{k_1' \cdots k_\ell'}.$$

(III. iii) To each directed line, assign a factor

$$k_j = \epsilon \nu_1 = \sum_{n=1}^{\infty} [\epsilon \exp(-\beta \omega_1')]^n = \frac{\epsilon \exp(-\beta \omega_1')}{1 - \epsilon \exp(-\beta \omega_1')},$$

where $k_1 = (\mathbf{k}_1, s_1)$ is the momentum and spin of the line and $\omega_j' = k_j^2/2m - g - \mu s_j H_0$.

(III. iv) Assign to each diagram an overall factor $\epsilon^{P_B}(1/S)$, where P_B is the number of permutations of bottom line momenta with respect to top line momenta in the product of the matrix elements $T_\ell\binom{k_1 \cdots k_\ell}{k_1 \cdots k_\ell}$, and S is the symmetry number of the diagram. The symmetry number is defined to be the number of permutations of labels of lines which leave the diagram topologically unchanged.

(III. v) Sum over all momenta and spins.

In order to obtain the distribution function which appears in Rule (III. iii), we have summed over infinite series of factors $\epsilon \exp(-\beta\omega_j)$ which appear when Eq. (III. 8) is substituted into Eq. (III. 12).

Some examples of *cluster* 0 *diagrams* are given in Fig. 4. Using Rules (III. i)—(III. v), we may associate the following algebraic expressions with the diagrams in Fig. 4:

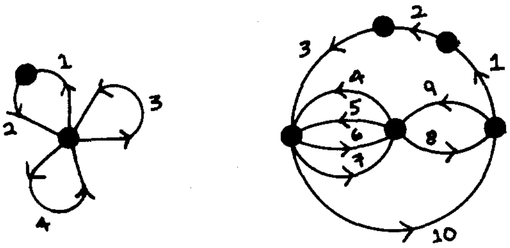


FIG. 4. Cluster 0-diagrams.

$$\text{Fig. }4(\text{a})=\frac{\epsilon}{2}\sum_{k_1\cdots k_4}\left[\prod_{i=1}^{4}(\epsilon\nu_i)\right]T_1\begin{pmatrix}k_2\\k_1\end{pmatrix}T_3\begin{pmatrix}k_1k_3k_4\\k_2k_3k_4\end{pmatrix},\qquad(\text{III. }14)$$

$$\text{Fig. }4(\text{b})=\frac{\epsilon^6}{4}\sum_{k_1\cdots k_{10}}\left[\prod_{i=1}^{10}(\epsilon\nu_i)\right]T_1\begin{pmatrix}k_3\\k_2\end{pmatrix}T_1\begin{pmatrix}k_2\\k_1\end{pmatrix}$$
$$\times T_3\begin{pmatrix}k_6k_7k_{10}\\k_4k_5\,k_3\end{pmatrix}T_3\begin{pmatrix}k_4k_5k_8\\k_6k_7k_9\end{pmatrix}T_2\begin{pmatrix}k_1\,k_9\\k_{10}\,k_9\end{pmatrix}.\qquad(\text{III. }15)$$

IV. DYNAMICAL CLUSTER FUNCTIONS

Ultimately, we wish to obtain an expression for the grand potential in terms of quantities we can calculate given the microscopic properties of the constituent particles in the system. Therefore, it is useful to expand the grand potential in terms of two-body operators, since the two body problem can usually be solved by one means or another.

We shall begin by first expanding the "dynamical" cluster functions $T_\ell\binom{k_1\cdots k_\ell}{k_1'\cdots k_\ell'}$ in terms of matrix elements of the binary operators, $R_\mu(\lambda_1\lambda_2)$ and the magnetic potential energy terms $\Delta H_j(\lambda)$. To evaluate $T_\ell\binom{k_1\cdots k_\ell}{k_1'\cdots k_\ell'}$, we insert Eq. (II. 35) into Eq. (III. 11) and then insert a complete set of unsymmetrized states between each binary operator and potential energy operator in the resulting expansion. {We must insert unsymmetrized states because the condition that no two neighboring binary operators can depend on the same pair of particles [cf. Eqs. (II. 13)–(II. 14)] destroys the symmetry properties of internal states of a cluster. However each cluster as a whole remains invariant under interchange of particles.} In this way we obtain an expansion for $T_\ell\binom{k_1\cdots k_\ell}{k_1'\cdots k_\ell'}$ in terms of unsymmetrized matrix elements of the binary operator and the magnetic potential energy operator. The matrix elements of the binary operators may be antisymmetrized by adding together various terms in the expression for $T_\ell\binom{k_1\cdots k_\ell}{k_1'\cdots k_\ell'}$ (cf. Ref. 15, p. 1050). As a result, we obtain the following expression for the dynamical cluster functions:

$$T_\ell\begin{pmatrix}k_1\cdots k_\ell\\k_1'\cdots k_\ell'\end{pmatrix}$$
$$=\sum_{Q=1}^{\infty}(\text{all different connected }Q\text{-vertex }\ell\text{ diagrams}).\qquad(\text{IV. }1)$$

A *connected Q-vertex ℓ diagram* is collection of Q R-vertices and ΔH vertices ordered from left to right and completely connected by internal wavy lines (wavy lines must begin and end at vertices and cannot be broken). Each *ℓ diagram* has ℓ external solid lines entering and ℓ external solid lines leaving. All lines, both wavy and solid, are directed to the left. Each R vertex has two lines entering and two lines leaving. Each ΔH vertex has one line entering and one line leaving. No wavy line double bonds may appear (two vertices cannot be joined by more than one wavy line). Two *connected Q-vertex ℓ diagrams* differ if they have different topological structure or if they have the same topological structure but the labeling of external lines differs and the temperature labeling of the R matrices differs.

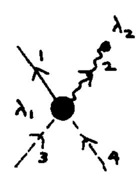
We may assign an algebraic expression to a *connected Q-vertex ℓ diagram* according to the following rules:

(IV. i) Label all the lines (solid and wavy) from 1 to n, where n is the number of lines (2ℓ lines will be solid and $n-2\ell$ lines will be wavy), and assign to the jth line a spin and momentum $k_j=(\mathbf{k}_j,s_j)$.

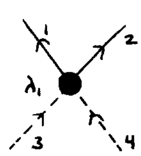
(IV. ii) Label the vertices from left to right from λ_1 to λ_Q.

(IV. iii) With each R vertex, associate a factor

$$=\theta(\lambda_3-\lambda_2)\theta(\lambda_2-\lambda_1)R\begin{pmatrix}k_1\,k_2\\k_3\,k_4\end{pmatrix}^{\lambda_2}_{\lambda_1}\quad\text{if }\lambda_3>\lambda_2,$$



$$=\theta(\beta-\lambda_2)\theta(\lambda_2-\lambda_1)R\begin{pmatrix}k_1\,k_2\\k_3\,k_4\end{pmatrix}^{\lambda_2}_{\lambda_1},$$



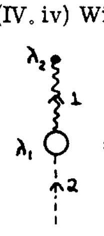
$$=\theta(\beta-\lambda_1)R\begin{pmatrix}k_1\,k_2\\k_3\,k_4\end{pmatrix}^{\beta}_{\lambda_1}.$$

In the above vertices, the dotted lines may be either solid or wavy. The matrix element $R\binom{k_1k_2}{k_3k_4}^{\lambda_2}_{\lambda_1}$ is defined

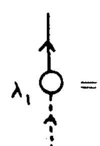
$$R\begin{pmatrix}k_1\,k_2\\k_3\,k_4\end{pmatrix}^{\lambda_2}_{\lambda_1}=\langle k_1k_2\,|\,R(\lambda_2,\lambda_1)\,|\,k_3k_4\rangle^{(s)}.$$

The quantity $\theta(x)$ is the Heaviside function and is defined $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$. The temperatures λ_3 and λ_2 are the temperatures of the vertices to which the outgoing lines attach.

(IV. iv) With each ΔH vertex, associate a factor



$$=\Delta H\begin{pmatrix}k_1\\k_2\end{pmatrix}_{\lambda_1}\theta(\lambda_2-\lambda_1),$$



$$=\Delta H\begin{pmatrix}k_1\\k_2\end{pmatrix}_{\lambda_1}\theta(\beta-\lambda_1),$$

where the dotted lines may be either solid or wavy, and the matrix element $\Delta H\binom{k_1}{k_2}_{\lambda_1}$ is defined

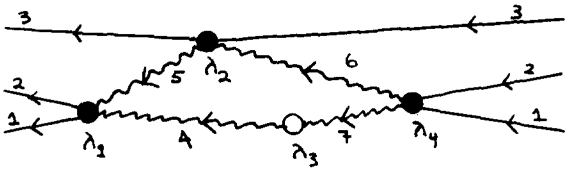


FIG. 5. Four-vertex three-diagram.

$$\Delta H\binom{k_1}{k_2}_{\lambda_1} = (-1)\langle k_1|\Delta H|k_2\rangle \exp[\lambda_1(\omega_1' - \omega_2')].$$

(IV. v) Multiply by an overall factor $\epsilon^{P_B+N_{\Delta H}}$, where $N_{\Delta H}$ is the number of magnetic potential energy vertices in the diagram and P_B is the number of permutations of bottom line momenta with respect to top line momenta in the product of the matrix elements $R\binom{k_1k_2}{k_3k_4}^{\lambda_2}_{\lambda_1}$ and $\Delta H\binom{k_1}{k_2}_{\lambda_1}$.

(IV. vi) Integrate over each temperature from 0 to β and sum over all internal momenta and spins.

By introducing Heaviside functions into the expressions for the binary and potential vertices, we have been able to simplify the limits of integration. This will prove useful later. We have also summed over certain classes of diagrams in doing this. The factor $\epsilon^{P_B+N_{\Delta H}}$ comes from the fact that when complete sets of states are inserted between operators in the expression for $\langle k_1\cdots k_\ell|U_\ell|k_1\cdots k_\ell\rangle$ the number of permutations of bottom row momenta with respect to top row momenta in the matrix elements $R\binom{k_1k_2}{k_3k_4}^{\lambda_2}_{\lambda_1}$ and $\Delta H\binom{k_1}{k_2}_{\lambda_1}$ is $\epsilon^{\ell+N_{\Delta H}}$. The resulting factor ϵ^ℓ cancels the factor ϵ^ℓ in Eq. (III. 11). In Fig. 5, we give an example of a *connected four-vertex three diagram*. The algebraic expression corresponding to the diagram in Fig. 5 is

$$\begin{aligned} \text{Fig. } 5 = \epsilon &\sum_{k_4\cdots k_7} \int_0^\beta \cdots \int_0^\beta d\lambda_1 \cdots d\lambda_4 \\ &\times R\binom{k_1\,k_2}{k_4\,k_5}^{\beta}_{\lambda_1} R\binom{k_5\,k_3}{k_6\,k_3}^{\lambda_1}_{\lambda_2} \Delta H\binom{k_4}{k_7}_{\lambda_3} R\binom{k_7\,k_6}{k_1\,k_2}^{\lambda_3}_{\lambda_4} \\ &\times \theta(\beta-\lambda_1)\theta(\lambda_1-\lambda_2)\theta(\beta-\lambda_2) \\ &\times \theta(\lambda_1-\lambda_3)\theta(\lambda_2-\lambda_3)\theta(\lambda_3-\lambda_4). \end{aligned} \qquad \text{(IV. 2)}$$

Let us note that if we draw Fig. 5 so that the ΔH vertex appears to the left of the middle R vertex, we obtain a different diagram.

V. GRAND POTENTIAL IN TERMS OF BINARY 0 DIAGRAMS

We shall now combine the results of Secs. IV and V, and obtain an expression for the grand potential in terms of matrix elements of the two-body operators $R_\mu(\lambda_1,\lambda_2)$ and magnetic potential energy operators, $\Delta H_j(\lambda)$. We can then use the results of scattering theory to investigate the properties of the grand potential.

To obtain the desired expression for $\Gamma(\beta,g,H_0,H_r)$, we must insert Eq. (IV. 1) into Eq. (III. 13). We then obtain the following definition:

$$\Gamma(\beta,g,H_0,H_r) = -\frac{1}{\beta}\sum_{Q=1}^{\infty}(\text{all different } Q\text{th order binary 0 diagrams}). \qquad \text{(V. 1)}$$

A Qth order *binary* 0 *diagram* is a collection of Q R-vertices and ΔH vertices completely connected by solid and wavy lines. R vertices have two lines entering and two lines leaving and ΔH vertices have one line entering and one line leaving. Wavy lines can only be directed to the left. Solid lines can be directed to the left and right. No wavy line double bonds can be formed. Two *binary* 0 *diagrams* differ if they have different topological structure, or if they have the same topological structure but different line types and temperature labeling of R vertices.

An algebraic expression may be associated with a Qth order *binany* 0 *diagram* according to the following rules:

(V. i) Label all lines from 1 to n, where n is the number of lines, and assign to the jth line a momentum and spin $k_j = (\mathbf{k}_j, s_j)$.

(V. ii) Label vertices from left to right from λ_1 to λ_Q.

(V. iii) With each R vertex associate a factor according to Rule (IV. iii).

(V. iv) With each ΔH vertex associate a factor according to Rule (IV. iv).

(V. v) Associate with each solid line a factor

$$\uparrow k_1 = \epsilon\nu_1 = \sum_{n=1}^{\infty}[\epsilon\exp(-\beta\omega_1')]^n = \frac{\epsilon\exp(-\beta\omega_1')}{1-\epsilon\exp(-\beta\omega_1')},$$

where $k_1 = (\mathbf{k}_1, s_1)$ is the momentum and spin of the line.

(V. vi) Multiply the entire expression by a factor $\epsilon^{P_B}\epsilon^{N_{\Delta H}}S^{-1}$ [cf. Rules (III. iv) and (IV. v)].

(V. vii) Integrate over the temperatures $\lambda_1,\ldots,\lambda_Q$ from 0 to β, and sum over all momenta and spins.

Some examples of *binary* 0 *diagrams* are displayed in Fig. 6. Algebraic expressions for the diagrams in Fig. 6 are given below:

$$\begin{aligned} \text{Fig. } 6(a) = \epsilon^5 &\sum_{k_1\cdots k_6} \int_0^\beta \cdots \int_0^\beta d\lambda_1 \cdots d\lambda_4 \\ &\times \theta(\beta-\lambda_1)\theta(\lambda_1-\lambda_2)\theta(\lambda_2-\lambda_3) \\ &\times \theta(\beta-\lambda_4)(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_5)(\epsilon\nu_6) R\binom{k_1\,k_2}{k_1\,k_3}^{\beta}_{\lambda_1} \\ &\times \Delta H\binom{k_3}{k_4}_{\lambda_2} R\binom{k_4\,k_5}{k_2\,k_6}^{\lambda_2}_{\lambda_3} \Delta H\binom{k_6}{k_5}_{\lambda_4}, \end{aligned} \qquad \text{(V. 2)}$$

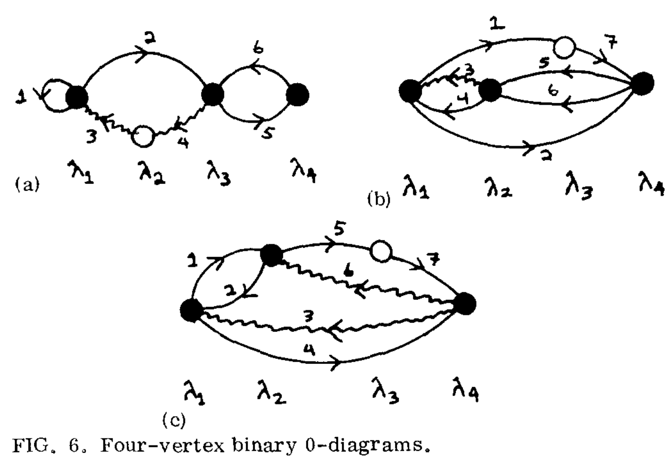


FIG. 6. Four-vertex binary 0-diagrams.

$$\text{Fig. 6(b)} = \epsilon^7 \sum_{k_1\cdots k_7} \int_0^\beta \cdots \int_0^\beta d\lambda_1 \cdots d\lambda_4 \times \theta(\beta-\lambda_1)\theta(\lambda_1-\lambda_2)\theta(\beta-\lambda_2) \times \theta(\beta-\lambda_4)(\epsilon\nu_1)(\epsilon\nu_4)(\epsilon\nu_2)(\epsilon\nu_5)(\epsilon\nu_6)(\epsilon\nu_7) \times R\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_1}^{\beta} R\begin{pmatrix}k_3 k_4\\ k_5 k_6\end{pmatrix}_{\lambda_2}^{\lambda_1} \Delta H\begin{pmatrix}k_7\\ k_1\end{pmatrix}_{\lambda_3} R\begin{pmatrix}k_5 k_6\\ k_2 k_7\end{pmatrix}_{\lambda_4}^{\beta}, \quad (\text{V. 3})$$

and

$$\text{Fig. 6(c)} = \epsilon^5 \sum_{k_1\cdots k_7} \int_0^\beta \cdots \int_0^\beta d\lambda_1 \cdots d\lambda_4 \theta(\beta-\lambda_1)\theta(\beta-\lambda_2) \times \theta(\beta-\lambda_3)\theta(\lambda_1-\lambda_2)\theta(\lambda_2-\lambda_4)(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_4) \times (\epsilon\nu_5)(\epsilon\nu_7) R\begin{pmatrix}k_1 k_4\\ k_2 k_3\end{pmatrix}_{\lambda_1}^{\beta} R\begin{pmatrix}k_2 k_5\\ k_1 k_6\end{pmatrix}_{\lambda_2}^{\beta} \Delta H\begin{pmatrix}k_7\\ k_5\end{pmatrix}_{\lambda_3} \times R\begin{pmatrix}k_3 k_6\\ k_4 k_7\end{pmatrix}_{\lambda_4}^{\lambda_2}. \quad (\text{V. 4})$$

VI. MATRIX ELEMENTS OF THE BINARY OPERATOR

We now wish to find an explicit expression for $R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$ in terms of the reaction matrix. If we take matrix elements of Eq. (II. 15), we obtain

$$R\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1} = -\frac{\partial}{\partial\lambda_2} W\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1}, \quad (\text{VI. 1})$$

where

$$W\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1} = {}_0\langle \mathbf{k}_1, s_1; \mathbf{k}_2, s_2 \,|\, W(\lambda_1,\lambda_2) \,|\, \mathbf{k}_3, s_3; \mathbf{k}_4, s_4\rangle_0^{(s)} \quad (\text{VI. 2})$$

and $W(\lambda_1, \lambda_2)$ is defined in Eq. (II. 16). Because we have a spherically symmetric interaction, the total z component of spin is not changed in the collision process and therefore cancels out of Eq. (VI. 2). We then obtain the following expression for $W\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$:

$$W\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1} = \exp\left[\lambda_1\left(\frac{k_1^2}{2m}+\frac{k_2^2}{2m}\right)\right] \exp\left[-\lambda_2\left(\frac{k_3^2}{2m}+\frac{k_4^2}{2m}\right)\right] \times {}_0\langle \mathbf{k}_1 s_1; \mathbf{k}_2 s_2 \,|\, \exp[-(\lambda_1-\lambda_2)H] \,|\, \mathbf{k}_3 s_3; \mathbf{k}_4 s_4\rangle_0^{(s)}. \quad (\text{VI. 3})$$

If we now insert a complete set of antisymmetrized eigenstates of H^2,

$$1^{(s)} = \tfrac{1}{4}\sum_{s_4 s_6} \int\int d\mathbf{k}_5\, d\mathbf{k}_6 \,|\mathbf{k}_5 s_5; \mathbf{k}_6 s_6\rangle_p^{(s)}\;{}_p^{(s)}\langle \mathbf{k}_5 s_5; \mathbf{k}_6, s_6|, \quad (\text{VI. 4})$$

into Eq. (VI. 3). [In Eq. (VI. 4) we assume that H^2 does not admit bound states. If it does they can be added (cf. Appendix B). Note also that one factor $\frac{1}{2}$ comes from normalization of the wavefunctions and another comes from the fact that the momentum integrations overcount the momentum states.] We then obtain

$$W\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1} = \delta(\mathbf{K}_{12}-\mathbf{K}_{34})\Big\{C^2(\mathbf{k}_{12})\,{}_0\langle \mathbf{k}_{12};\, s_1, s_2\, \mathbf{k}_{34};\, s_3, s_4\rangle_0^{(s)} + mC^2(\mathbf{k}_{12})\exp\left(-\frac{\lambda_2}{m}(k_{34}^2-k_{12}^2)\right)\, {}_0^{(s)}\langle \mathbf{k}_{12};\, s_1, s_2 |A| \mathbf{k}_{34}; s_3, s_4\rangle_0$$
$$\times P\left(\frac{1}{k_{12}^2-k_{34}^2}\right) + mC^2(\mathbf{k}_{34})\exp\left(-\frac{\lambda_1}{m}(k_{34}^2-k_{12}^2)\right)\,{}_0\langle \mathbf{k}_{12}; s_1, s_2 |A| \mathbf{k}_{34}; s_3, s_4\rangle_0^{(s)}\, P\left(\frac{1}{k_{34}^2-k_{12}^2}\right)$$
$$+\frac{m^2}{2}\sum_{s_5 s_6}\int d\mathbf{k}_{56} C^2(\mathbf{k}_{56})\exp\left(\frac{\lambda_1}{m}(k_{12}^2-k_{56}^2)\right)\exp\left(-\frac{\lambda_2}{m}(k_{34}^2-k_{56}^2)\right)\,{}_0\langle \mathbf{k}_{12}; s_1, s_2 |A| \mathbf{k}_{56}; s_5, s_6\rangle_0^{(s)}$$
$$\times {}_0^{(s)}\langle \mathbf{k}_{56}; s_5, s_6 |A| \mathbf{k}_{34}; s_3, s_4\rangle_0\, P\left(\frac{1}{k_{56}^2-k_{12}^2}\right) P\left(\frac{1}{k_{56}^2-k_{34}^2}\right)\Big\}. \quad (\text{VI. 5})$$

We can simplify Eq. (VI. 5) to some extent. From Appendix B, we note that

$$C(\mathbf{k}_{12})\,{}_0\langle \mathbf{k}_{34}; s_3, s_4 | \mathbf{k}_{12}; s_1, s_2\rangle_0 \equiv \langle s_3, s_4| \sum_{\ell, m} y_{\ell,m}(\mathbf{k}_{34}) y_{\ell,m}^*(\mathbf{k}_{12}) \frac{\cos[\delta_\ell(k_{12})]}{k_{12}k_{34}}(k_{12}-k_{34})\,| s_1, s_2\rangle \quad (\text{VI. 6})$$

and

$$C(\mathbf{k}_{12})\,{}_0\langle \mathbf{k}_{34}; s_3, s_4 |A| \mathbf{k}_{12}; s_1, s_2\rangle_0 \equiv \langle s_3, s_4| \sum_{\ell, m} y_{\ell,m}(\mathbf{k}_{34}) y_{\ell,m}^*(\mathbf{k}_{12}) \frac{2}{\pi}\frac{\cos[\delta_\ell(k_{12})]}{k_{12}} \langle k_{34}\ell |A| k_{12}\ell\rangle\, | s_1 s_2\rangle. \quad (\text{VI. 7})$$

Furthermore we note that the principal part satisfies the relation

$$P\left(\frac{1}{x}\right)P\left(\frac{1}{y}\right) = \frac{1}{y-x}\left[P\left(\frac{1}{x}\right)-P\left(\frac{1}{y}\right)\right] + \pi^2\delta(x)\delta(y). \quad (\text{VI. 8})$$

If we use Eq. (B20), and assume that ${}_0\langle 0\ell | k\ell\rangle_p = 0$ (when this is not true we can simply add another term onto the equations below), we obtain, after some algebra,

$$W\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix}_{\lambda_2}^{\lambda_1} = \delta(\mathbf{k}_{12}-\mathbf{k}_{34})\Big\{{}_0\langle \mathbf{k}_{12}; s_1 s_2 | \mathbf{k}_{34}; s_3 s_4\rangle_0^{(s)} + mC^2(\mathbf{k}_{12})\exp\left(-\frac{\lambda_2}{m}(k_{34}^2-k_{12}^2)\right)\,{}_0^{(s)}\langle \mathbf{k}_{12}; s_1 s_2 |A| \mathbf{k}_{34}; s_3 s_4\rangle_0\, P\left(\frac{1}{k_{12}^2-k_{34}^2}\right)$$
$$+ mC^2(\mathbf{k}_{34})\exp\left(-\frac{\lambda_1}{m}(k_{34}^2-k_{12}^2)\right)\,{}_0\langle \mathbf{k}_{12}; s_1, s_2 |A| \mathbf{k}_{34}; s_3, s_4\rangle_0^{(s)}\left[P\left(\frac{1}{k_{34}^2-k_{12}^2}\right) + \tfrac{1}{2}m^2\left(\frac{1}{k_{12}^2-k_{34}^2}\right)\sum_{s_5 s_6}\int d\mathbf{k}_{56}\right.$$
$$\times C^2(\mathbf{k}_{56})\exp\left[\frac{\lambda_1}{m}(k_{12}^2-k_{34}^2)\right]\exp\left[-\frac{\lambda_2}{m}(k_{34}^2-k_{56}^2)\right]\,{}_0\langle \mathbf{k}_{12}; s_1 s_2 |A| \mathbf{k}_{56}; s_5 s_6\rangle_0^{(s)}\;{}_0^{(s)}\langle \mathbf{k}_{56}; s_5, s_6 |A| \mathbf{k}_{34}; s_3, s_4\rangle_0$$
$$\times\left.\left[P\left(\frac{1}{k_{56}^2-k_{12}^2}\right) - P\left(\frac{1}{k_{56}^2-k_{34}^2}\right)\right]\right]. \quad (\text{VI. 9})$$

We now can obtain an expression for $R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$ by differentiating with respect to λ_2. We find

$$R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}=\delta(\mathbf{K}_{12}-\mathbf{K}_{34})\Big\{-C^2(\mathbf{k}_{12})\,^{(s)}_{0}\langle\mathbf{k}_{12};\sigma_1,\sigma_2|A|\mathbf{k}_{34};\sigma_3,\sigma_4\rangle_0\exp\Big(-\frac{\lambda_2}{m}(k_{34}^2-k_{12}^2)\Big)+\frac{m}{2}\sum_{\sigma_5\sigma_6}\int d\mathbf{k}_{56}$$

$$\times C^2(\mathbf{k}_{56})\frac{(k_{34}^2-k_{56}^2)}{(k_{12}^2-k_{34}^2)}\exp\Big(\frac{\lambda_1}{m}(k_{12}^2-k_{56}^2)\Big)\exp\Big(-\frac{\lambda_2}{m}(k_{34}^2-k_{56}^2)\Big)\,_0\langle\mathbf{k}_{12};\sigma_1\sigma_2|A|\mathbf{k}_{56};\sigma_5\sigma_6\rangle_0^{(s)}$$

$$\times\,^{(s)}_{0}\langle\mathbf{k}_{56};\sigma_5\sigma_6|A|\mathbf{k}_{34};\sigma_3\sigma_4\rangle_0\Big[P\Big(\frac{1}{k_{56}^2-k_{12}^2}\Big)-P\Big(\frac{1}{k_{56}^2-k_{34}^2}\Big)\Big]\Big\}. \tag{VI. 10}$$

Equation (VI. 10) can be simplified further. If we use Eq. (VI. 8) we may show that

$$\frac{(k_{34}^2-k_{56}^2)}{(k_{12}^2-k_{34}^2)}\Big[P\Big(\frac{1}{k_{56}^2-k_{12}^2}\Big)-P\Big(\frac{1}{k_{56}^2-k_{34}^2}\Big)\Big]=P\Big(\frac{1}{k_{12}^2-k_{56}^2}\Big). \tag{VI. 11}$$

Combining Eqs. (VI. 10) and (VI. 11), we finally obtain the following expression for matrix elements of the binary operator:

$$R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}=-\delta(\mathbf{K}_{12}-\mathbf{K}_{34})\Big\{C^2(\mathbf{k}_{12})\,_0\langle\mathbf{k}_{12};\sigma_1\sigma_2|A|\mathbf{k}_{34};\sigma_3\sigma_4\rangle_0^{(s)}\exp\Big(-\frac{\lambda_2}{m}(k_{34}^2-k_{12}^2)\Big)+\frac{m}{2}\sum_{\sigma_5\sigma_6}\int d\mathbf{k}_{56}C^2(\mathbf{k}_{56})$$

$$\times\exp\Big(\frac{\lambda_1}{m}(k_{12}^2-k_{34}^2)\Big)\exp\Big(-\frac{\lambda_2}{m}(k_{34}^2-k_{56}^2)\Big)P\Big(\frac{1}{k_{56}^2-k_{12}^2}\Big)\,_0\langle\mathbf{k}_{12};\sigma_1\sigma_2|A|\mathbf{k}_{56};\sigma_5\sigma_6\rangle_0^{(s)}\,_0\langle\mathbf{k}_{56};\sigma_5\sigma_6|A|\mathbf{k}_{34};\sigma_3,\sigma_4\rangle_0^{(s)}\Big\}. \tag{VI. 12}$$

For the purposes of the subsequent discussion, it is convenient to reintroduce center of mass coordinates into Eq. (VI. 12). We then obtain

$$R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}=-\Big[C^2(\mathbf{k}_{12})\,_0\langle k_1,k_2|A|k_3,k_4\rangle_0^{(s)}\exp[-\lambda_2(\omega_1+\omega_2-\omega_3-\omega_4)]+\tfrac{1}{2}\sum_{k_5,k_6}C^2(\mathbf{k}_{56})\exp[\lambda_1(\omega_1+\omega_2-\omega_5-\omega_6)]$$

$$\times\exp[-\lambda_2(\omega_3+\omega_4-\omega_5-\omega_6)]P\Big(\frac{1}{\omega_5+\omega_6-\omega_1-\omega_2}\Big)\,_0\langle k_1k_2|A|k_5k_6\rangle_0^{(s)}\,_0\langle k_5k_6|A|k_3k_4\rangle_0^{(s)}\Big], \tag{VI. 13}$$

where $\omega_i=k_i^2/2m$.

There are some features in the expression for $R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$ which are interesting to point out. The first term on the right-hand side of Eq. (VI. 13) depends only on one temperature, while the second term depends on two temperatures. The first term has the structure of a vertex we would encounter in perturbation theory. Indeed, if we relax the restriction on wiggly line double bonds, replace $C^2(\mathbf{k}_{12})\,_0\langle k_1k_2|A|k_2k_4\rangle_0^{(s)}$ by $_0\langle k_1k_2|V|k_3k_4\rangle_0^{(s)}$ in the first term on the right-hand side of Eq. (VI. 13), and set the second term in Eq. (VI. 13) equal to zero, we obtain the perturbation expansion of the grand potential. Because of this similarity, we expect that some terms in the strong coupling expansion of $\Gamma(\beta,g,H_0,H_r)$ will have the same structure as some terms in the perturbation expansion of $\Gamma(\beta,g,H_0,H_r)$. In fact, as we shall see, the terms which perturbation theorists use to describe density and spin fluctuations in He^3, also appear in the strong coupling expansion.

VII. GRAND POTENTIAL IN TERMS OF THE REACTION MATRIX

We now wish to expand the grand potential in a form which shows explicitly the structure of the temperature dependence of each term. We therefore must take into account, explicitly, the structure of the matrix elements $R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$ and $\Delta H\binom{k_1}{k_2}_{\lambda_1}$.

As we have pointed out in Sec. VI, the matrix element $R\binom{k_1k_2}{k_3k_4}_{\lambda_2}^{\lambda_1}$ breaks into terms with quite different temperature dependences. We therefore will represent each of these terms by separate vertices. In Rules (IV. iii) and (IV. iv), we associated Heaviside functions, which reflected the temperature ordering of various vertices, with the vertices themselves. Now, in order to be able to treat the solid and wavy lines on an equal basis, we shall instead associate the Heaviside functions with the lines. Eventually, we will want to add together diagrams with the same topological structure but which differ in their line types.

We can obtain an expression for $\Gamma(\beta,g,H_0,H_r)$, in which temperature dependences of solid and wavy lines are treated in an identical manner, if we expand the Heaviside functions in the expressions for the *binary* 0 *diagrams* using the identity

$$\theta(\beta-\lambda_1)\theta(\beta-\lambda_2)=\theta(\beta-\lambda_1)\theta(\lambda_1-\lambda_2)+\theta(\beta-\lambda_2)\theta(\lambda_2-\lambda_1). \tag{VII. 1}$$

As a result of doing this, each *binary* 0 *diagram* will yield one or more diagrams in which the vertices are ordered with respect to one another in a well-defined way. [Notice that in Fig. 6(b), for example, the vertices with temperatures λ_3 and λ_4 have no definite order with respect to the other two vertices.] We shall not expand the *binary* 0 *diagrams* completely, however. We shall only expand them until we obtain all diagrams which cannot be deformed into one another without changing the direction of at least one solid line.

The ΔH vertices will also separate into two parts. The ΔH vertices are inhomogeneous vertices. That is, they do not conserve the momentum of the lines entering and leaving the vertex. This can be made explicit if we expand the cosine in Eq. (II. 1) in terms of exponentials; $\cos\mathbf{k}_0\cdot\mathbf{r}=\frac{1}{2}[\exp(i\mathbf{k}_0\cdot\mathbf{r})+\exp(-i\mathbf{k}_0\cdot\mathbf{r})]$. Then the ΔH vertices can be written as a sum of two terms; one of which increases the momentum of the incoming line by a factor $\mathbf{k}_0$, and the other which decreases it by a factor $\mathbf{k}_0$.

If we carry out the above changes, we obtain the following expression for $\Gamma(\beta, g, H_0, H_r)$:

$$\Gamma(\beta, g, H_0, H_r) = -\frac{1}{\beta}\sum_{Q=0}^{\infty}(\text{all different } Q\text{th order } A\text{-matrix 0 diagrams}). \qquad \text{(VII. 2)}$$

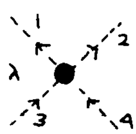
A Qth order, *A-matrix* 0 *diagram is* a collection of Q A-vertices, D vertices, and ΔH vertices ordered from left to right and completely connected by solid and wavy lines. A vertices and D vertices (D stands for "double") each have two lines entering and two lines leaving. ΔH vertices have only one line entering and one line leaving. Wavy lines must be directed to the left, while solid lines may be directed either to the left or right. D vertices with two solid lines must be placed so that solid lines are directed to the right. No wavy line double bonds may appear *except* internally in the D vertices. Two Qth order *A-matrix* 0 *diagrams* differ if they have different topological structure; or if they have the same topological structure but the lines are of different types or directions, or the D vertices have different temperature labeling.

Algebraic expressions can be associated with the Qth order *A-matrix* 0 *diagrams* according to the following rules:

(VII. ii) Label each line from 1 to n, where n is the number of lines, and associate with the jth line a momentum and spin $k_j = (\mathbf{k}_j, \sigma_j)$.

(VII. ii) Label the vertices from left to right from λ_1 to λ_Q. (A vertices and ΔH vertices require only one label, but D vertices require two labels. One temperature label of a D vertex is determined by its horizontal position in the diagram. The other is determined by the type of lines that leave it and the temperature of the vertices to which they attach [cf. Rule (VII. v)].

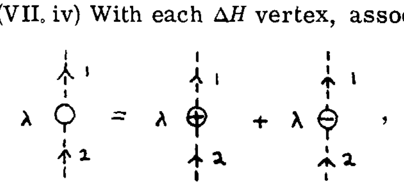
(VII. iii) With each A vertex associate a factor



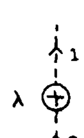
$$= A\begin{pmatrix} k_1 k_2 \\ k_3 k_4 \end{pmatrix} = -C^2(\mathbf{k}_{12})_0 \langle k_1 k_2 | A | k_3 k_4 \rangle_0^{(s)}$$

where the dotted lines can be either wavy or solid lines.

(VII. iv) With each ΔH vertex, associate a factor

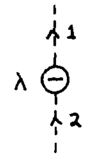


where



$$= +\frac{\mu}{2}\sigma_1 H_r \delta_{\sigma_1,\sigma_2}\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_0)$$

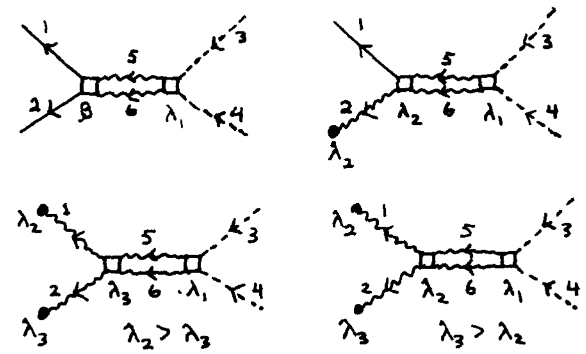
and



$$= +\frac{\mu}{2}\sigma_1 H_r \delta_{\sigma_1,\sigma_2}\delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_0),$$

where the dotted lines may be either solid or wavy.

(VII. v) To each D vertex assign temperature labels according to the following conventions:



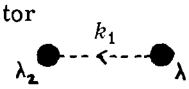
In the above vertices, the right-most temperatures are assigned according to Rule (VII. ii). The left-most temperatures are assigned according to the nature of the outgoing lines and the temperature of the vertices to which they attach. For a D vertex with two outgoing solid lines, the left-most temperature is β. For D vertices with one outgoing wavy line the temperature of the left-most vertex is that of the vertex to which the wavy line attaches. For D vertices with two outgoing wavy lines which attach to vertices with temperatures λ_2 and λ_3, the temperature of the left-most vertex can be either λ_2 or λ_3. In general, both possibilities occur and lead to different expressions.

To each of the above D vertices assign a factor

$$D\begin{pmatrix} k_1 k_2 \\ k_5 k_6 \\ k_3 k_4 \end{pmatrix} = -C^2(\mathbf{k}_{56})_0 \langle k_1 k_2 | A | k_5 k_6 \rangle_0^{(s)}\, {}_0\langle k_5 k_6 | A | k_3 k_4 \rangle_0^{(s)} \times P\left(\frac{1}{\omega_5' + \omega_6' - \omega_1' - \omega_2'}\right)$$

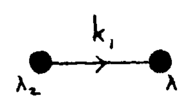
The factor $\frac{1}{2}$ in Eq. (VI. 13) is now contained in the symmetry factor S^{-1} of Rule (VII. ix).

(VII. vi) With each left directed line, associate a factor



$$= \theta(\lambda_2 - \lambda_1)\exp[-(\lambda_2 - \lambda_1)\omega_1'],$$

and with each right directed line associate a factor



$$= \theta(\lambda_2 - \lambda_1)\exp[(\lambda_2 - \lambda_1)\omega_1'],$$

where $\omega_1' = k_1^2/2m - g - \mu\sigma_1 H_0$. The dotted lines may be either solid or wavy.

(VII. vii) To each A vertex or ΔH vertex with no lines leaving or entering on the left, assign a factor $\theta(\beta - \lambda_1)$, where λ_1 is the temperature of the vertex. To each A vertex, ΔH vertex, or D vertex with no lines leaving or entering on the right, assign a factor $\theta(\lambda_1)$, where λ_1 is the temperature of the vertex.

$$\times D\begin{pmatrix} k_1 k_{11} \\ k_3\ k_4 \\ k_5 k_{10}\end{pmatrix} D\begin{pmatrix} k_5 k_9 \\ k_6 k_7 \\ k_2 k_8\end{pmatrix} \Delta H_+\begin{pmatrix} k_2 \\ k_1\end{pmatrix} A\begin{pmatrix} k_8 k_{10} \\ k_9 k_{11}\end{pmatrix}, \qquad \text{(VII. 3)}$$

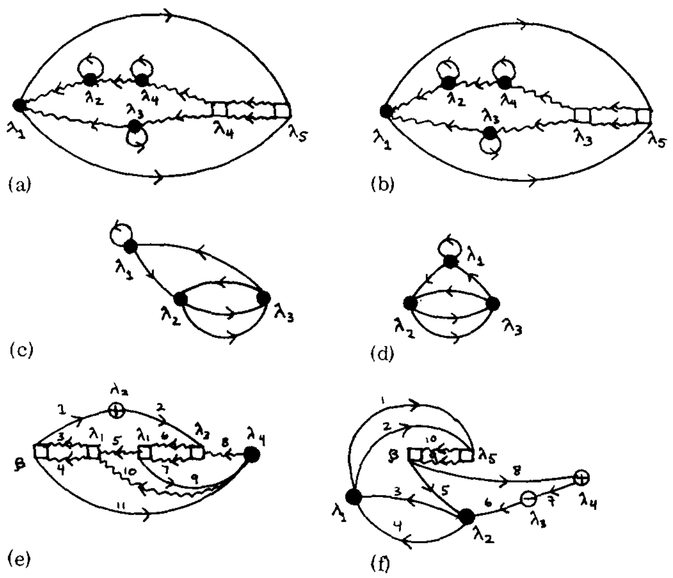


FIG. 7. A-matrix 0-diagrams.

$$\text{Fig. } 7(\text{f}) = \epsilon^8 \sum_{k_1\cdots k_{10}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\lambda_1\cdots d\lambda_5 (\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_3)$$
$$\times(\epsilon\nu_4)(\epsilon\nu_5)(\epsilon\nu_6)(\epsilon\nu_7)(\epsilon\nu_8)\theta(\lambda_1-\lambda_2)\theta(\lambda_2-\lambda_3)$$
$$\times\theta(\lambda_3-\lambda_4)\theta(\beta-\lambda_4)\theta(\lambda_1-\lambda_5)\theta(\beta-\lambda_5)\theta(\beta-\lambda_2)$$
$$\times\theta(\lambda_5)\theta(\lambda_4)\theta(\beta-\lambda_1)\exp[-(\lambda_1-\lambda_2)(\omega_3'+\omega_4')]$$
$$\times\exp[(\beta-\lambda_2)\omega_5']\exp[-(\lambda_2-\lambda_3)\omega_6']$$
$$\times\exp[-(\lambda_3-\lambda_4)\omega_7']\exp[(\beta-\lambda_4)\omega_8']$$
$$\times\exp[-(\beta-\lambda_5)(\omega_9'+\omega_{10}')]\exp[-(\lambda_5-\lambda_1)(\omega_1'+\omega_2')]$$
$$\times\ A\begin{pmatrix} k_1 k_2 \\ k_3 k_4\end{pmatrix} A\begin{pmatrix} k_3 k_4 \\ k_5 k_6\end{pmatrix} \Delta H_-\begin{pmatrix} k_6 \\ k_7\end{pmatrix} \Delta H_+\begin{pmatrix} k_7 \\ k_8\end{pmatrix} D\begin{pmatrix} k_5\ k_8 \\ k_9 k_{10} \\ k_1\ k_2\end{pmatrix}. \qquad \text{(VII. 4)}$$

(VII. vii) Associate with each solid line, a distribution function

$$k_1 = (\epsilon\nu_1) = \epsilon\exp(-\beta\omega_1')/[1-\epsilon\exp(-\beta\omega_1')].$$

(VII. ix) Multiply the entire expression by a factor $\epsilon^{P_B}\epsilon^{N_{\Delta H}}S^{-1}$, where S is the symmetry number of the diagram, $N_{\Delta H}$ is the number of ΔH vertices in the diagram and P_B is the number of permutations of top line momenta with respect to bottom line momenta in the product of the matrix elements $A\binom{k_1k_2}{k_3k_4}$, $\Delta H\binom{k_1}{k_2}$, and $D\begin{pmatrix}k_1k_2\\k_3k_4\\k_5k_6\end{pmatrix}$.

(VII. x) Sum over all momenta and spins. Integrate over all temperatures from $-\infty$ to ∞.

To obtain the energies ω_1' in Rule (VII. vi), rather than just kinetic energies, we have added back the magnetic field terms and chemical potential terms which cancelled in Sec. VI. It is important to note that wavy line double bonds occur internally in the D vertices but cannot occur between D vertices and A vertices. In Fig. 7, we give some examples of A-matrix 0 diagrams. We note that the diagrams in Fig. 7(a) and 7(b) are *not* identical because of the way the left-most temperature of the D vertex is assigned. Furthermore, the diagrams in Fig. 7(c) and 7(d) are not identical because one of the solid lines is directed differently. Algebraic expressions for the diagrams in Figs. 7(e) and 7(f) are given below:

$$\text{Fig. } 7(\text{e}) = (\epsilon^5/4)\sum_{k_1\cdots k_{11}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\lambda_1\cdots d\lambda_4$$
$$\times(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_9)(\epsilon\nu_{11})\theta(\beta-\lambda_2)\theta(\beta-\lambda_4)\theta(\lambda_1-\lambda_4)$$
$$\times\theta(\beta-\lambda_1)\theta(\lambda_1-\lambda_3)\theta(\lambda_2-\lambda_3)\theta(\lambda_3-\lambda_4)\theta(\lambda_4)$$
$$\times\exp[(\beta-\lambda_4)\omega_{11}']\exp[-(\beta-\lambda_1)(\omega_3'+\omega_4')]$$
$$\times\exp[-(\lambda_1-\lambda_3)(\omega_6'+\omega_7')]\exp[(\lambda_2-\lambda_3)\omega_2']$$
$$\times\exp[-(\lambda_3-\lambda_4)\omega_8']\exp[-(\lambda_1-\lambda_4)(\omega_{10}'-\omega_9')]$$
$$\times\exp[(\beta-\lambda_2)\omega_1']$$

We shall now find it useful to distinguish between three different types of *A-matrix* 0 *diagrams*:

Definition (VII. a) Type I A-matrix 0 diagrams contain no double bonds or D vertices.

Definition (VII. b) Type II A matrix 0 diagrams contain at least one solid line or mixed line double bond but no D vertices.

Definition (VII. c) Type III A-matrix 0 diagrams contain at least one D vertex.

As we shall see in a subsequent paper, Type I A-matrix 0 diagrams can be added together to obtain diagrams analogous to perturbation theory diagrams with similar structure. Type II A-matrix 0 diagrams are unsymmetric in that for each diagram with a solid line or mixed line double bond there is no diagram with similar topological structure, but a wavy line double bond in place of the solid line or mixed line double bond. Therefore, Type II A-matrix 0 diagrams cannot be added together to yield diagrams with an analog in perturbation theory. Type III A-matrix 0 diagrams obviously have no analog in perturbation theory.

VIII. CONCLUSION

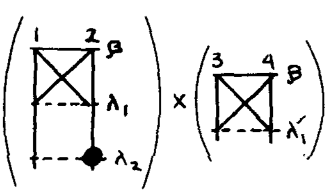
In the previous sections, we obtained an expression for the grand potential, $\Gamma(\beta, g, H_0, H_r)$, of a strongly coupled Fermi fluid, in which the two-body interactions were described solely in terms of the reaction matrix, $\langle k_1k_2|A|k_3k_4\rangle^{(s)}$; and the "propagation" of each particle between interactions was described in terms of a "propagator" of the form $\theta(\lambda)\exp(\pm\lambda\omega')$. In addition, we have included in our expressions the effect of an external spatially varying magnetic field in an exact way. The expressions we have obtained are well behaved even for particles with an infinite hard core.

As we have seen, the quantum binary expansion contains much more structure and much less symmetry than the quantum perturbation expansion. One reason is the exclusion of wavy line double bonds from the binary expansion. The exclusion of wavy line double bonds, and the inclusion of mixed line or solid line double bonds can be understood physically. When two particles with hard cores undergo a collision, they cannot recollide unless they first undergo an intermediate collision with a third particle in the medium. The exclusion of wavy line double bonds is then a dynamical effect which occurs both in the classical and quantum binary expansion. However, for a system of identical particles with degenerate wavefunctions, there can be an exchange between one or both of the particles and particles in the medium. When this occurs an *apparent* recollision becomes possible, as seen from the appearance of mixed line or solid double bonds. However, it is purely a quantum mechanical effect.

Perhaps the most striking feature about the *A-matrix 0 diagrams* is that they contain a subclass of diagrams which are highly symmetric (the Type I diagrams) and which have analogs in perturbation theory. As we shall see in a subsequent paper, a subset of the Type I diagrams contain all the terms necessary to describe spin and density fluctuations in liquid He^3. These terms are easily resummed and give a highly coherent contribution to the thermodynamic properties of a strongly coupled Fermi fluid.

APPENDIX A

We wish to show, by means of an example, how Eq. (II. 20) can be used to obtain a product of connected ladder diagrams by summing over an infinite number of unconnected ladder diagrams. We shall do this by considering the inverse problem. We shall take a given product of connected ladder diagrams and show that it yields an infinite number of unconnected ladder diagrams with well-defined topological structure. Consider the following product to two-particle connected ladder diagrams:



$$
\begin{aligned}
&= \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^\beta d\lambda_1' \, R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2) R_{34}(\beta,\lambda_1') \\
&= \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda} d\lambda_1' \, R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2) R_{34}(\beta,\lambda_1') \\
&\quad + \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_1' \int_0^{\lambda_1'} d\lambda_2 \, R_{12}(\beta,\lambda_1) R_{34}(\beta,\lambda_1')\Delta H_2(\lambda_2) \\
&\quad + \int_0^\beta d\lambda_1' \int_0^{\lambda_1'} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{34}(\beta,\lambda_1') R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2).
\end{aligned}
\tag{A1}
$$

To obtain the last part of Eq. (A1), we have split the integration over λ_1' into three parts and have made a simple change in the order of integration. Furthermore, we have used the fact that R_{12} and ΔH_2 commute with R_{34}.

Let us consider one of the integrals in Eq. (A1) and use Eq. (II. 20). We wish to obtain an expansion for (A1) in terms of products of binary operators whose temperatures are ordered from right to left from 0 to β. Clearly the pair functions in Eq. (A1) are not in that form. If we use Eq. (II. 20) the first integral on the rhs of Eq. (A1) can be written

$$
\begin{aligned}
&\int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_1' \, R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2) R_{34}(\beta,\lambda_1') \\
&\quad = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_1' \, R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2) \\
&\quad \times \Big[1 + \int_{\lambda_2}^\beta ds \, R_{34}(\beta,s)\Big] R_{34}(\lambda_2,\lambda_1') \\
&\quad = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_1' \, R_{12}(\beta,\lambda_1)\Delta H_2(\lambda_2) R_{34}(\lambda_2,\lambda_1') \\
&\quad + \int_0^\beta ds \int_0^s d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_1' R_{34}(\beta,s) R_{12}(\beta,\lambda_1) \\
&\quad \times \Delta H_2(\lambda_2) R_{34}(\lambda_2,\lambda_1') \\
&\quad + \int_0^\beta d\lambda_1 \int_0^{\lambda_1} ds \int_0^s d\lambda_2 \int_0^{\lambda_2} d\lambda_1' \, R_{12}(\beta,\lambda_1) R_{34}(\beta,s) \\
&\quad \times \Delta H_2(\lambda_2) R_{34}(\lambda_2,\lambda_1').
\end{aligned}
\tag{A2}
$$

Let us consider integrals of the following type:

$$
\begin{aligned}
&\int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{12}(\beta,\lambda_1) R_{34}(\beta,\lambda_2) \\
&\quad = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{12}(\beta,\lambda_1)\Big[1 + \int_{\lambda_1}^\beta ds \, R_{34}(\beta,s) R_{34}(\lambda_1,\lambda_2) \\
&\quad = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{12}(\beta,\lambda_1) R_{34}(\lambda_1,\lambda_2) \\
&\quad + \int_0^\beta ds \int_0^s d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{34}(\beta,s) R_{12}(\beta,\lambda_1) R_{34}(\lambda_1,\lambda_2) \\
&\quad = \int_0^\beta d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{12}(\beta,\lambda_1) R_{34}(\lambda_1,\lambda_2) \\
&\quad + \int_0^\beta ds_1 \int_0^{s_1} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{34}(\beta,s_1) R_{12}(s_1,\lambda_1) R_{34}(\lambda_1,\lambda_2) \\
&\quad + \int_0^\beta ds_2 \int_0^{s_2} ds_1 \int_0^{s_1} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, R_{12}(\beta,s_2) R_{34}(s_2,s_1) \\
&\quad \times R_{12}(s_1,\lambda_1) R_{34}(\lambda_1,\lambda_2) + \cdots .
\end{aligned}
\tag{A3}
$$

To obtain the rhs of Eq. (A3), we have repeatedly

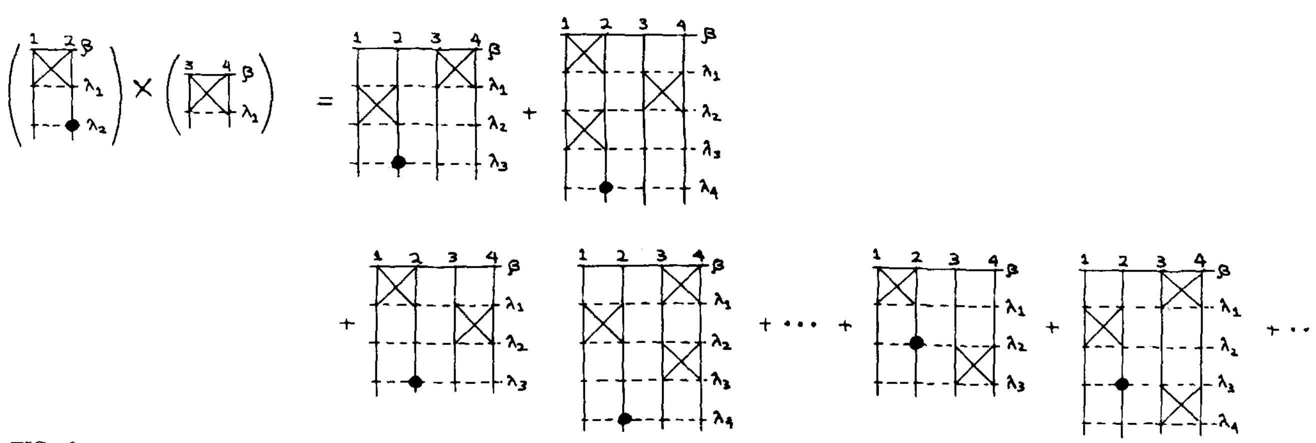


FIG. 8. Decomposition of a product of two-particle connected ladder diagrams into an infinite number of unconnected four-particle ladder diagrams.

applied Eq. (II. 20). If we now combine Eqs. (A1), (A2), and (A3), we see that the product of three-particle connected ladder diagrams in Eq. (A1) can be written as an infinite sum of unconnected four-particle ladder diagrams. In Fig. 8, we display some of them. Note that the circle can only appear on the bottom line or with a single (34) cross below it.

APPENDIX B: TWO-BODY EIGENSTATES

In this appendix we wish to outline relevant properties of the eigenstates of the two-body Hamiltonian. The two-body Hamiltonian may be written in the form:

$$H^2=\frac{k_1^2}{2m}+\frac{k_2^2}{2m}+V(|\mathbf{r}_{12}|)=\frac{\mathbf{K}_{12}^2}{4m}+\frac{k_{12}^2}{m}+V(|\mathbf{r}_{12}|), \tag{B1}$$

where $\mathbf{K}_{12}=\mathbf{k}_1+\mathbf{k}_2$ is the center of mass momentum and $\mathbf{k}_{12}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$ is the relative momentum. The Hamiltonian separates into a part which describes free center of mass motion and the relative motion.

Because we are considering identical fermions, eigenstates of both H_0^2 and H^2 must be antisymmetric under exchange of particles. Eigenstates of H_0^2 may be written

$$|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0^{(s)}=|\mathbf{k}_1,s_1;\mathbf{k}_2,s_2\rangle_0+\epsilon|\mathbf{k}_2s_2;\mathbf{k}_1s_1\rangle_0, \tag{B2}$$

and eigenstates of H^2 may be written

$$\begin{aligned}|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_p^{(s)}&=C(\mathbf{k}_{12})\Big\{|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0^{(s)}+\sum_{s_5s_6}\iint d\mathbf{k}_5\,d\mathbf{k}_6\\&\times|\mathbf{k}_5,s_5;\mathbf{k}_6,s_6\rangle_0^{(s)}P\left(\frac{1}{\omega_1+\omega_2-\omega_3-\omega_4}\right)\\&\times{}_0\langle\mathbf{k}_5s_5;\mathbf{k}_6,s_6|A|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0\end{aligned} \tag{B3}$$

(cf. Ref. 16, Secs. 4.3–4.5). In Eq. (B3) we have written standing wave eigenstates of H^2. The matrix element ${}_0\langle\mathbf{k}_5,s_5;\mathbf{k}_6,s_6|A|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0$ is the reaction matrix and P denotes principal part. The normalization constant, $C(\mathbf{k}_{12})$, only appears for standing wave states. When using travelling wave states, defined in terms of the T matrix, $C(\mathbf{k}_{12})$ does not appear (cf. Ref. 17, Sec. 7.1.3). The states $|\mathbf{k}_1,s_1;\mathbf{k}_2,s_2\rangle_0^{(s)}$ and $|\mathbf{k}_1,s_1;\mathbf{k}_2,s_2\rangle_p^{(s)}$ are not completely normalized, but must be multiplied by a factor $2^{-1/2}$.

Since the Hamiltonian splits into a center of mass part and a relative part, so will the wavefunctions:

$$|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0^{(s)}=|\mathbf{K}_{12}\rangle|\mathbf{k}_{12};s_1,s_2\rangle_0^{(s)} \tag{B4}$$

where

$$|\mathbf{k}_{12};s_1s_2\rangle_0^{(s)}=|\mathbf{k}_{12};s_1s_2\rangle_0+\epsilon|-\mathbf{k}_{12};s_2s_1\rangle_0; \tag{B5}$$

and

$$|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_p^{(s)}=|\mathbf{K}_{12}\rangle_0|\mathbf{k}_{12};s_1,s_2\rangle_p^{(s)}, \tag{B6}$$

where

$$\begin{aligned}|\mathbf{k}_{12};s_1,s_2\rangle_p^{(s)}&=C(\mathbf{k}_{12})\Big\{|\mathbf{k}_{12};s_1,s_2\rangle_0^{(s)}+m\sum_{s_5s_6}\int d\mathbf{k}_{56}\\&\times|\mathbf{k}_{56};s_5,s_6\rangle_0^{(s)}P\left(\frac{1}{k_{12}^2-k_{56}^2}\right)\\&\times{}_0\langle\mathbf{k}_{56};s_5s_6|A|\mathbf{k}_{12};s_1s_2\rangle_0\Big\}.\end{aligned} \tag{B7}$$

In Eq. (B7), we have changed from coordinates $\mathbf{k}_1$ and $\mathbf{k}_2$ to coordinates $\mathbf{K}_{12}$ and $\mathbf{k}_2$, and we have used the fact that ${}_0\langle\mathbf{K}_{56}|\mathbf{K}_{12}\rangle_0=\delta(\mathbf{K}_{56}-\mathbf{k}_{12})$. (The reaction matrix depends only on relative coordinates.)

In Eq. (B3), we have ignored the possibility of bound states. If the Hamiltonian admits bound states, then we must add an extra term onto Eq. (B3) to obtain a complete set (cf. Ref. 17, pp. 159 and 205).

Let us now consider some relevant properties of the unsymmetrized eigenstates of H^2. We can then easily write down explicit expressions for the antisymmetrized states.

In the position representation, we may write

$$\begin{aligned}&{}_0\langle\mathbf{r}_3,s_3;\mathbf{r}_4,s_4|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_p\\&={}_0\langle\mathbf{R}_{34}|\mathbf{K}_{12}\rangle_0\delta_{s_1,s_2}\delta_{s_2,s_1}\,{}_0\langle\mathbf{r}_{34}|\mathbf{k}_{12}\rangle,\end{aligned} \tag{B8}$$

where

$${}_0\langle\mathbf{R}_{34}|\mathbf{K}_{12}\rangle_0=\frac{1}{(2\pi)^{3/2}}\exp(i\mathbf{K}_{34}\cdot\mathbf{R}_{12}) \tag{B9}$$

and

$${}_0\langle\mathbf{r}_{34}|\mathbf{k}_{12}\rangle_p=\left(\frac{2}{\pi}\right)^{1/2}\sum_{\ell=0}^{\infty}\frac{(2\ell+1)}{4\pi}(i)^\ell\frac{{}_0\langle r_{34}\ell|k_{12}\ell\rangle_p}{k_{12}r}P_\ell(\mathbf{r}_{34}\cdot\mathbf{k}_{12}). \tag{B10}$$

The normalization of Eqs. (B9) and (B10) is written for particles in an infinite box. The wavefunction ${}_0\langle r_{34}\ell|k_{12}\ell\rangle_p$ is defined

$$\begin{aligned}&{}_0\langle r_{34}\ell|k_{12}\ell\rangle_p\\&=\cos[\delta_\ell(k_{12})]\left[{}_0\langle r_{34}\ell|k_{12}\ell\rangle_0+\left(\frac{2m}{\pi}\right)\int_0^\infty dk\,k\,{}_0\langle r_{34}\ell|k\ell\rangle_0\right.\\&\left.\times P\left(\frac{1}{k_{12}^2-k^2}\right){}_0\langle k\ell|A|k_{12}\ell\rangle_0\right].\end{aligned} \tag{B11}$$

The free particle wavefunction, ${}_0\langle k_{34}\ell|k_{12}\ell\rangle$, is defined in terms of spherical Bessel functions as

$${}_0\langle r_{34}\ell|k_{12}\ell\rangle_0=k_{12}r_{34}j_\ell(k_{12}k_{34}). \tag{B12}$$

The free particle wavefunction in the position representation, ${}_0\langle\mathbf{r}_3,s_3;\mathbf{r}_4s_4|\mathbf{k}_1s_1;\mathbf{k}_2s_2\rangle_0$ can be written in an analogous manner, but in Eq. (B8) ${}_0\langle\mathbf{r}_{34}|\mathbf{k}_{12}\rangle_0$ replaces ${}_0\langle\mathbf{r}_{34}|\mathbf{k}_{12}\rangle_p$ and in Eq. (B10) ${}_0\langle r_{34}\ell|k_{12}\ell\rangle_0$ replaces ${}_0\langle r_{34}\ell|k_{12}\ell\rangle_p$. The reaction matrix ${}_0\langle k\ell|A|k_{12}\ell\rangle_0$ in Eq. (B11) has been defined so that

$$\begin{aligned}&{}_0\langle\mathbf{k}|A|\mathbf{k}'\rangle_0\\&=\left(\frac{2}{\pi}\right)\sum_{\ell,m}\frac{1}{k'}Y_{\ell m}(\hat{k})Y_{\ell m}^*(\hat{k}')\,{}_0\langle k\ell|A|k'\ell\rangle_0,\end{aligned} \tag{B13}$$

where the functions $Y_{\ell m}(\hat{k})$ are spherical harmonics dependent on the angles made by the vector $\hat{k}$ with respect to an arbitrary axis (we use the conventions of Ref. 18). The factor $\cos[\delta_\ell(k_{12})]$ is the contribution to the ℓth partial wave coming from the constant $C(\underset{\sim}{k}_{12})$.

The functions ${}_0\langle r\ell|k\ell\rangle_0$ and ${}_0\langle r\ell|k\ell\rangle_p$ satisfy the differential equations

$$\frac{\partial^2}{\partial r^2}\,{}_0\langle r\ell|k\ell\rangle_0-\frac{\ell(\ell+1)_0\langle r\ell|k\ell\rangle_0}{r^2}+k^2\,{}_0\langle r\ell k\ell\rangle_0=0 \tag{B14}$$

and

$$\frac{\partial^2}{\partial r^2}\,{}_0\langle r\ell|k\ell\rangle_p-\frac{\ell(\ell+1)_0\langle r\ell|k\ell\rangle_p}{r^2}+[k^2-mV(|\mathbf{r}|)]_0\langle r\ell|k\ell\rangle_p=0, \tag{B15}$$

respectively. We may use Eqs. (B11), (B14), and (B15) to obtain the following expression for the reaction matrix in terms of the exact two-body wavefunction and the potential

$$ {}_0\langle k\ell | A | k'\ell\rangle_0 = \frac{m}{k\cos[\delta_\ell(k')]}\int_0^\infty dr\, {}_0\langle k\ell | r\ell\rangle_0 V(r) {}_0\langle r\ell | k'\ell\rangle_p. \tag{B16} $$

Diagonal matrix elements of the reaction matrix ${}_0\langle k\ell | A | k\ell\rangle_0$ are related to scattering phase shifts $\delta_\ell(k)$. If we note that

$$ {}_0\langle r\ell | k\ell\rangle_0 \underset{kr\to\infty}{\to} \sin(kr - \ell\pi/2), \tag{B17} $$

$$ {}_0\langle r\ell | k\ell\rangle_p \underset{kr\to\infty}{\to} \cos[\delta_\ell(k)]\sin(kr - \ell\pi/2) + \sin[\delta_\ell(k)]\cos(kr - \ell\pi/2), \tag{B18} $$

and

$$ {}_0\langle r\ell | k\ell\rangle_0 \underset{r\to 0}{\to} \frac{(kr)^{\ell+1}}{(2\ell+1)!!}, \tag{B19} $$

then we may show, using Eq. (B14), (B15), and (B16) that

$$ {}_0\langle k\ell | A | k\ell\rangle_0 = -\tan\delta_\ell(k) - \delta_{\ell,0}\frac{{}_0\langle 0\ell | k\ell\rangle_p}{\cos[\delta_\ell(k)]} \tag{B20} $$

where $\delta_{\ell,0}$ denotes a Kronecker delta function.

The position representation is useful for obtaining properties of the reaction matrix. However, in our expressions for the grand potential, we will need expressions for the antisymmetrized eigenstates in the momentum representation. To obtain these, it is convenient to introduce the spin exchange operator $\frac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ denote Pauli spin matrices. The effect of the exchange operator acting on states $|s_1, s_2\rangle$ is

$$ \tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|s_1,s_2\rangle = |s_2,s_1\rangle. \tag{B21} $$

Equation (B21) is easy to verify. Let us expand the states $|s_1,s_2\rangle$ in terms of eigenstates of the total spin $\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$ and the total z component of spin $S_z = s_1 + s_2 = \frac{1}{2}(\sigma_{z_1}+\sigma_{z_2})$. The operator $\mathbf{S}^2$ has eigenvalues $S(S+1)$ where $S=1$ or 0. The operator S_z has eigenvalues 1, 0, or -1. If we denote the eigenstates of $\mathbf{S}^2$ and S_z by $|S,S_z\rangle$, we may write

$$ |\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,1\rangle, \tag{B22} $$

$$ |\tfrac{1}{2},-\tfrac{1}{2}\rangle = (2)^{-1/2}[|1,0\rangle - |00\rangle], \tag{B23} $$

$$ |-\tfrac{1}{2},\tfrac{1}{2}\rangle = (2)^{-1/2}[|1,0\rangle + |0,0\rangle], \tag{B24} $$

$$ |-\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle. \tag{B25} $$

If we now note that

$$ \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = 2S(S+1) - 3, \tag{B26} $$

then using Eqs. (B22)—(B26), we may easily verify Eq. (B21).

We now use the spin exchange operator to write an expression for antisymmetric eigenstates of H^2:

$$ {}_0\langle \mathbf{k}_3, s_3; \mathbf{k}_4, s_4 | \mathbf{k}_1, s_1; \mathbf{k}_2 s_2\rangle_p^{(s)} = {}_0\langle \mathbf{K}_{34} | \mathbf{K}_{12}\rangle_0\, {}_0\langle \mathbf{k}_{34}; s_3 s_4 | \mathbf{k}_{12}; s_1, s_2\rangle_p^{(s)}, \tag{B27} $$

where

$$ \begin{aligned} {}_0\langle \mathbf{k}_{34}; s_3\, s_4 | \mathbf{k}_{12}; s_1, s_2\rangle_p^{(s)} &= {}^{(s)}_{0}\langle \mathbf{k}_{34}; s_3 s_4 | \mathbf{k}_{12}; s_1, s_2\rangle_p \\ &= \langle s_3, s_4 | \sum_{\ell,m}\Big\{ Y_{\ell,m}(\hat{\mathbf{k}}_{34})Y^*_{\ell,m}(\hat{\mathbf{k}}_{12}) + \frac{\epsilon}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)Y_{\ell,m}(-\hat{\mathbf{k}}_{34}) \\ &\quad\times Y^*_{\ell,m}(+\hat{\mathbf{k}}_{12})\Big\} \times \frac{\cos[\delta\,(k_{12})]}{k_{12}k_{34}}\Big[\delta(k_{12}-k_{34}) + \left(\frac{2k_{34}m}{\pi}\right) \\ &\quad\times P\left(\frac{1}{k_{12}^2-k_{34}^2}\right)\langle k_{34}\ell | A | k_{12}\ell\rangle\Big]\; |s_1, s_2\rangle. \end{aligned} \tag{B28} $$

It is possible to find explicit expressions for the reaction matrix for certain types of potential. For example, if we consider a finite repulsive core of height V and radius a [$V(r)=V$ for $r<a$ and $V(r)=0$ for $r>a$], then we find

$$ \langle k'\ell | A | k\ell\rangle = \frac{mVk}{k'(k'^2-z^2)}\frac{(zF_\ell(k'a)F_{\ell-1}(za) - k'F_{\ell-1}(k'a)F_\ell(za))}{(kF_\ell(za)G_{\ell-1}(ka) - zG_\ell(ka)F_{\ell-1}(za))}, \tag{B29} $$

where $F_\ell(ka) = kaj_\ell(ka)$ and $G_\ell(ka) = kan_\ell(ka)$ [$j_\ell(ka)$ and $n_\ell(ka)$ are spherical Bessel functions, and $z^2 = k^2 - mV$.

Equation (B29) gives a well-defined result even for an infinite hard core. Indeed, one finds that

$$ \langle k'\ell | A | k\ell\rangle \underset{V\to\infty}{\to} -\frac{k}{k'}\frac{F_\ell(k'a)}{G_\ell(ka)} + O\left(\frac{1}{V}\right). \tag{B30} $$

However, one must be careful in taking the limit $V\to\infty$. As Lee and Yang and Mohling both point out,[19] if one takes the limit too soon the wavefunction in Eq. (B3) appears not to form a complete set. One can see the difficulty by looking at the operator

$$ W(\lambda_1,\lambda_2) = \exp(\lambda_1 H_0)\exp[-(\lambda_1-\lambda_2)H]\exp(-\lambda_2 H_0), \tag{B31} $$

where $\lambda_1 \geq \lambda_2$ and $H = H_0 + V$. If we take the limits $\lambda_1 \to \lambda_2$ and $V\to\infty$ in different order, we obtain

$$ \lim_{\lambda_1\to\lambda_2}\lim_{V\to\infty} W(\lambda_1,\lambda_2) = 0 \tag{B32} $$

and

$$ \lim_{V\to\infty}\lim_{\lambda_1\to\lambda_2} W(\lambda_1,\lambda_2) = 1. \tag{B33} $$

Since the limit $V\to\infty$ is unphysical, we shall always assume that V is very large but finite and that the reaction matrix is well approximated by the first term in Eq. (B30). Then we should not have any difficulty concerning the completeness of our states.

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Nonlinear response of equilibrium strongly coupled Fermi fluids. II. Fourier expansion and partial resummation of expectation values—polarization diagrams

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This is Paper II of a series of three papers in which a self-consistent propagator resummation of self-energy effects in a strongly coupled Fermi fluid is performed. In the present paper, a generating function for the expectation values of arbitrary one- and two-body operators is introduced and written in the form of a cluster expansion. An explicit expression is written for the magnetization of a strongly coupled Fermi fluid in the presence of a constant and a spatially varying external field, and rules are given for evaluating it in terms of a reaction matrix expansion. A Fourier expansion is performed on a subclass of the diagrams contributing to the grand potential and the magnetization, and a self-consistent resummation of self-energy effects due to both the medium and the external spatially varying field is performed. It is found that traditional perturbation theory techniques for summing self-energy effects cannot be applied to all terms in a reaction matrix expansion. The effect of the external fields on the polarization diagrams is discussed.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as RI, we derived an exact expression for the grand potential of a strongly interacting Fermi fluid, in the presence of a magnetic field with a constant and a spatially varying part. Contributions to the grand potential due to two-body interactions were expressed entirely in terms of the reaction matrix, and temperature dependent single particle propagators.

In the present paper, we shall generalize the results of RI to the expressions for expectation values by introducing a generating function for the expectation values of single-particle and two-particle operators. We shall then focus on the expressions for the magnetization and the grand potential.

Normally when considering the response of a system to a weak external magnetic field, only linear terms in the external field are retained in the microscopic expressions for the magnetization.[2] One assumes that for a weak field the nonlinear terms must be much smaller than the linear terms and therefore can be neglected. However, as we shall see, nonlinear terms can give rise to self-energy effects and therefore can be accompanied by a "secular" (polynomial) dependence on the inverse temperature. In the limit of zero temperature, these terms need not be small, and some other argument must be found for neglecting them. For the system we are considering, there will also be self-energy effects due to the interaction of particles with the medium. If the self-energy effects due to interaction with the external field are small compared to those due to interaction with the medium, then the linear approximation can be made.

Part of the purpose of the present series of papers is to introduce a method of resumming self-energy effects (secular terms in the inverse temperature, β) without at the same time introducing undefined energy denominators in our expressions, as happens when generalized Hartree—Fock methods are used (cf. the Introduction of RI). One can avoid undefined energy denominators by performing propagator resummations of the type used in quantum field theory. For equilibrium properties, one uses a technique introduced by Matsubara[3–5] to make a Fourier expansion of various thermodynamic properties. One can then resum all diagrams containing self-energy effects, and all quantities remain well defined.

Because of the appearance of D vertices and the exclusion of certain types of double bond structures the Matsubara technique does not apply to all terms in the reaction matrix expansion for the grand potential (or to expectation values in general). However, it does apply to the subclass of terms in the reaction matrix expansion which contains no double bond structures or D vertices. These terms include the polarization diagrams which are used to describe spin and density fluctuations in liquid He^3. The polarization diagrams have been studied by Brinkman and Englesberg[6] using perturbation theory, and by Reichl and Tuttle[7] using reaction matrices and generalized Hartree—Fock resummation of self-energy effects. They have never been studied in the presence of an external spatially varying magnetic field (Brinkman and Englesberg include a constant external magnetic field). We shall therefore use the Matsubara technique to study the behavior of the polarization diagrams in the presence of a spatially varying external field. We can then make comparisons between various methods of treating the polarization diagrams. In a subsequent paper, we shall introduce an alternative propagator formulation of thermodynamic properties, which can be used even in the presence of double bond structures or D vertices.

We shall begin in Sec. II by introducing a generating function for the expectation values of one- and two-body operators, and we shall consider the particular case of the magnetization.

In Sec. III, we use the similarity between the grand potential, $\Gamma(\beta, g, H_0, H_r)$, and the magnetization, $\langle M(\mathbf{r})\rangle$, to write an expression for $\langle M(\mathbf{r})\rangle$ in terms of *A-matrix* 0_M *diagrams*. These are analogous to the *A-matrix* 0 *diagrams* introduced in RI, Sec. VII for the grand potential.

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In Sec. IV, we restrict our consideration to a subclass of *A-matrix* 0 *diagrams* and 0_M *diagrams* (the Type I diagrams). These diagrams are the only ones which have counterparts in perturbation theory. They can be expressed in terms of propagators which are periodic functions of β. The expressions for the grand potential and the magnetization obtained using only the Type I diagrams [we denote them as $\Gamma(\beta, g, H_0, H_r)_{\rm I}$ and $\langle M(\mathbf{r})\rangle_{\rm I}$, respectively] can then be expanded in a Fourier series and the self-energy effects easily identified.

In Sec. V, we write our expressions for $\Gamma(\beta, g, H_0, H_r)_{\rm I}$ and $\langle M(\mathbf{r})\rangle_{\rm I}$ in terms of a Fourier series and in Sec. VI we discuss the form of the self-energy structures appearing in the expressions for $\Gamma(\beta, g, H_0, H_r)_{\rm I}$ and $\langle M(\mathbf{r})\rangle_{\rm I}$.

In Sec. VII, we self consistently resum the self-energy effects in the expressions for $\Gamma(\beta, g, H_0, H_r)_{\rm I}$ and $\langle M(\mathbf{r})\rangle_{\rm I}$ and in Sec. VIII we apply our results to the case of polarization diagrams. Finally, in Sec. IX we make some concluding remarks.

II. GENERATING FUNCTION

We wish to find a generating function for the expectation value of arbitrary one- and two-body operators

$$\hat{0}_1^N = \sum_{i=1}^{N} \hat{0}_i \tag{II. 1}$$

and

$$\hat{0}_2^N = \sum_{i<j}^{N(N-1)/2} \hat{0}_{ij}. \tag{II. 2}$$

The expectation values of these operators are defined

$$\langle 0_{1(2)}\rangle = \exp[\beta\Gamma(\beta, g, H_0, H_r)] \times \sum_{N=0}^{\infty} \mathrm{Tr}_N \exp[-\beta(H^{N'} + \Delta H^N)]\, \hat{0}^N_{1(2)}, \tag{II. 3}$$

where $\hat{0}^N_{1(2)}$ can denote either $\hat{0}_1^N$ or $\hat{0}_2^N$. The definition of all quantities in Eq. (II. 3) have been given in RI, Secs. II. and III, and will not be repeated here.

Let us now introduce the following generating function:

$$\begin{aligned} L(\Theta, \beta, g, H_0, H_r) &\equiv \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{k_1\cdots k_N\\ k_1'\cdots k_N'}} \prod_{i=1}^{N} \exp(-\beta\omega_i') \\ &\times \langle k_1, \ldots, k_N | \exp(\beta H_0^{N'}) \exp[-\beta(H^{N'} + \Delta H^N)] | \\ &\times k_1', \ldots, k_N'\rangle^{(s)} \left(\prod_{j=1}^{N} \Theta_{j'j}\right), \end{aligned} \tag{II. 4}$$

where we use the notation

$$\prod_{j=1}^{N} \Theta_{j'j} = \Theta_{1'1} \times \Theta_{2'2} \times \cdots \times \Theta_{N'N}. \tag{II. 5}$$

The quantities $\Theta_{jj'}$ will be defined below. In terms of the generating function, $\langle 0_1\rangle$ is defined

$$\langle 0_1\rangle = \sum_{k_a'k_a} \left[\frac{\partial}{\partial\Theta_{a'a}} L(\Theta, \beta, g, H_0, H_r)\right]_{\Theta_{jj'} \to \delta_{jj'}} \times \exp[\beta\Gamma(\beta, g, H_0, H_r)] \tag{II. 6}$$

and $\langle 0_2\rangle$ is defined

$$\langle 0_2\rangle = \sum_{k_a'k_a} \sum_{k_b'k_b} \left[\frac{\partial}{\partial\Theta_{b'b}} \frac{\partial}{\partial\Theta_{a'a}} L(\Theta, \beta, g, H_0, H_r)\right]_{\Theta_{jj'} \to \delta_{jj'}} \times \exp[\beta\Gamma(\beta, g, H_0, H_r)]. \tag{II. 7}$$

In Eq. (II. 6) the differentiation is first performed and then in the resulting expression $\partial\Theta_{j'j}/\partial\Theta_{a'a}$ is defined

$$\frac{\partial\Theta_{j'j}}{\partial\Theta_{a'a}} = \delta_{k_j,k_a} \langle k_j' | \hat{0}_1 | k_j\rangle \delta_{k_j',k_a'}. \tag{II. 8}$$

The remaining factors $\Theta_{j'j}$ are then set equal to delta functions; i.e., $\Theta_{j'j} \to \delta_{j'j}$. Similarly, in Eq. (II. 7) the differentiation is first performed and in the resulting expression the quantity $(\partial\Theta_{j'j}/\partial\Theta_{a'a})(\partial\Theta_{i'i}/\partial\Theta_{b'b})$ is defined

$$\frac{\partial\Theta_{j'j}}{\partial\Theta_{a'a}} \frac{\partial\Theta_{i'i}}{\partial\Theta_{b'b}} = \delta_{k_j,k_a} \delta_{k_j',k_a'} \delta_{k_i,k_b} \delta_{k_i',k_b'} \langle k_{j'} k_{i'} | \hat{0}_2 | k_j k_i\rangle^{(s)}. \tag{II. 9}$$

If the cluster functions of RI, Sec. III, are substituted into Eq. (II. 4), the generating function can be written as the exponential of a function $B(\Theta, \beta, g, H_0, H_r)$

$$L(\Theta, \beta, g, H_0, H_r) = \exp[B(\Theta, \beta, g, H_0, H_r)], \tag{II. 10}$$

where

$$B(\Theta, \beta, g, H_0, H_r) = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{k_1\cdots k_l\\ k_1'\cdots k_l'}} \left(\prod_{i=1}^{l} \exp(-\beta\omega_i)\right) \times U^{(s)}\begin{pmatrix} k_1 \cdots k_l \\ k_1' \cdots k_l' \end{pmatrix} \left(\prod_{j=1}^{l} \Theta_{j'j}\right). \tag{II. 11}$$

The derivation of Eqs. (II. 10) and (II. 11) is completely analogous to that used in RI to obtain grand potential from the grand partition function.

If we now combine Eqs. (II. 6), (II. 7), and (II. 10), we obtain

$$\langle 0_1\rangle = \exp(\beta\Gamma) \sum_{k_a'k_a} \left[\frac{\partial B}{\partial\Theta_{a'a}} e^B\right]_{\Theta_{jj'} \to \delta_{jj'}} \tag{II. 12}$$

and

$$\langle 0_2\rangle = \exp(\beta\Gamma) \sum_{k_a'k_a} \sum_{k_b'k_b} \left[\left(\frac{\partial^2 B}{\partial\Theta_{b'b}\partial\Theta_{a'a}} + \frac{\partial B}{\partial\Theta_{b'b}} \frac{\partial B}{\partial\Theta_{a'a}}\right) e^B\right]_{\Theta_{j'j} \to \delta_{j'j}}. \tag{II. 13}$$

However,

$$[e^B]_{\Theta_{jj'} \to \delta_{jj'}} = \exp(-\beta\Gamma) \tag{II. 14}$$

and, therefore, we obtain

$$\langle 0_1\rangle = \sum_{k_a k_a'} \left[\frac{\partial B}{\partial\Theta_{a'a}}\right]_{\Theta_{j'j} \to \delta_{j'j}} \tag{II. 15}$$

and

$$\langle 0_2\rangle = \sum_{k_a k_a'} \sum_{k_b k_b'} \left[\frac{\partial^2 B}{\partial\Theta_{b'b}\partial\Theta_{a'a}} + \frac{\partial B}{\partial\Theta_{b'b}} \frac{\partial B}{\partial\Theta_{a'a}}\right]_{\Theta_{j'j} \to \delta_{j'j}}. \tag{II. 16}$$

We see from Eqs. (II. 15) and (II. 16) that the expectation values of one-particle and two-particle operators can be written entirely in terms of antisymmetrized cluster expansions.

If we wish to find an expression for the expectation value of the magnetization, we simply write

$$\langle M(\mathbf{r})\rangle = \sum_{k_a' k_a} \left[\frac{\partial B(\Theta, \beta, g, H_0, H_r)}{\partial \Theta_{a'a}}\right]_{\Theta_{j'j} = \delta_{j'j}}, \qquad \text{(II. 17)}$$

where we let

$$\frac{\partial \Theta_{j'j}}{\partial \Theta_{a'a}} = \delta_{\mathbf{k}_j, \mathbf{k}_a} \delta_{\mathbf{k}_j', \mathbf{k}_a'} \langle \mathbf{k}_j' | \mathbf{r} \rangle \langle \mathbf{r} | \mathbf{k}_j \rangle \mu s_j \delta_{s_j s_a} \delta_{s_j' s_a'}, \qquad \text{(II. 18)}$$

where μ is the magnetic moment and $_j$ is the z component of spin (cf. RI, Sec. II, for a complete definition of the notation).

III. MAGNETIZATION IN TERMS OF THE REACTION MATRIX

We may now follow the same steps as in RI and obtain an expression for the magnetization in terms of the reaction matrix. The result is

$$\langle M(\mathbf{r})\rangle = \sum_{Q=1}^{\infty} (\text{all different } Q\text{th order } A\text{-matrix } 0_M \text{ diagrams}). \qquad \text{(III. 1)}$$

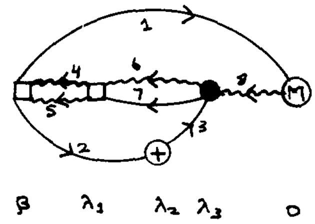
A Qth order *A-matrix* 0_M *diagram* contains one M vertex and a collection of Q A-vertices, D-vertices, and ΔH vertices. The vertices are ordered from left to right, ending with the M vertex, which always appears at the extreme right; and they are entirely interconnected by solid and wavy lines. A vertices and D vertices each have two lines entering and two lines leaving. ΔH vertices and the M vertex only have one line entering and one line leaving. Wavy lines must be directed to the left, while solid lines may be directd either to the left or right. D vertices with two solid lines leaving must appear at the extreme left so that the solid lines which leave the vertex are directed to the right. No wavy line double bonds may appear *except* internally in the D vertices. Two Qth order *A-matrix* 0_M *diagrams* differ if they have different topological structure, or if they have the same topological structure, but the lines are of different types or directions, or the D vertices have different temperature labeling.

Rules for evaluating the Qth order A-matrix 0_M diagrams are given in Appendix A.

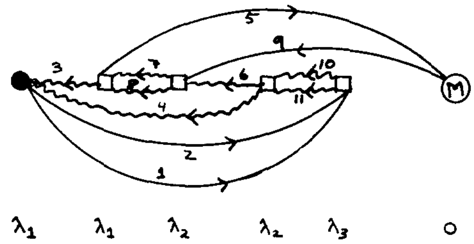
Some examples of *A-matrix* 0_M *diagrams* are given in Fig. 1. Algebraic expressions corresponding to the diagrams in Fig. 1 are given below:

$$\text{Fig. 1(a)} = \epsilon^6 \sum_{k_1 \cdots k_8} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 d\lambda_3\, \theta(\beta-\lambda_2)\,\theta(\beta-\lambda_1)$$
$$\times \theta(\lambda_2-\lambda_3)\,\theta(\lambda_1-\lambda_3)\,\theta(\lambda_3)\exp(\beta\omega_1')$$
$$\times \exp[-(\beta-\lambda_1)(\omega_4'+\omega_5')]\exp[(\beta-\lambda_2)\omega_2']$$
$$\times \exp[(\lambda_2-\lambda_3)\omega_3']\exp[-(\lambda_1-\lambda_3)(\omega_6'+\omega_7')]$$
$$\times \exp(-\lambda_3\omega_8')\,(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_7)(\epsilon\nu_3)$$
$$\times D\begin{pmatrix} k_1k_2 \\ k_4k_5 \\ k_6k_7 \end{pmatrix} \Delta H_+ \begin{pmatrix} k_3 \\ \\ k_2 \end{pmatrix} A \begin{pmatrix} k_6k_7 \\ \\ k_3k_8 \end{pmatrix} M \begin{pmatrix} k_8 \\ \\ k_1 \end{pmatrix}, \qquad \text{(III. 2)}$$

$$\text{Fig. 1(b)} = \epsilon^5 \sum_{k_1 \cdots k_{11}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 d\lambda_3\, \theta(\beta-\lambda_1)\,\theta(\lambda_1-\lambda_2)$$
$$\times \theta(\lambda_2-\lambda_3)\,\theta(\lambda_1-\lambda_3)\,\theta(\lambda_1)\,\theta(\lambda_2)(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_5)(\epsilon\nu_9)$$



(a)



(b)

FIG. 1. A-matrix 0_M diagrams.

$$\times \exp[-(\lambda_1-\lambda_2)\omega_4']\exp[(\lambda_1-\lambda_3)(\omega_1'+\omega_2')]$$
$$\times \exp[-(\lambda_1-\lambda_2)(\omega_7'+\omega_8')]\exp(\lambda_1\omega_5')\exp(-\lambda_2\omega_9')$$
$$\times \exp[-(\lambda_2-\lambda_3)(\omega_{10}'+\omega_{11}')]\, A\begin{pmatrix} k_1k_2 \\ k_3k_4 \end{pmatrix}$$
$$\times D\begin{pmatrix} k_3k_5 \\ k_7k_8 \\ k_6k_9 \end{pmatrix} D\begin{pmatrix} k_4k_6 \\ k_{10}k_{11} \\ k_1k_2 \end{pmatrix} M\begin{pmatrix} k_9 \\ \\ k_5 \end{pmatrix}. \qquad \text{(III. 3)}$$

We note that just as for the case of the grand potential, the expression for the magnetization can also be divided into Type I, Type II, and Type III 0_M diagrams (cf. RI Sec. VII). Rules (A. i)–(A. xi) are the same as those used to evaluate A-matrix 0 diagrams except that we have generalized them to include the possibility of M vertices. The rules for evaluating expectation values of one- and two-body operators will be essentially the same as Rules (A. i)–(A. xi) except that we must generalize them to include vertices corresponding to the operators $\hat{0}_i$ and $\hat{0}_{ij}$.

IV. TYPE I A-MATRIX 0 DIAGRAMS AND 0_M DIAGRAMS

We shall now look in more detail at the way in which the external magnetic field affects the grand potential and the magnetization. For simplicity, we shall only study the Type I 0 diagrams and 0_M diagrams since these can be studied using Matsubara techniques.

We first define the following quantities:

$$\Gamma(\beta, g, H_0, H_r)_{\rm I} = -\frac{1}{\beta}\sum(\text{all different Type I 0 diagrams}) \qquad \text{(IV. 1)}$$

and

$$\langle M(\mathbf{r})\rangle_{\rm I} = \sum (\text{all different Type I } 0_M \text{ diagrams}). \qquad \text{(IV. 2)}$$

We remember that Type I 0 diagrams and 0_M diagrams contain no double bond structures and no D vertices. The Type I diagrams have the property that for each Type I diagram with one or more wavy lines, there is another Type I diagram with the same topological structure but with a wavy line replaced by a solid line.

From Rule (A. ix) (see Appendix A) we know that solid lines with momentum and spin k_j, directed to the left, receive a factor $\epsilon\nu_j$ while wavy lines directed to the left receive a factor 1. If we add together all diagrams with the same topological structure and line directions, but different line types, we lose the distinction between solid and wavy lines. We can then associate a factor $(1+\epsilon\nu_j)$ to all lines of momentum and spin k_j directed to the left, and a factor $\epsilon\nu_j$ to all lines of momentum and spin k_j directed to the right.

We next can use the simple device, introduced by Matsubara,[3-5] to add together all diagrams with the same topological structure regardless of the direction of lines. We introduce the propagator

$$\mathcal{D}_j(\lambda_1-\lambda_2)=[(1+\epsilon\nu_j)\theta(\lambda_1-\lambda_2)+(\epsilon\nu_j)\theta(\lambda_2-\lambda_1)] \times\exp[-(\lambda_1-\lambda_2)\omega_j']. \quad (\text{IV. }3)$$

In terms of the propagator $\mathcal{D}_j(\lambda_1-\lambda_2)$, we can write the following expression for $\Gamma(\beta,g,\mathcal{H}_0,\mathcal{H}_r)$ and $\langle M(\mathbf{r})\rangle_{\text{I}}$:

$$\Gamma(\beta,g,\mathcal{H}_0,\mathcal{H}_r)_{\text{I}}=-\frac{1}{\beta}\sum(\text{all different Type I contracted } 0 \text{ diagrams}) \quad (\text{IV. }4)$$

and

$$\langle M(\mathbf{r})\rangle_{\text{I}}=\sum(\text{all different Type I contracted } 0_M \text{ diagrams}). \quad (\text{IV. }5)$$

A *Type I contracted* 0 *diagram* contains a collection of A vertices and ΔH vertices completely connected by directed solid lines. A *Type I contracted* 0_M *diagram* contains one M vertex and a collection of A vertices and ΔH vertices completely connected by directed solid lines An A matrix has two lines entering and two lines leaving, while ΔH vertices and the M vertex have one line entering and one line leaving. The M vertex must appear on the extreme right, but the A vertices and ΔH vertices can have any desired order with respect to one another. No double bonds of any type can appear in a *Type I contracted* 0 *diagram* or 0_M *diagram*. Two *Type I contracted* 0 *diagrams* differ if they have different topological structure.

Algebraic expressions may be associated with the *Type I contracted* 0 *diagrams* or 0_M *diagrams* according to the rules in Appendix B.

Examples of *contracted* 0_M *diagrams* are given in Fig. 2. Algebraic expressions corresponding to the diagrams in Fig. 2(a) and 2(b) are given below:

$$\text{Fig. 2(a)}=\epsilon^4\sum_{k_1\cdots k_6}\int_0^\beta\int_0^\beta\int_0^\beta d\lambda_1\,d\lambda_2\,d\lambda_3(\epsilon\nu_6)(1+\epsilon\nu_5)$$
$$\times\exp[-\lambda_3(\omega_5'-\omega_6')\mathcal{D}_1(\lambda_1-\lambda_2)\mathcal{D}_2(\lambda_2-\lambda_1)$$
$$\times\mathcal{D}_3(\lambda_2-\lambda_3)\mathcal{D}_4(\lambda_3-\lambda_2)$$

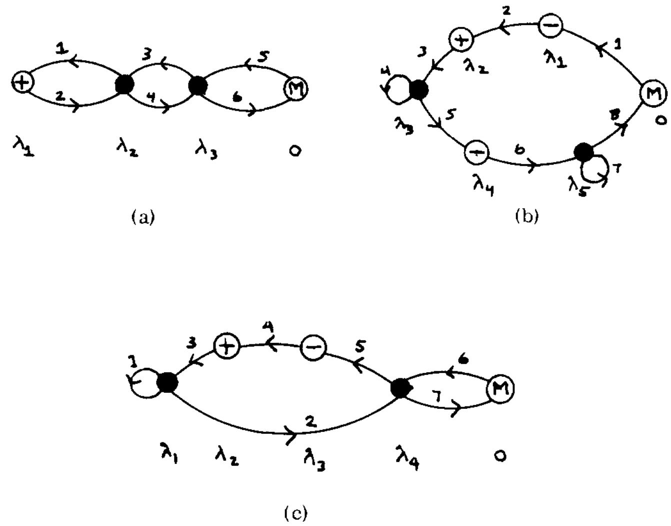


FIG. 2. Contracted 0_M diagrams.

$$\times\Delta H_+\begin{pmatrix}k_2\\k_1\end{pmatrix}A\begin{pmatrix}k_1k_4\\k_2k_3\end{pmatrix}A\begin{pmatrix}k_3k_6\\k_4k_5\end{pmatrix}M\begin{pmatrix}k_5\\k_6\end{pmatrix}, \quad (\text{IV. }6)$$

$$\text{Fig. 2(b)}=\epsilon^8\sum_{k_1\cdots k_8}\int_0^\beta\cdots\int_0^\beta d\lambda_1\cdots d\lambda_5$$
$$\times\exp(-\lambda_1\omega_1')\exp(+\lambda_5\omega_8')(1+\epsilon\nu_1)(\epsilon\nu_8)$$
$$\times(\epsilon\nu_4)(\epsilon\nu_7)\mathcal{D}_6(\lambda_5-\lambda_4)$$
$$\mathcal{D}_2(\lambda_2-\lambda_1)\mathcal{D}_3(\lambda_3-\lambda_2)\mathcal{D}_5(\lambda_4-\lambda_3)$$
$$\times\Delta H_-\begin{pmatrix}k_2\\k_1\end{pmatrix}\Delta H_+\begin{pmatrix}k_3\\k_2\end{pmatrix}\Delta H_-\begin{pmatrix}k_6\\k_5\end{pmatrix}A\begin{pmatrix}k_4k_5\\k_4k_3\end{pmatrix}A\begin{pmatrix}k_7k_8\\k_7k_6\end{pmatrix}$$
$$\times M\begin{pmatrix}k_1\\k_8\end{pmatrix}. \quad (\text{IV. }7)$$

Rules (B. i)–(B. ix) (see Appendix B) can be used to evaluate both the *Type I* 0 *diagrams* and the *Type I* 0_M *di agrams*.

V. FOURIER EXPANSION OF GRAND POTENTIAL AND MAGNETIZATION

We can now expand the grand potential and the magnetization in a Fourier series. If we use Eq. (IV. 3), we may show that $\mathcal{D}_j(\lambda)$ is a periodic functin of λ with period 2β. We can write it therefore as a Fourier seires

$$\mathcal{D}_j(\lambda)=\frac{1}{\beta}\sum_{n_j=-\infty}^{\infty}\frac{\exp(iz_j\lambda)}{(iz_j+\omega_j')}, \quad (\text{V. }1)$$

where $z_j=n_j\pi/\beta$ (n is an odd integer for fermions). We also note that

$$\int_0^\beta d\lambda\exp\left[i(\pi\lambda/\beta)\left(\sum_i n_i\right)\right]=\beta\delta\left(\sum_i n_i\right), \quad \text{for } \sum_i n_i \text{ even.} \quad (\text{V. }2)$$

If we substitute Eq. (V. 1) into the expressions for $\Gamma(\beta,g,\mathcal{H}_0,\mathcal{H}_r)_{\text{I}}$ and $\langle M(\mathbf{r})\rangle_{\text{I}}$ obtained by using Rules (B. i)–(B. ix), and integrate over temperature, we obtain the rules presented in Appendix C for evaluating the Type I contracted 0 *diagrams* and 0_m *diagrams* in terms of a Fourier transformed expression.

Algebraic expressions for the *contracted* 0_M *diagrams* in Fig. 2(a) and Fig. 2(b) obtained using Rules (C. i)—(C. x) (see Appendix C) are given below:

$$\text{Fig. 2(a)} = \frac{\epsilon^4}{\beta^2}\sum_{k_1\cdots k_6}\int_0^\beta d\lambda_3 \sum_{n_1,n_3}(\epsilon\nu_6)(1+\epsilon\nu_5)\exp[-\lambda_3(\omega_5'-\omega_6')$$
$$\times\left(\frac{1}{iz_1+\omega_1'}\right)\left(\frac{1}{iz_1+\omega_2'}\right)\left(\frac{1}{iz_3+\omega_3'}\right)\left(\frac{1}{iz_3+\omega_4'}\right)$$
$$\times\Delta H_+\binom{k_2}{k_1}A\binom{k_1k_4}{k_2k_3}A\binom{k_3k_6}{k_4k_5}M\binom{k_5}{k_6}, \qquad (\text{V. }3)$$

$$\text{Fig. 2(b)} = \frac{\epsilon^8}{\beta}\sum_{k_1\cdots k_8}\sum_{n_2}\int_0^\beta\int_0^\beta d\lambda_1 d\lambda_5 \exp[-\lambda_1(\omega_1'+iz_2)$$
$$\times\exp[\lambda_5(\omega_8'-iz_6)\,(1+\epsilon\nu_1)(\epsilon\nu_8)(\epsilon\nu_4)(\epsilon\nu_7)$$
$$\times\left(\frac{1}{iz_2+\omega_2'}\right)\left(\frac{1}{iz_2+\omega_6'}\right)\left(\frac{1}{iz_2+\omega_3'}\right)\left(\frac{1}{iz_2+\omega_5'}\right)$$
$$\times\Delta H_-\binom{k_2}{k_1}\Delta H_+\binom{k_3}{k_2}\Delta H_-\binom{k_6}{k_5}$$
$$\times A\binom{k_4k_5}{k_4k_3}A\binom{k_7k_8}{k_7k_6}M\binom{k_1}{k_8}. \qquad (\text{V. }4)$$

VI. SELF-ENERGY STRUCTURES

When we study equilibrium systems at low temperatures we must take care to remove *self-energy structures* which appear in expressions for the thermodynamic properties. The reason is that *self-energy structures* give rise to terms with polynomial dependence on β. (One can show this by doing the temperature integrations for diagrams with self-energy structures.) These polynomial terms can destroy the convergence of expansions for the thermodynamic properties.

Self-energy structures are collections of vertices which can be removed from a diagram by cutting two lines of the same momentum, spin, and energy (k, z). An *irreducible self-energy structure* has no internal line with the same energy, momentum, and spin as the lines which enter and leave the *self-energy structure*.

Irreducible self-energy structures contain both ΔH vertices and A vertices. Some examples are given in Fig. 3. We may use the rules of Sec. V to write the sum of all possible irreducible self-energy structures which can appear on a line of momentum and spin k_1 and energy z_1 as

$$M_+(z_1, \mathbf{k}_1+\mathbf{k}_0, s_1) + M_-(z_1\mathbf{k}_1-\mathbf{k}_0, s_1) + \Sigma(\mathbf{k}_1, z_1, \mathbf{k}_0, s_1)$$
$$= \sum [\text{all different irreducible self-energy structures of momentum, spin, and energy, } (k_1, z_1)]. \qquad (\text{VI. }1)$$

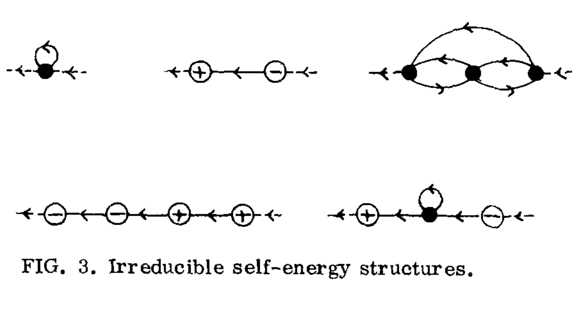


FIG. 3. Irreducible self-energy structures.

The first vertex a line touches upon entering an irreducible self-energy structure will be a ΔH_+ vertex, a ΔH_- vertex, or an A vertex. The three functions in Eq. (VI. 1) distinguish between these three cases. They are defined as follows:

$$M_\pm(z_1, \mathbf{k}_1\pm\mathbf{k}_0, s_1) = \frac{(\frac{1}{2}\mu H_r)^2}{iz_1+\omega'(\mathbf{k}_1\pm\mathbf{k}_0)+M_\pm(z_1,\mathbf{k}_1\pm 2\mathbf{k}_0, s_1)+\Sigma(\mathbf{k}_1+\mathbf{k}_0, z_1, s_1)} \qquad (\text{VI. }2)$$

and

$\Sigma(k_1, z_1, s_1) = \sum$ (all different Type I contracted irreducible 1 diagrams with at least one A vertex and an equal number of ΔH_+ and ΔH_- vertices which cannot be cut into two parts by cutting one line). (VI. 3)

A Type I *contracted* 1 *diagram* is the same as a Type I *contracted* 0 *diagram* except that it contains one external line leaving, and it is not possible to cut it into two pieces by cutting an internal line with the same values (k_1, z_1) as the external lines.

Algebraic expressions are assigned to Type I contracted 1 diagrams according to Rules (B. i)—(B. ix) or (C. i)—(C. xi) but no factors are assigned to the external lines.

In addition to self-energy structures, there will be lines which contain unequal numbers of ΔH_+ and ΔH_- vertices. We must also resum self-energy structures on these lines but we must be careful to do it in such a way that we do not over count the original lines. Let us consider the case of a line with n ΔH_- vertices and $n+1$ ΔH_+ vertices. We can sum over all such lines and obtain a single resummed line containing one ΔH_+ vertex and resummed propagators. We shall use the following convention. Denote the momentum spin and energy of the line entering the collection of ΔH vertices by (k_1, z_1). We will choose the extra ΔH_+ vertex to be that ΔH_+ vertex for which a line (k_1, z_1) enters, but no line (k_1, z_1) ever appears again after it leaves the ΔH_+ vertex. An example of this choice is given in Fig. 4. Because of this convention, the resummed propagators on either side of the ΔH_+ vertex may be different.

We now can give the following convention for resumming a line with m ΔH_- vertices and n ΔH_+ vertices such that $n-m=l$. If we sum over all lines containing m ΔH_- vertices and n ΔH_+ vertices, we obtain a resummed line with $|l|\,\Delta H_\pm$ vertices. The line entering the first $\Delta H_\pm$ vertex is assigned a propagator

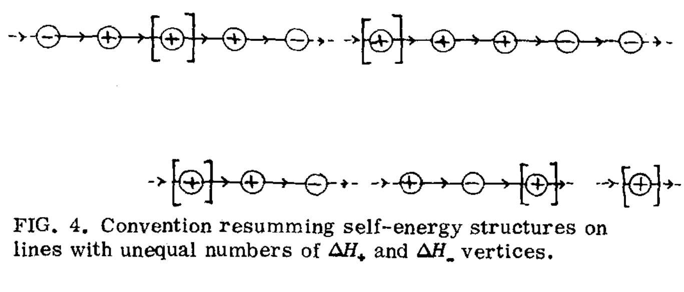


FIG. 4. Convention resumming self-energy structures on lines with unequal numbers of ΔH_+ and ΔH_- vertices.

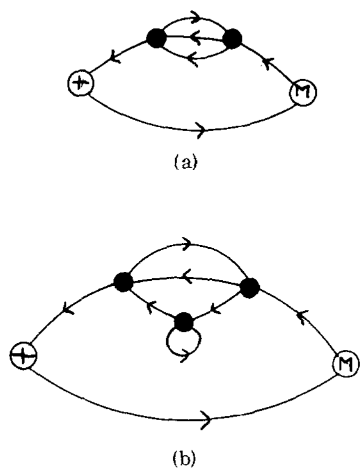


FIG. 5. Contracted 0_M diagrams with self-energy structures.

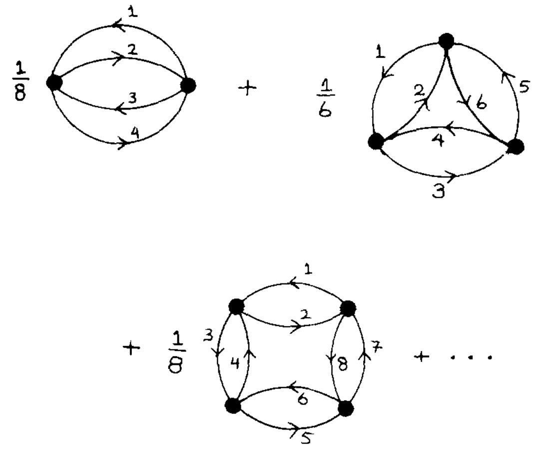


FIG. 6. Sum of polarization diagrams and their symmetry numbers.

$$[iz_1 + \omega'(k_1) + M_+(z_1, \mathbf{k}_1 + \mathbf{k}_0, s_1) + M_-(z_1, \mathbf{k}_1 - \mathbf{k}_0, s_1)$$

$$+ \Sigma(\mathbf{k}_1, z_1, \mathbf{k}_0, s_1)]^{-1}.$$

The line connecting the jth $\Delta H_\pm$ vertex to the $(j+1)$st $\Delta H_\pm$ vertex or to an A vertex is assigned a propagator

$$[iz_1 + \omega'(\mathbf{k}_1 \pm j\mathbf{k}_0) + M_\pm(z_1, \mathbf{k}_1 \pm (j+1)\mathbf{k}_0, s_1)$$

$$+ \Sigma(\mathbf{k}_1 \pm j\mathbf{k}_0, z_1, \mathbf{k}_0, s_1)]^{-1}.$$

VII. RESUMMATION OF THE GRAND POTENTIAL AND THE MAGNETIZATION

As we have noted in Sec. VI, at low temperatures resummation of self-energy effects is essential for the convergence of perturbation or binary expansions of thermodynamic quantities. In liquid He^3 it leads to the Landau quasiparticle interpretation of the liquid. However, any attempt to resum the self-energy structures in the Type I *A-matrix* 0 *diagrams* leads to difficulties because of the exclusion of double bond structures.

We can see this from the diagrams in Fig. 5. Figure 5(a) has a solid line double bond and therefore is not included among the Type I diagrams. Figure 5(b) is a Type I diagram. It has no double bond, but if we remove the single vertex *self-energy structure*, we obtain a diagram with a double bond. In order to resum systematically propagators in the Type I A-matrix 0 diagrams or 0_m diagrams, the double bond structures must be included because they form the "skeleton" of many of the Type I diagrams which contain self-*energy* structures.

We can resum all self-energy effects in Type I A-matrix 0 diagrams and 0_m diagrams if in the resummed diagrams we allow double bond structures, but explicitly subtract a "bare" double bond for each resummed double bond that appears. We then obtain the following expressions for $\Gamma(\beta, g, H_0, H_r)_{\mathrm{I}}$ and $\langle M(\mathbf{r})\rangle_{\mathrm{I}}$:

$$\Gamma(\beta, g, H_0, H_r)_{\mathrm{I}} = \frac{1}{\beta}\sum \text{(all different irreducible 0 diagrams)} \tag{VII.1}$$

and

$$\langle M(\mathbf{r})\rangle_{\mathrm{I}} = \sum \text{(all different irreducible } 0_m \text{ diagrams)}. \tag{VII.2}$$

An irreducible 0 diagram (0_m diagram) is the same as a contracted 0 diagram (0_m diagram) except that double bonds are allowed, but self-energy structures are not.

Algebraic expressions may be associated to the irreducible 0 diagrams and 0_m diagrams according to the rules in Appendix D.

We can now make some general comments about the overall dependence of $\langle M(\mathbf{r})\rangle$ on $\mathbf{r}$. Because the A vertices conserve momentum one can show, by explicitly summing over all momentum delta functions in a diagram, that a diagram containing m ΔH_+ vertices and n ΔH_- vertices will have a spatial dependence $\exp[i(m-n)\times\mathbf{k}_0\cdot\mathbf{r}]$. Since any values of m and n are possible, the magnetization $\langle M(\mathbf{r})\rangle$ will depend on *all* harmonics of the parameter $\mathbf{k}_0$. If we turn off the field H_r, we will still have a constant magnetization due to the presence of the term $s_j\mu H_0$ in the energies ω_j'. The grand potential $\Gamma(\beta, g, H_0, H_r)$ will only depend on equal numbers of ΔH_+ and ΔH_- vertices because of overall momentum conservation in the 0 diagrams.

In the next section, we shall apply the above results to the case of the polarization diagrams.

VIII. POLARIZATION DIAGRAMS

We can now use the rules of Appendix D to calculate the contribution to the grand potential coming from the polarization diagrams. The polarization diagrams describe spin and density fluctuations in the Fermi fluid. Brinkman and Engelsberg[6] have calculated the effect of spin and density fluctuations on the grand potential and heat capacity using perturbation theory and including a constant external magnetic field. Their expressions can only be considered as phenomenological if they are applied to liquid He^3, because for a system of particles with a large repulsive core expansions in powers of the potential are divergent. Our expressions for the polarization diagrams involve only reaction matrices and are well behaved for particles with large repulsive cores. Furthermore we shall include the effect of both a constant external magnetic field and a spatially varying external magnetic field. Expressions for the polarization diagrams in terms of reaction matrix expansions have been studied by Reichl and Tuttle.[7] In these papers, the self-energy effects are summed using generalized Hartree—Fock techniques and as a result energy denominators are introduced which are not well defined in the thermodynamic limit where the free particle energy spectrum can become continuous. We shall show below that the Matsubara method does not lead to the appearance of undefined energy denominators.

In Fig. 6, we have drawn the polarization diagrams, together with their symmetry numbers, up to fourth order. If we use the rules in Appendix D, we obtain the following expression for the infinite sum of polarization diagrams:

$$\Gamma(\beta, g, H_0, H_r)_{\rm pol} = \sum_{p=2}^{\infty} \frac{1}{2p}\frac{\epsilon^p}{\beta^{2p}} \sum_{k_1\cdots k_{2p}} \sum_{n_1\cdots n_{2p}} \left[\beta\delta_{n_1+n_{2p}, n_2+n_{2p-1}} A\begin{pmatrix}1, & 2p\\ 2, & 2p-1\end{pmatrix}\right]\left[\beta\delta_{n_2+n_3, n_1+n_4} A\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}\right]$$
$$\times\cdots\times\left[\beta\delta_{n_{2p-2}+n_{2p-1}, n_{2p-3}+n_{2p}} A\begin{pmatrix}2p-2, & 2p-1\\ 2p-3, & 2p\end{pmatrix}\right]\left(\frac{1}{iz_1+\omega'(k_1)+S(z_1,k_1,H_r)}\right)\left(\frac{1}{iz_2+\omega'(k_2)+S(z_2,k_2,H_r)}\right)$$
$$\times\cdots\times\left(\frac{1}{iz_{2p}+\omega'(k_{2p})+S(z_{2p},k_{2p},H_r)}\right) - \Gamma_{(2)}(\beta, g, H_0, H_r)_{\rm pol}, \tag{VIII. 1}$$

where

$$S(z_1, k_1, H_r) \equiv M_+(z_1, \mathbf{k}_1+\mathbf{k}_0, s_1, H_r) + M_-(z_1, \mathbf{k}_1-\mathbf{k}_0, s_1, H_r) + \Sigma(z_1, k_1, H_r). \tag{VIII. 2}$$

We have suppressed everywhere the dependence on H_0 since it only appears in $\omega'(k)$. In Eq. (VIII. 1), the second order contribution $\Gamma_{(2)}(\beta, g, H_0, H_r)_{\rm pol}$ has been subtracted because it is counted twice in the sum. If we now note that the A matrices conserve momentum (cf. Sec. VI and Appendix B of RI), we can sum over all even numbered momenta $\mathbf{k}_j$ and even numbered integers n_j to obtain

$$\Gamma(\beta, g, H_0, H_r)_{\rm pol} = \sum_{p=2}\frac{1}{2p}\frac{\epsilon^p}{\beta^p}\sum_{s_1\cdots s_{2p}}\sum_{\mathbf{k}_1,\mathbf{k}_3,\ldots,\mathbf{k}_{2p-1}}\sum_{\mathbf{q}}\sum_{n_1,n_3,\ldots,n_{2p-1}}\sum_{l} A\begin{pmatrix}\mathbf{k}_1, s_1; & \mathbf{k}_{2p-1}+\mathbf{q}, s_{2p}\\ \mathbf{k}_1+\mathbf{q}, s_2; & \mathbf{k}_{2p-1}, s_{2p-1}\end{pmatrix} A\begin{pmatrix}\mathbf{k}_1+\mathbf{q}, s_2; & \mathbf{k}_3, s_3\\ \mathbf{k}_1, s_1; & \mathbf{k}_3+\mathbf{q}, s_4\end{pmatrix}$$
$$\times\cdots\times A\begin{pmatrix}\mathbf{k}_{2p-3}+\mathbf{q}, s_{2p-2}; & \mathbf{k}_{2p-1}, s_{2p-1}\\ \mathbf{k}_{2p-3}, s_{2p-2}; & \mathbf{k}_{2p-1}+\mathbf{q}, s_{2p}\end{pmatrix}\times\left(\frac{1}{iz_1+\omega'_{s_1}(\mathbf{k}_1)+S(z,\mathbf{k}_1,s_1,H_r)}\right)$$
$$\times\left(\frac{1}{iz_1+i\pi l/\beta+\omega'_{s_2}(\mathbf{k}_1+\mathbf{q})+S(z_1+l\pi/\beta,\mathbf{k}_1+\mathbf{q}, s_2, H_r)}\right)\times\cdots\times\left(\frac{1}{iz_{2p-1}+\omega'_{s_{2p-1}}(\mathbf{k}_{2p-1})+S(z_{2p-1},\mathbf{k}_{2p-1}, s_{2p-1}, H_r)}\right)$$
$$\times\left(\frac{1}{iz_{2p-1}+i\pi l/\beta+\omega'_{s_{2p}}(\mathbf{k}_{2p-1})+S(z_{2p-1}+\pi l/\beta, \mathbf{k}_{2p-1}+\mathbf{q}, s_{2p}, H_r)}\right) - \Gamma_{(2)}(\beta, g, H_0, H_r)_{\rm pol}, \tag{VIII. 3}$$

where $\mathbf{q}$ is the momentum transfer and l is an even integer. The sum over l is taken over all even integers.

In the present paper, we are most concerned about the way in which self-energy effects from the external field and the medium affect the polarization diagrams. As has been discussed in Ref. 7, p. 196, if the reaction matrix is peaked fairly sharply about small values of momentum transfer $\mathbf{q}$ (and if H_r and $\mathbf{k}_0$ are not too large), then the momenta $\mathbf{k}_1, \mathbf{k}_3, \ldots, \mathbf{k}_{2p-1}$ in Eq. (VIII. 3) will be restricted to the vicinity of the Fermi surface. Furthermore, if the reaction matrix $A\binom{\mathbf{k}_1+\mathbf{q}, s_2; \mathbf{k}_3, s_3}{\mathbf{k}_1, s_1; \mathbf{k}_3+\mathbf{q}, s_4}$ is a slowly varying function of $\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_3 = \cos\theta_{13}$, then we can expand the reaction matrix in a Legendre series involving the angle θ_{13}. To first approximation, we can keep only the first term in the Legendre series and write

$$A\begin{pmatrix}\mathbf{k}_1+\mathbf{q}, s_2; \mathbf{k}_3, s_3\\ \mathbf{k}_1, s_1; \mathbf{k}_3+\mathbf{q}, s_4\end{pmatrix} \sim A(k_f, \mathbf{q}: s_2, s_3; s_1, s_4), \tag{VIII. 4}$$

where k_f is the Fermi momentum.

For the purposes of the present discussion we shall make this approximation here. We can then compare our results with those of Brinkman and Engelsberg and those of Reichl and Tuttle. We hope to discuss the accuracy of this approximation in greater detail in later papers.

If we use the approximation in Eq. (VIII. 4), we can write Eq. (VIII. 3) in the form

$$\Gamma(\beta, g, H_0, H_r)_{\rm pol} = \sum_{p=2}^{\infty}\frac{1}{2p}\frac{\epsilon^p}{\beta^p}\sum_{s_1, s_{2p}}\sum_{\mathbf{q}}\sum_{l} A(k_f, \mathbf{q}: s_1, s_{2p}; s_2, s_{2p-1}) A(k_f\mathbf{q}: s_2, s_3; s_1, s_4)$$
$$\times\cdots\times A(k_f\mathbf{q}: s_{2p-2}, s_{2p-1}; s_{2p-3}, s_{2p})\,\chi(\mathbf{q}, l; l_1, s_2, H_r)\chi(\mathbf{q}, l; s_3, s_4, H_r)$$
$$\times\cdots\times\chi(\mathbf{q}, l; s_{2p-1}, s_{2p}, H_r) - \Gamma_{(2)}(\beta, g, H_0, H_r)_{\rm pol}, \tag{VIII. 5}$$

where

$$\chi(\mathbf{q},l;s_1,s_2,H_r)\equiv\sum_{\mathbf{k}_1}\sum_{n_1}\left(\frac{1}{iz_1+\omega'_{s_1}(\mathbf{k}_1)+S(z_1,\mathbf{k}_1,s_1,H_r)}\right)\left(\frac{1}{iz_1+i\pi l/\beta+\omega'_{s_2}(\mathbf{k}_1+\mathbf{q})+S(z_1+\pi l/\beta,\mathbf{k}_1+\mathbf{q},s_2,H_r)}\right) \quad \text{(VIII. 6)}$$

corresponds to a propagator of a particle hole pair with total momentum $\mathbf{q}$, "energy" $i\pi l/\beta$, and total spin s_1+s_2. Self-energy effects due to the medium and the external fields have been included.

In order to evaluate Eq. (VIII. 6), we must find the poles of the particle propagators appearing there. This, in general, will involve some sort of approximation. We shall consider the simplest approximation which illustrates the effect of the external fields on the propagators. If spin and density fluctuations in the medium give the dominant contribution to the grand potential then they will also give a large contribution to the self-energy $\Sigma(z_1,k_1,s_1,H_r)$. However, for the present purposes we shall neglect contributions to $\Sigma(z_1,k_1,s_1,H_r)$ from the polarization diagrams and approximate it by the lowest order contribution

$$\Sigma(z_1,\mathbf{k}_1,s_1,H_r)\sim\sum_{k_2}A\begin{pmatrix}k_1k_2\\k_1k_2\end{pmatrix}(\epsilon\nu(k_2))\equiv A(k_1). \quad \text{(VIII. 7)}$$

If we had included in $\Sigma(z_1,\mathbf{k}_1,s_1,H_r)$ an infinite sum of polarization diagrams then it would depend on the energy iz. The self-energies $M_\pm(z_1,\mathbf{k}_1\pm\mathbf{k}_0,s_i,H_r)$ are essentially continued fractions. We shall assume that the field H_r is not too large and that $M_\pm(z_1,\mathbf{k}_1\pm\mathbf{k}_0,s_1,H_r)$ can be approximated by the following expression

$$M_\pm(z_1,\mathbf{k}_1\pm\mathbf{k}_0,s_1,H_r)=\frac{(\frac{1}{2}\mu H_r)^2}{(iz_1+\omega'_{s_1}(\mathbf{k}_1\pm\mathbf{k}_0)+A(k_1))}\,. \quad \text{(VIII. 8)}$$

Then the propagators in Eq. (VIII. 6) can be written in the form

$$(iz_1+\omega'(k_1)+S(z_1,k_1,H_r))^{-1}=\left[iz_1+\omega'(k_1)+A(k_1)+(\tfrac{1}{2}\mu H_r)^2\left(\frac{1}{iz_1+\omega'_{s_1}(\mathbf{k}_1+\mathbf{k}_0)+A(k_1)}+\frac{1}{iz_1+\omega'_{s_1}(\mathbf{k}_1-\mathbf{k}_0)+A(k_1)}\right)\right]^{-1}. \quad \text{(VIII. 9)}$$

To find the poles of the propagator in Eq. (VIII. 9), we must solve a cubic equation. Let us assume that the applied field has a very long wavelength. Then $k_0^2/m\ll H_r\mu$ and we can write the propagator in the form

$$(iz_1+\omega'(k_1)+S(z_1,k_1,H_r))^{-1}\approx\left(\frac{(iz_1+\omega'_{s_1}(\mathbf{k}_1+\mathbf{k}_0)+A(k_1))(iz_1+\omega'_{s_1}(\mathbf{k}_1-\mathbf{k}_0)+A(k_1))}{(iz_1+\omega'(k_1)+A(k_1)+k_0^2/m)(iz_1+\omega'(k_1)+A(k_1)-i\mu H_r/\sqrt{2})(iz_1+\omega'(k_1)+A(k_1)+i\mu H_r/\sqrt{2})}\right). \quad \text{(VIII. 10)}$$

We see that if we include the lowest order effects due to the spatially varying external field, and assume that the external field has a long wavelength, the poles of the propagator are shifted from their value when $H_r=0$ by amounts which depend independently on the wavelength of the external field and the external field itself. The shift due to the external field is imaginary.

If we use the expression for the propagator given in Eq. (VIII. 10) and use standard techniques for evaluating $\chi(\mathbf{q}l;s_1s_2,H_r)$ (cf. Ref. 5, Sec. 13) we obtain the following expression:

$$\begin{aligned}\chi(\mathbf{q}l;s_1s_2H_r)\equiv\Bigg(&\nu(\Delta+C)\frac{(\Delta+C-\Delta_+)(\Delta+C-\Delta_-)(\xi+\Delta+C-\Delta'_+)(\xi+\Delta+C-\Delta'_-)}{(C-B)(C+B)(\xi+\Delta-\Delta')(\xi+\Delta-\Delta'+C-B)(\xi+\Delta+C-\Delta'+B}\\&+\nu(\Delta+B)\frac{(\Delta+B-\Delta_+)(\Delta+B-\Delta_-)(\Delta+B+\xi-\Delta'_+)(\Delta+B+\xi-\Delta'_-)}{(B-C)(2B)(\xi+\Delta-\Delta'+B-C)(\xi+\Delta-\Delta')(\xi+\Delta-\Delta'+2B)}\\&+\nu(\Delta-B)\frac{(\Delta-B-\Delta_+)(\Delta-B-\Delta_-)(\xi+\Delta-B-\Delta'_+)(\xi+\Delta-B-\Delta'_-)}{(C-B)(2B)(\xi+\Delta-\Delta'-B-C)(\xi+\Delta-\Delta'-2B)(\xi+\Delta-\Delta')}\\&+\nu(\Delta'+C-\xi)\frac{(\Delta'+C-\Delta_+-\xi)(\Delta'+C-\xi-\Delta_-)(\Delta'+C-\Delta'_+)(\Delta'+C-\Delta'_-)}{(\Delta'-\Delta-\xi)(\Delta'-\Delta-\xi+C-B)(\Delta'-\Delta-\xi+C+B)(C-B)(C+B)}\\&+\nu(\Delta'+B-\xi)\frac{(\Delta'+B-\xi-\Delta_+)(\Delta'+B-\xi-\Delta_-)(\Delta'+B-\Delta'_+)(\Delta'+B-\Delta'_-)}{(\Delta'-\Delta-\xi+B-C)(\Delta'-\Delta-\xi)(\Delta'-\Delta-\xi+2B)(B-C)(2B)}\\&+\nu(\Delta'-B-\xi)\frac{(\Delta'-B-\xi-\Delta_+)(\Delta'-B-\xi-\Delta_-)(\Delta'-B-\Delta'_+)(\Delta'-B-\Delta'_-)}{(\Delta'-\Delta-\xi-B-C)(\Delta'-\Delta-\xi-2B)(\Delta'-\Delta-\xi)(B+C)(2B)}\Bigg),\end{aligned} \quad \text{(VIII. 11)}$$

where $\xi=i\pi l/\beta$,

$$\Delta_\pm=\omega'_{s_1}(\mathbf{k}_1\pm\mathbf{k}_0)+A(k_1), \quad \text{(VIII. 12)}$$

$$\Delta=\omega'_s(k_1)+A(k_1), \quad \text{(VIII. 13)}$$

$$\Delta'=\omega'_{s_2}(\mathbf{k}_1+\mathbf{q})+A_{s_2}(\mathbf{k}_1+\mathbf{q}), \quad \text{(VIII. 14)}$$

$$C=k_0^2/m, \quad \text{(VIII. 15)}$$

$$B=i\mu H_r/\sqrt{2}, \quad \text{(VIII. 16)}$$

$$\Delta'_\pm=\omega'_{s_2}(\mathbf{k}_1+\mathbf{q}\pm\mathbf{k}_0)+A_{s_2}(\mathbf{k}_1+\mathbf{q}). \quad \text{(VIII. 17)}$$

We can now make some comments about the particle hole propagator in Eq. (VIII.11). First we notice that if $s_1 = s_2$, the constant external field $\mathcal{H}_0$ everywhere cancels out of the expression for $\chi(q, l; s_1 s_2 \mathcal{H}_r)$ whereas if $s_1 = -s_2$ it contributes. The case $s_1 = s_2$ corresponds to a particle hole pair with z component of spin equal to zero. The case $s_1 = -s_2$ corresponds to a particle hole pair with z component of spin equal to one. In contrast to this the spatially varying external field always contributes regardless of spin. The expression for the propagator involves no undefined energy denominators. Self-energy effects merely shift the number and value of the poles in the particle hole propagator. The contribution to the propagator due to the amplitude of the external field $\mathcal{H}_r$ is imaginary. However, the overall expression for $\Gamma(\beta, g, \mathcal{H}_0, \mathcal{H}_r)_{\text{pol}}$ will be real since we started with a real expression.

If Eq. (VIII.11) is substituted into Eq. (VIII.5), the contribution to $\Gamma(\beta, g, \mathcal{H}_0, \mathcal{H}_r)$ coming from spin and density fluctuations of total spin 1 will separate from those with total spin zero and we can write Eq. (VIII.5) in the form of a logarithm which depends on the propagators as is done in Ref. 6.

IX. CONCLUSION

In the present paper, we have introduced a general method of writing a cluster expansion for the expectation values of arbitrary one- and two-body operators. We then restricted our attention to the magnetization and grand potential and performed a propagator resummation (of the Matsubara type) on a subset of terms in the Fourier expansion of the magnetization and grand potential. We studied the effect of a spatially varying external magnetic field on the polarization diagrams. We found that a constant magnetic field has no effect on particle hole pairs of spin zero and merely shifts the energy of the particle hole paris of spin one, while a spatially varying field strongly affects both. Application of a spatially varying field will probably destroy the coherence of spin and density fluctuations.

For equilibrium quantities the only single particle propagator method used for summing self-energy effects is due to Matsubara. But as we have seen, the Matsubara method cannot be applied to all terms in a reaction matrix expansion. We therefore would like to find a method which enables us to include systematically D vertex contributions to the self-energy. Such a method will be discussed in a subsequent paper.

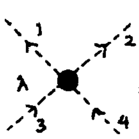
APPENDIX A

Algebraic expressions can be associated to the Qth order *A-matrix* 0_M *diagrams* according to the following rules:

(A.i) Label each line from 1 to n, where n is the number of lines, and associate with the jth line a momentum and spin $k_j = (\mathbf{k}_j, s_j)$.

(A.ii) Label the vertices from left to right from λ_1 to λ_Q, and assign to the M vertex a temperature $\lambda = 0$ [cf. RI Rule (VII.ii)].

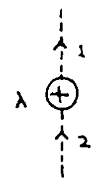
(A.iii) With each A vertex associate a factor



$$= A\begin{pmatrix}k_1 k_2\\ k_3 k_4\end{pmatrix} \equiv -C^2(\mathbf{k}_{12})\, {}_0\langle k_1, k_2 | A | k_3, k_4\rangle_0$$

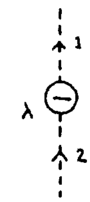
where the dotted lines can stand for either wavy or solid lines.

(A.iv) With the ΔH_+ and ΔH_- vertices associate factors



$$= +\frac{\mu}{2}\mathcal{H}_r\, \delta_{s_1,s_2} s_1\, \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_0)$$

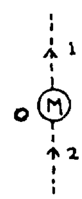
and



$$= +\frac{\mu}{2}\mathcal{H}_r\, \delta_{s_1,s_2} s_1\, \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_0),$$

where the dotted lines may be either solid or wavy.

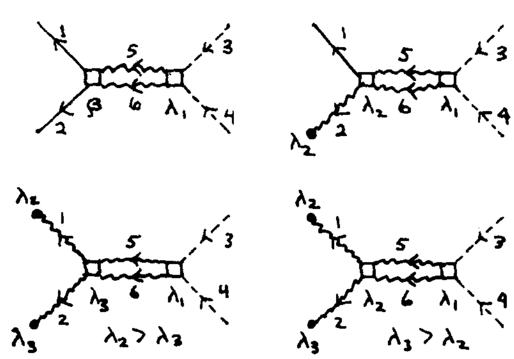
(A.v) With each M vertex associate a factor



$$= M\begin{pmatrix}k_1\\ k_2\end{pmatrix} = \epsilon\mu s_1 \delta_{s_1,s_2}\left(\frac{1}{2\pi}\right)^3 \exp[-i(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r}],$$

where the dotted lines may be either solid or wavy.

(A.vi) The temperature labels of the D vertices are assigned according to the types of lines which leave the vertices, as indicated in the following diagrams:

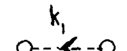


[cf. RI Rule (VII.v) for a more complete discussion]. To each of the above D vertices, assign a factor

$$D\begin{pmatrix} k_1k_2 \\ k_5k_6 \\ k_3k_4 \end{pmatrix} = -C^2(\mathbf{k}_{56})\, {}_0\langle k_1k_2|A|k_5k_6\rangle_0^{(s)}\, {}_0\langle k_5k_6|A|k_3k_4\rangle_0^{(s)}$$

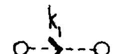
$$\times P\left(\frac{1}{\omega_5' + \omega_6' - \omega_1' - \omega_2'}\right).$$

(A. vii) With each left directed line, associate a factor



$$= \theta(\lambda_2 - \lambda_1) \exp[-(\lambda_2 - \lambda_1)\omega_1']$$

and with each right directed line associate a factor

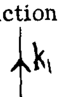


$$= \theta(\lambda_2 - \lambda_1) \exp[(\lambda_2 - \lambda_1)\omega_1'],$$

where $\omega_1' = k_1^2/2m - g - \mu\ {}_1H_0$. The dotted lines may be solid or wavy. If the right most vertex is an M vertex then $\lambda_1 = 0$.

(A. viii) To each A vertex or ΔH vertex with no lines entering or leaving on the left, assign a factor $\theta(\beta - \lambda_1)$, where λ_1 is the temperature of the vertex. With each A vertex, D vertex, or ΔH vertex with no line entering or leaving on the right, associate a factor $\theta(\lambda_1)$ where λ_1 is the temperature of the vertex.

(A. ix) Associate with each solid line, a distribution function



$$= \epsilon\nu_1 = \epsilon \exp(-\beta\omega_1')/(1 - \epsilon\exp(-\beta\omega_1')).$$

(A. x) Multiply the entire expression by a factor $\epsilon^{N_{\Delta H}}\epsilon^{P_B}S^{-1}$ where $N_{\Delta H}$ is the number of ΔH vertices, P_B is the number of permutations of bottom line momenta with respect to top line momenta in the product of the various matrix elements, and S is the symmetry number of the diagram.

(A. xi) Sum over all momenta and spins. Integrate over all temperatures from $-\infty$ to ∞.

APPENDIX B

Algebraic expressions can be associated to the Type I contracted 0 diagrams or 0_m diagrams according to the following rules:

(B. i) Label each line from 1 to n, where n is the number of lines, and associate with the jth line a momentum and spin $k_j = (\mathbf{k}_j, s_j)$.

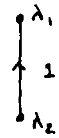
(B. ii) Label the vertices from λ_1 to λ_Q, where Q is the number of ΔH and A vertices and assign to the M vertex a temperature $\lambda = 0$.

(B. iii) With each A vertex associate a factor according to Rule (A. iii).

(B. iv) With each ΔH_+ and ΔH_- vertex associate factors according to Rule (A. iv).

(B. v) With each M vertex associate a factor according to Rule (A. v.).

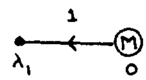
(B. vi) With each line of momentum and spin $k_1 = (\mathbf{k}_1, s_1)$ which begins and ends on either a $\Delta H_\pm$ vertex or A vertex, associate a factor



$$= \mathcal{D}_1(\lambda_1 - \lambda_2),$$

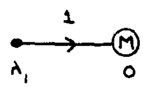
where $\mathcal{D}_1(\lambda_1 - \lambda_2)$ is defined in Eq. (IV. 3). If $\lambda_1 = \lambda_2$, associate a factor $(\epsilon\nu_1)$.

(B. vii) With each left directed line which attaches to the M vertex associate a factor



$$= \exp(-\lambda_1\omega_1')(1 + \epsilon\nu_1)$$

and with each right directed line which attaches to the M vertex associate a factor



$$= \exp(\lambda_1\omega_1')(\epsilon\nu_1).$$

(B. viii) Multiply the entire expression by a factor $\epsilon^{N_{\Delta H}}\epsilon^{P_B}S^{-1}$ where $\epsilon^{N_{\Delta H}}$ is the number of ΔH vertices in the diagram, P_B is the number of permutations of bottom row momenta with respect to top row momenta, and S is the symmetry number of the diagram.

(B. ix) Sum over all momenta and spins and integrate over all temperatures from 0 to β.

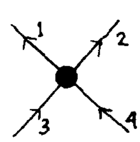
By changing the limits of integration from $-\infty$ to ∞ back to 0 to β, we have removed the necessity of Rule (A. viii).

APPENDIX C

Fourier transformed expressions for the Type I, contracted 0 diagrams or 0_m diagrams may be obtained using the following rules:

(C. i) Label each line from 1 to n, where n is the number of lines, and associate with the jth line a momentum and spin $k_j = (\mathbf{k}_j, s_j)$ and an energy $z_j = \pi n_j/\beta$.

(C. ii) Associate with each A vertex that is not directly connected to the M vertex by a line, a factor



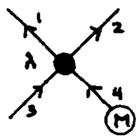
$$= \beta\delta_{n_1+n_2, n_3+n_4} A\begin{pmatrix} k_1k_2 \\ k_3k_4 \end{pmatrix}.$$

(C. iii) With each M vertex associate a factor according to Rule (A. v).

(C. iv) Associate with each $\Delta H_\pm$ vertex, that is not directly connected to the M vertex by a line, a factor

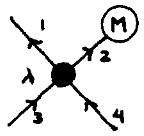
$$= -\beta\delta_{n_1, n_2}\Delta H_\pm\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

(C. v) With each A vertex which is directly connected to the M vertex by one or two lines, associate a temperature, λ, and a factor



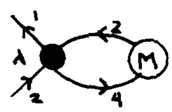
$$= \exp[-\lambda(\omega_4' - iz_3 + iz_1 + iz_2)]A\begin{pmatrix} k_1k_2 \\ k_3k_4 \end{pmatrix}$$

or



$$= \exp[\lambda(\omega_2' - iz_1 + iz_3 + iz_4)]A\begin{pmatrix} k_1k_2 \\ k_3k_4 \end{pmatrix}$$

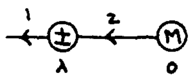
or



$$= \exp[-\lambda(\omega_2' - \omega_4' + iz_1 - iz_3)]A\begin{pmatrix} k_1k_4 \\ k_3k_2 \end{pmatrix},$$

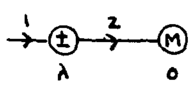
depending on the particular case considered.

(C. vi) With each $\Delta H_\pm$ vertex which is directly connected to the M vertex by one or two lines, associate a factor



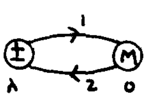
$$= -\exp[-\lambda(\omega_2' + iz_1)\Delta H_\pm\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

or



$$= -\exp[\lambda(\omega_2' + iz_1)]\Delta H_\pm\begin{pmatrix} k_2 \\ k_1 \end{pmatrix}$$

or



$$= -\exp[+\lambda(\omega_1' - \omega_2')]\Delta H_\pm\begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

depending on the particular case considered.

(C. vii) With each line of momentum and spin $k_1 = (\mathbf{k}_1, s_1)$ and energy z_1 which begins and ends on either a $\Delta H_\pm$ vertex or an A vertex, associate a factor

$$= 1/(iz_1 + \omega_1'),$$

where $z_1 = \pi n_1/\beta$ (n_1 an odd integer). If a line begins and ends on the same vertex, associate with it a factor $(\epsilon\nu_1)$.

(C. viii) With each line (k_1, z_1) that attaches to the M vertex, assign a factor $(1+\epsilon\nu_1)$ if it is directed to the left and a factor $(\epsilon\nu_1)$ if it is directed to the right.

(C. ix) Assign to each diagram an overall factor $\epsilon^{N_{\Delta H}}\epsilon^{P_B}S^{-1}$ [cf. Rule (A. x)].

(C. x) Integrate over the temperatures of the vertices that connect to the M vertex from 0 to β.

(C. xi) Sum over all momenta and spin and sum over all odd integers n_j from $-\infty$ to ∞. Multiply by a factor $1/\beta$ for each line in the diagram that does not connect to an M vertex.

APPENDIX D

We may associate algebraic expressions with the *irreducible 0 diagrams* and 0_m *diagrams* according to the following rules:

(D. i) Label each line from 1 to n, where n is the number of lines, and assign to the jth line a momentum, spin, and energy (k_j, z_j).

(D. ii) With each M vertex associate a factor according to Rule (A. v).

(D. iii) Associate with each A vertex a factor according to Rule (C. ii) or (C. v).

(D. iv) Associate with each $\Delta H_\pm$ vertex a factor according to Rules (C. iv) or (C. vi).

(D. v) Associate with each line with labels (k_1, z_1) containing no $\Delta H_\pm$ vertices a factor

$$[(iz_1 + \omega'(k_1) + M_+(z_1, \mathbf{k}_1 + \mathbf{k}_0, s_1) + M_-(z_1, \mathbf{k}_1 - \mathbf{k}_0, s_1)$$
$$+ \Sigma(\mathbf{k}_1, z_1, \mathbf{k}_0, s_1))]^{-1}.$$

(D. vi) If a line contains a sequence of $l\,\Delta H_+$ vertices or $l\,\Delta H_-$ vertices, then with the line entering the first $\Delta H_\pm$ vertex associate the propagator in Rule (D. v). With the line connecting the jth $\Delta H_\pm$ vertex to the $(j+1)st$ $\Delta H_\pm$ vertex or with an A vertex, associate a factor

$$[(iz_1 + \omega_1'(\mathbf{k}_1 \pm i\mathbf{k}_0, s_1) + M_\pm(z_1, \mathbf{k}_1 \pm (j+1)\mathbf{k}_0, s_1)$$
$$+ \Sigma(\mathbf{k}_1 \pm j\mathbf{k}_0, z_1, \mathbf{k}_0, s_1)]^{-1}.$$

(D. vii) For each double bond structure, subtract a factor

$$= \left(\frac{1}{iz_1 + \omega_1'}\right)\left(\frac{1}{iz_2 + \omega_2'}\right).$$

(D. viii) Assign factors according to Rules (C. viii)–(C. xi).

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Propagator techniques for equilibrium strongly coupled Fermi fluids

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This is Paper III of a series of three papers in which a self-consistent propagator resummation of self-energy effects in a strongly coupled Fermi fluid is performed. In the present paper, a Laplace transformation of the reaction matrix expansion of the grand potential and the magnetization is obtained, and a self-consistent propagator resummation of self-energy effects due both to the medium and to constant and spatially varying external magnetic fields is performed. The procedure for resumming polarization diagrams is discussed.

I. INTRODUCTION

In two previous papers, hereafter referred to as RI[1] and RII,[2] we obtained exact expressions for the grand potential and for the magnetization of a strongly coupled Fermi fluid in the presence of a constant and a spatially varying external magnetic field. In RI, we wrote the expressions for the grand potential in terms of temperature dependent single particle propagators and reaction matrices. In RII, we found expressions for the expectation values of one- and two-body operators in terms of cluster expansions and we wrote an expression for the magnetization in terms of temperature dependent single particle propagators and reaction matrices. However, we found that the usual perturbation theory techniques used to sum self-energy effects in the single particle propagators could only be applied to a small subclass of terms in the expressions for the grand potential and the magnetization.

In this paper, we wish to discuss an alternative *single particle propagator* method for self consistently resumming all self-energy effects in the reaction matrix expansion of equilibrium quantities. We begin in Sec. II by writing the Laplace transform of the single particle propagators and then the Laplace transform of the grand potential and the magnetization. In Sec. III, we define and discuss the self-energy structures that appear in the *A-matrix* 0 *diagrams* and 0_M *diagrams* and we explicitly resum them. In Sec. IV we show how the collective effects due to spin and density fluctuations appear in our expressions for the grand potential and we explicitly resum their contribution to the grand potential. In Sec. V, we make some concluding remarks.

II. LAPLACE TRANSFORM OF THE GRAND POTENTIAL AND THE MAGNETIZATION

In RI, Sec. VIII, we obtained an expression for the grand potential, $\Gamma(\beta, g, H_0, H_r)$, in terms of *A-matrix* 0 *diagrams* and in RII, Sec. III, we obtained an expression for the magnetization $\langle M(\mathbf{r})\rangle$ in terms of *A-matrix* 0_M *diagrams*. Both the *A*-matrix 0 diagrams and 0_M diagrams were defined entirely in terms of reaction matrices and temperature dependent single particle propapators. We now wish to write an expression for the Laplace transform of the grand potential and the magnetization.

We shall first Laplace transform the single particle propagators. We may write $\theta(\lambda)\exp(\pm\lambda\omega)$ in the form

$$\theta(\lambda)\exp(\pm\lambda\omega)=\frac{1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty} ds\,\exp(-s\lambda)\frac{1}{s\pm\omega}$$

$$=\frac{1}{2\pi i}\oint_c ds\,\exp(-s\lambda)\frac{1}{s\pm\omega}. \qquad \text{(II.1)}$$

In Eq. (II.1), γ is a positive number such that $\gamma>|\omega|$. The Bromwich contour in Eq. (II.1) encloses both poles of $1/(s\pm\omega)$ (cf. Fig. 1). We can always choose γ large enough so that the condition $\gamma>|\omega|$ is fulfilled (cf. Ref. 3, Sec. 14).

If we substitute Eq. (II.1) into the expressions for the *A-matrix* 0 *diagrams* and 0_M *diagrams*, we find that each line is described by a different "energy" s_i. However, the values of s_i for various lines are related by the way the particles interact with one another. If we perform the temperature integrations in the various 0 *diagrams* and 0_M *diagrams* and make use of the relation

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\,\exp[-\lambda(s_1+s_2-s_3-s_4)]=\delta(s_1+s_2-s_3-s_4)] \qquad \text{(II.2)}$$

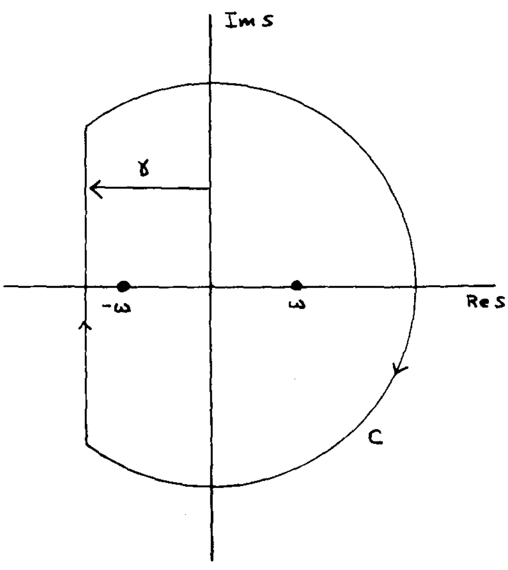

FIG. 1. Bromwich contour for the propagators.

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we obtain conservation relations between the energies of various lines in a given diagram which are governed by the topology of the diagram. [Note that if we choose s_i to be pure imaginary, Eq. (II.2) reduces to the usual definition of a Dirac delta function.]

The temperature integrations in the expressions for $\Gamma(\beta,g,H_0,H_r)$ and $\langle m(\mathbf{r})\rangle$ exhibit a sort of "causality" in that they are ordered in a well defined way from small values of λ to larger values of λ. When we substitute Eq. (II.1) into the expressions for $\Gamma(\beta,g,H_0,H_r)$ and $\langle m(\mathbf{r})\rangle$, we must find a way to keep track of the ordering or "causality" of the temperature integrations. This can be done if we redefine the propagator in Eq. (II.1) as

$$\theta(\lambda)\exp(\pm\lambda\omega)=\lim_{\delta\to 0}\frac{1}{2\pi i}\oint_c ds\exp(-s\lambda)\frac{1}{s\pm\omega-\delta}. \qquad \text{(II.3)}$$

We then stipulate that only those poles contribute which depend on $+\delta$ (i.e., poles of the form $s=\pm\omega+\delta$). With this convention, we can unambiguously reproduce the expressions for the grand potential and magnetization which are obtained by performing the temperature integrations directly.

We can now write the rules for obtaining the Laplace transformed expressions for the grand potential, $\Gamma(g,\beta,H_0,H_r)$, and the magnetization $\langle m(\mathbf{r})\rangle$.

$$\Gamma(\beta,g,H_0,H_r)$$
$$=-\frac{1}{\beta}\sum_{Q=1}^{\infty}(\text{all different } Q\text{th order } A\text{-matrix 0 diagrams}) \qquad \text{(II.4)}$$

and

$$\langle m(\mathbf{r})\rangle$$
$$=\sum_{Q=1}^{\infty}(\text{all different } Q\text{th order } A\text{-matrix } 0_M \text{ diagrams}). \qquad \text{(II.5)}$$

An *A-matrix 0 diagram* contains a collection of Q $\Delta H_\pm$ vertices, A vertices and D vertices. An *A-matrix 0_M diagram* contains one M vertex and a collection of Q $\Delta H_\pm$ vertices, A vertices, and D vertices. The vertices are ordered from left to right, and are entirely interconnected by solid and wavy lines. If a diagram contains an M vertex (a 0_M diagram) then it appears at the extreme right. A vertices and D vertices each have two lines entering and two lines leaving. The M vertex and $\Delta H_\pm$ vertices only have one line entering and one line leaving. Wavy lines must be directed to the left, while solid lines may be directed either to the left or right. D vertices with two solid lines leaving must be placed so that the solid lines which leave the vertex are directed to the right. No wavy line double bonds may appear *except* internally in the D vertices. Two Qth *order A matrix* 0 *diagrams* or 0_M *diagrams* differ if they have different topological structure; or if they have the same topological structure, but the lines are of different types or directions, or the D vertices have different temperature labeling.

Algebraic expressions for the *A matrix* 0 *diagrams* and the *A matrix 0_M diagrams* may be obtained according to the rules in Appendix A.

Two examples of *A matrix 0_M diagrams* are given in Fig. 2. Algebraic expressions corresponding to the diagrams in Fig. 2 are given below:

Fig. 2(a)

$$=\epsilon^3\sum_{k_1\cdots k_7}\left(\frac{1}{2\pi i}\right)^4\int\cdots\int ds_L\,ds_1\cdots ds_5$$
$$\times\exp(-\beta s_L)\,\delta(s_L-s_1-s_2-s_3-s_4)\,\delta(-s_5+s_1+s_3+s_4)$$
$$\times(\epsilon\nu_1)(\epsilon\nu_2)(\epsilon\nu_7)(\epsilon\nu_5)\,A\begin{pmatrix}k_1k_2\\k_6k_7\end{pmatrix}D\begin{pmatrix}k_6k_7\\k_3k_4\\k_1k_5\end{pmatrix}m\begin{pmatrix}k_5\\k_2\end{pmatrix}\left(\frac{1}{s_L-4\delta}\right)$$
$$\times\left(\frac{1}{s_1+\omega_1'-\delta}\right)\left(\frac{1}{s_2+\omega_2'-\delta}\right)$$
$$\times\left(\frac{1}{s_3-\omega_3'-\delta}\right)\left(\frac{1}{s_4-\omega_4'-\delta}\right)\left(\frac{1}{s_5-\omega_5'-\delta}\right), \qquad \text{(II.6)}$$

Fig. 2(b)

$$=\epsilon^3\sum_{k_1\cdots k_9}\left(\frac{1}{2\pi i}\right)^5\int\cdots\int ds_1\cdots ds_7$$
$$\times\exp[-\beta(s_1+s_2+s_3+s_4)]\,\delta(s_3+s_4-s_5-s_6)$$
$$\times\delta(-s_7+s_1+s_5+s_6)(\epsilon\nu_8)(\epsilon\nu_2)(\epsilon\nu_7)$$
$$\times D\begin{pmatrix}k_1k_2\\k_3k_4\\k_8k_9\end{pmatrix}D\begin{pmatrix}k_8k_9\\k_5k_6\\k_1k_7\end{pmatrix}m\begin{pmatrix}k_7\\k_2\end{pmatrix}\left(\frac{1}{s_1+\omega_1'-\delta}\right)$$
$$\times\left(\frac{1}{s_3-\omega_3'-\delta}\right)\left(\frac{1}{s_4-\omega_4'-\delta}\right)\left(\frac{1}{s_5-\omega_5'-\delta}\right)$$
$$\times\left(\frac{1}{s_6-\omega_6'-\delta}\right)\left(\frac{1}{s_7-\omega_7'-\delta}\right)\left(\frac{1}{s_2+\omega_2'-\delta}\right). \qquad \text{(II.7)}$$

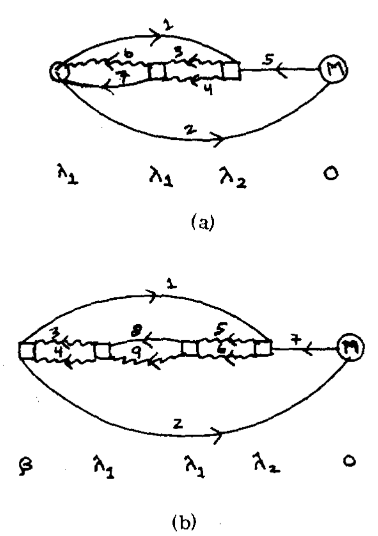


FIG. 2. A-matrix 0_M diagrams.

III. REMOVAL OF SELF-ENERGY STRUCTURES

We can now discuss the *self-energy structures* that appear in the expressions for the grand potential $\Gamma(\beta g \mathcal{H}_0 \mathcal{H}_r)$ and the magnetization $\langle m(\mathbf{r})\rangle$. The discussion is somewhat similar to that of RII except that now our diagrams are ordered from left to right and we need not exclude any diagrams from our expressions for $\Gamma(\beta, g, \mathcal{H}_0 \mathcal{H}_r)$ and $\langle m(\mathbf{r})\rangle$.

Since the diagrams have left-right ordering we must be careful in the way we define the *self-energy structures*. *Self-energy structures* are now defined to be those parts of an *A-matrix* 0 *diagram* or an *A-matrix* 0_M *diagram* which can be removed by cutting two *identically* directed lines with the same momentum, energy, and spin (k_1, s_1). They have no momentum indices in common with the diagram to which they attach except for the momentum of the attaching lines. An *irreducible self-energy structure* cannot be cut in half by cutting an internal line with the same momentum, energy, and spin as the lines which enter and leave the *self-energy structure*. *Self-energy structures* appear because of conservation of momentum and energy at each vertex or group of vertices.

Self-energy structures composed of $\Delta H_{\pm}$ vertices were discussed in some detail in RII, and therefore we will not discuss them again here. In Figs. 3(a) and 3(b) we give some examples of self-energy structures which contain A vertices and D vertices. In Fig. 3(c) we have an example of an object which is *not* a self-energy structure, because the lines which enter and leave it do not have the same energy [cf. Rules (A.ix) and (A.x) in Appendix A]. The dotted lines in Fig. 3 may be either solid or wavy.

Before we resum the self-energy structures in the *A-matrix* 0 *diagrams* and 0_M *diagrams* we shall make one further simplification. We shall add together all diagrams for which one can replace wavy lines by solid lines and not change anything but the dependence of the diagram on the factor $(\epsilon\nu_1)$. We then obtain diagrams which contain left-directed wavy and dotted lines, and right-directed solid lines. The dotted lines are represented by a factor $(1+\epsilon\nu_1)$ and are the sum of left-directed wavy and solid lines. We then can draw the dotted lines as left-directed solid lines since this presents no ambiguities.

We now can resum the self-energy structures that appear in the A-matrix 0 *diagrams* and *A-matrix* 0_M *diagrams*. We then obtain the following expressions for $\Gamma(\beta, g, \mathcal{H}_0, \mathcal{H}_r)$ and $\langle m(\mathbf{r})\rangle$:

$$\Gamma(\beta, g, \mathcal{H}_0, \mathcal{H}_r) = -\frac{1}{\beta}\sum(\text{all different irreducible } A\text{-matrix 0 diagrams}) \tag{III.1}$$

and

$$\langle m(\mathbf{r})\rangle = \sum(\text{all different irreducible } A\text{-matrix } 0_M \text{ diagrams}). \tag{III.2}$$

An *irreducible A-matrix* 0 *diagram* (or 0_M *diagram*) is the same as an *A-matrix* 0 *diagram* (or 0_M *diagram*) except for the following changes: (1) no self-energy structures may appear; (2) all left-directed lines are solid, except internal lines of a D vertex or lines beginning and ending on vertices with the same temperature label; (3) solid line double bonds are allowed; and (4) a given line can only contain ΔH_+ vertices or ΔH_- vertices but not a combination of them.

An algebraic expression may be associated with the irreducible *A-matrix* 0 *diagrams* and 0_M *diagrams* according to Rules (A.i)—(Axii) except that Rule (A.vii) is replaced to read:

(III.vii) (a) With each left-directed solid line (k_1, s_1) which connects an M vertex, A vertex or D vertex on the right to an A vertex, D vertex, or $\Delta H_{\pm}$ vertex on the left, associate a factor

$$\frac{(1+\epsilon\nu_1)}{[s_1 - \omega_1' - (1+\epsilon\nu_1)(\Sigma_L(k_1, s_1) + M_-^L(s_1, \mathbf{k}_1 - \mathbf{k}_0, \mathscr{s}_1 \mathcal{H}_r) + M_+^L(s_1, \mathbf{k}_1 + \mathbf{k}_0, \mathscr{s}_1, \mathcal{H}_r)) - \delta]}.$$

(b) With each right-directed solid line (k_1, s_1) which connects an A vertex or D vertex on the left to an M vertex, A vertex, D vertex, or $\Delta H_{\pm}$ vertex on the right, associate a factor

$$\frac{(\epsilon\nu_1)}{[s_1 + \omega_1' + (\epsilon\nu_1)(\Sigma_R(k_1, s_1) + M_-^R(s_1, \mathbf{k}_1 - \mathbf{k}_0, \mathscr{s}_1, \mathcal{H}_r) + M_+^R(s_1 \mathbf{k}_1 \pm \mathbf{k}_0, \mathscr{s}_1 \mathcal{H}_r)) - \delta]}.$$

(c) With each left-directed solid line which attaches one $\Delta H_{\pm}$ vertex to another $\Delta H_{\pm}$ vertex or to an A vertex or D vertex, associate a factor

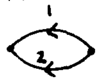
$$\frac{(1+\epsilon\nu_1)}{[s_1 - \omega_1' - (1+\epsilon\nu_1)(\Sigma_L(k_1, s_1) + M_{\pm}^L(s_1, \mathbf{k}_1 + \mathbf{k}_0, \mathscr{s}_1, \mathcal{H}_r)) - \delta]}.$$

(d) With each right-directed solid line which attaches one $\Delta H_{\pm}$ vertex to another $\Delta H_{\pm}$ vertex or to an A vertex, D vertex, or M vertex, associate a factor

$$\frac{(\epsilon\nu_1)}{[s_1 + \omega_1' + (\epsilon\nu_1)(\Sigma_R(k_1, s_1) + M_{\pm}^R(s_1, \mathbf{k}_1 \pm \mathbf{k}_0, \mathscr{s}_1, \mathcal{H}_r)) - \delta]}.$$

(e) To each internal wavy line (k_1, s_1) of a D vertex assign a factor $1/(s - \omega_1' - \delta)$.

(f) For each solid line double bond subtract a factor



$$= \frac{1}{(s_1-\omega_1'-\delta)(s_2-\omega_2'-\delta)}\,.$$

(g) For each line attaching to vertices with same label, assign a factor $(\epsilon\nu_1)$ if it is solid and no factor if it is wavy.

The quantities $M_+^{L(R)}(s_1,\mathbf{k}_1+\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r)$, $M_-^{L(R)}(s_1,\mathbf{k}_1-\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r)$, and $\Sigma_{L(R)}(s_1,k_1,\mathcal{H}_r)$ appearing in Rule (III.vii) are defined as follows:

$$M_\pm^L(s_1,\mathbf{k}_1\pm\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r)=\frac{(\frac{1}{2}\mu\mathcal{H}_r)^2}{[s_1-\omega'_1(\mathbf{k}_1\pm\mathbf{k}_0)-(1+\epsilon\nu_1)(M_\pm^L(s_1,\mathbf{k}_1\pm 2\mathbf{k}_0),\mathscr{s}_1,\mathcal{H}_r)+\Sigma_L(s_1,\mathbf{k}_1\pm\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r))-\delta]}, \qquad \text{(III.3)}$$

$$M_\pm^R(s_1,\mathbf{k}_1,\pm 2\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r)=\frac{(\frac{1}{2}\mu+_r)^2}{[s_1+\omega'_{\mathscr{s}_1}(\mathbf{k}_1\pm\mathbf{k}_0)+(\epsilon\nu_1)(M_\pm^R(s_1,\mathbf{k}_1\pm 2\mathbf{k}_0)\mathscr{s}_1,\mathcal{H}_r)+\Sigma_R(s_1,\mathbf{k}_1\pm\mathbf{k}_0,\mathscr{s}_1,\mathcal{H}_r))-\delta]}, \qquad \text{((III.4)}$$

and

$\Sigma_{L(R)}(s_1,k_1)=\sum$ (all different irreducible A-matrix 1 diagrams with left (right) directed external lines of momentum spin and energy (s_1,k_1) which contain at least one A vertex or D vertex and an equal number of ΔH_+ and ΔH_- vertices.) (III.5)

An *irreducible A-matrix* 1 *diagram* is the same as an *irreducible A-matrix* 0 *diagram* except that it has one external incoming line and one external outgoing line. The external lines are identically directed. Algebraic expressions may be associated to the *irreducible A-matrix* 1 *diagrams* in the same way as for *A-matrix* 0 *diagrams* except that no factors are assigned to external lines.

Some examples of *irreducible A-matrix* 0_M *diagrams* are given in Fig. 4. Algebraic expressions for the diagrams in Fig. 4 are given below:

$$\text{Fig. 4(a)}=\epsilon^2\left(\frac{1}{2\pi i}\right)^3\int ds_L\,ds_1\cdots ds_4\exp(-\beta s_L)\delta(s_L-s_1-s_2)\,\delta(s_1+s_2-s_3-s_4)(\epsilon\nu_5)A\begin{pmatrix}k_5k_2\\k_5k_1\end{pmatrix}A\begin{pmatrix}k_1k_4\\k_2k_3\end{pmatrix}M\begin{pmatrix}k_3\\k_4\end{pmatrix}\left(\frac{1}{s_L-4\delta}\right)$$

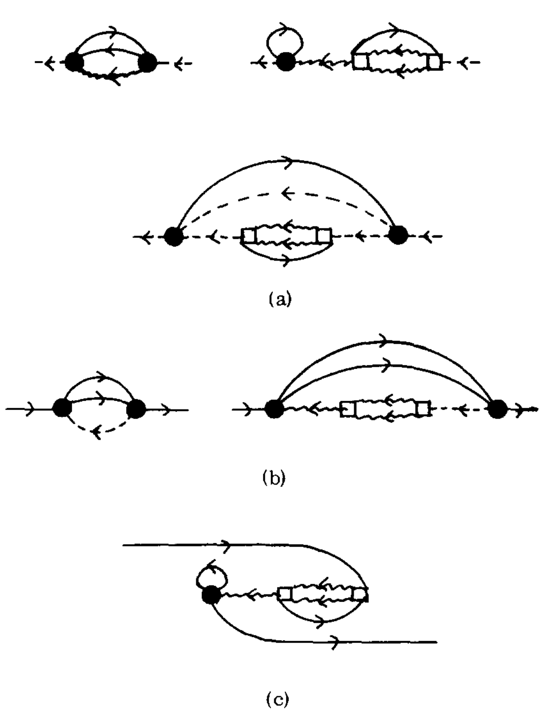


FIG. 3. Structures appearing in the A-matrix 0_M diagrams: (a) left directed self-energy structures, (b) right directed self-energy structures, (c) not a self-energy structure.

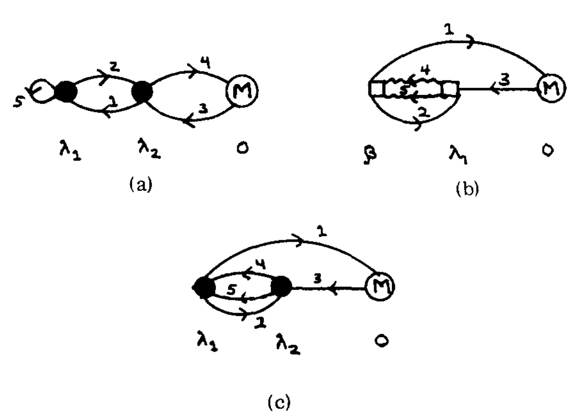


FIG. 4. Irreducible A-matrix 0_M diagrams.

$$\times\left(\frac{(1+\epsilon\nu_1)}{s_1-\omega_1'-(1+\epsilon\nu_1)\mathcal{S}_L(k_1,s_1)-\delta}\right)\left(\frac{(\epsilon\nu_2)}{s_2+\omega_2'+(\epsilon\nu_2)\mathcal{S}_R(k_2s_2)-\delta}\right)$$

$$\times\left(\frac{(1+\epsilon\nu_3)}{s_3-\omega_3'-(1+\epsilon\nu_3)\mathcal{S}_L(k_3,s_3,)-\delta}\right)\left(\frac{(\epsilon\nu_4)}{s_4+\omega_4'+(\epsilon\nu_4)\mathcal{S}_R(k_4,s_4)-\delta}\right), \tag{III.6}$$

$$\text{Fig. 4(b)}=\epsilon\left(\frac{1}{2\pi i}\right)^4\int ds_1\cdots ds_5\exp[-\beta(s_1+s_2+s_4+s_5)]\delta(s_4+s_5+s_2-s_3)D\begin{pmatrix}k_1k_2\\k_4k_5\\k_3k_2\end{pmatrix}M\begin{pmatrix}k_3\\k_1\end{pmatrix}$$

$$\times\left(\frac{(\epsilon\nu_1)}{s_1+\omega_1'+(\epsilon\nu_1)\mathcal{S}_R(k_1,s_1)-\delta}\right)\left(\frac{(\epsilon\nu_2)}{s_2+\omega_2'+(\epsilon\nu_2)\mathcal{S}_R(k_1s_1)-\delta}\right)$$

$$\times\left(\frac{(1+\epsilon\nu_3)}{s_3-\omega_3'-(1+\epsilon\nu_3)\mathcal{S}_L(k_3,s_3)-\delta}\right)\left(\frac{1}{s_4-\omega_4'-\delta}\right)\left(\frac{1}{s_5-\omega_5'-\delta}\right), \tag{III.7}$$

$$\text{Fig. 4(c)}=\epsilon^4\left(\frac{1}{2\pi i}\right)^4\int ds_L\,ds_1\cdots ds_5\exp(-\beta s_L)\delta(s_L-s_1-s_2-s_4-s_5)\delta(s_4+s_5+s_2-s_3)$$

$$\times A\begin{pmatrix}k_1k_2\\k_4k_5\end{pmatrix}A\begin{pmatrix}k_4k_5\\k_2k_3\end{pmatrix}M\begin{pmatrix}k_3\\k_1\end{pmatrix}\left\{\left(\prod_{j=1}^{2}\left(\frac{(\epsilon\nu_j)}{(s_i+\omega_i-(\epsilon\nu_i)\mathcal{S}_R(k_i,s_i)-\delta)}\right)\left(\frac{(1+\epsilon\nu_3)}{s_3-\omega_3-(1+\epsilon\nu_3)\mathcal{S}_L(k_3,s_3)-\delta}\right)\right.\right.$$

$$\left.\left.\times\left(\prod_{j=4}^{5}\frac{(1+\epsilon\nu_j)}{(s_j-\omega_j-(1+\epsilon\nu_j)\mathcal{S}_L(k_j,s_j)-\delta)}\right)-\left(\frac{1}{s_4-\omega_4'-\delta}\right)\left(\frac{1}{s_5-\omega_5'-\delta}\right)\right\}. \tag{III.8}$$

In Eqs. (III.6)–(III.8), the self-energy $\mathcal{S}_{L(R)}(k_1,s_1)$ is defined

$$\mathcal{S}_{L(R)}(k_1,s_1)\equiv\Sigma_{L(R)}(k_1,s_1,H_r)+M_-^{L(R)}(s_1,\mathbf{k}_1-\mathbf{k}_0,\sigma_1,H_r)+M_+^{L(R)}(s_1,\mathbf{k}_1+\mathbf{k}_0,\sigma_1,H_r). \tag{III.9}$$

IV. POLARIZATION DIAGRAMS

In RII, Sec. VIII, we studied the contribution to the grand potential due to polarization diagrams composed entirely of A vertices (with $\Delta H_\pm$ vertex self-energy structures included). We found that, using the Matsubara technique, we could not include contributions to the polarization diagrams coming from D vertices. Such contributions are important if we want to sum D vertices into the vertices of the polarization diagrams or if we want to include D vertex effects in the self-energy structures. The inclusion of such effects would probably be necessary for realistic calculations of the thermodynamic properties of liquid He^3.

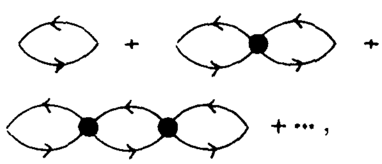
We would like to show how one can sum the polarization diagrams in the Laplace transform expansion of the grand potential. For simplicity we shall turn off the external fields (set $H_0=H_r=0$) since we are primarily interested in the structure of the polarization diagrams themselves. Furthermore we shall suppress the spin dependence in the diagrams. Inclusion of the spin dependence simply causes the polarization diagrams to split into a part due to density fluctuations and a part due to spin fluctuations, but does not change the basic structure of the diagrams.

Some of the lower order polarization diagrams which contribute to the grand potential are displayed in Fig. 5. We first note that all horizontal chains of "bubbles" can be summed, if we assume that the A vertices depend only on the momentum transfer, $\mathbf{q}$ (cf. RII, Sec. VIII),

$$A\begin{pmatrix}\mathbf{k}_1; & \mathbf{k}_3+\mathbf{q}\\ \mathbf{k}_1+\mathbf{q}; & \mathbf{k}_3\end{pmatrix}\sim A(\mathbf{q}).$$

We then can define

$\chi(\mathbf{q},E)=$ (IV.1)



where

$$\chi(\mathbf{q},E)=\frac{\chi_0(\mathbf{q},E)}{1-\epsilon A(\mathbf{q})\chi_0(\mathbf{q},E)} \tag{IV.2}$$

and

$$\chi_0(Q,E)=\left(\frac{1}{2\pi i}\right)\int ds_1\sum_{k_1}\left(\frac{(1+\epsilon\nu_{\mathbf{k}_1})}{(s_1-\omega(\mathbf{k}_1)-(1+\epsilon\nu_{\mathbf{k}_1})\Sigma_L(\mathbf{k}_1,s_1)-\delta}\right)$$

$$\times\left(\frac{\epsilon\nu_{\mathbf{k}_1+\mathbf{q}}}{E-s_1+\omega(\mathbf{k}_1+\mathbf{q})+(\epsilon\nu_{\mathbf{k}_1+\mathbf{q}})\Sigma_R(\mathbf{k}_1+\mathbf{q},E-s_1)-\delta}\right), \tag{IV.3}$$

where $E=s_1+s_2$. The poles of the propagator $\chi(\mathbf{q},E)$ contain collective mode effects due to the spin and density fluctuations (cf. Ref. 4 for a discussion of the structure of Eq. (IV.2) for the case of an electron gas). We could generalize Eq. (IV.3) even further by summing D vertices and higher order terms involving D vertices and A vertices into the vertices in Eq. (IV.1).

If we perform the sum over all horizontal chains of "bubbles" then the sum over polarization diagrams reduces to a sum over step diagrams as indicated in Fig. 6, where each line with label j in a step diagram corresponds to a propagator $\chi(\mathbf{q},E_j)$ and each vertex corresponds to a factor $A(\mathbf{q})$.

We can now write the expression for the grand potential in the form of an infinite sum which can be evaluated systematically:

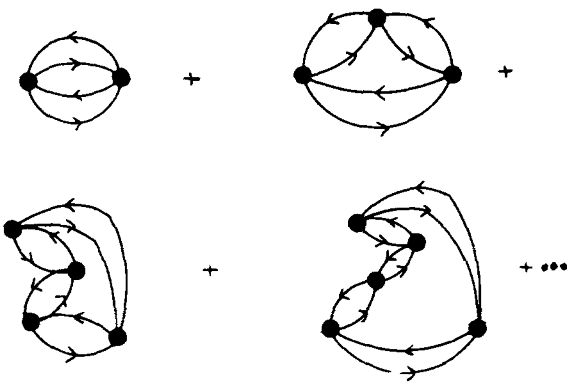


FIG. 5. Some polarization diagrams which contribute to the grand potential.

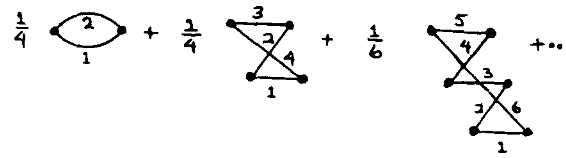


FIG. 6. After summation over chains of horizontal bubbles, the polarization diagrams reduce to step diagrams.

$$\Gamma(\beta,g)_{\rm pol}=\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{1}{2\pi i}\right)^{2n}\int\cdots\int dE_1\cdots dE_{2n}\sum_{\mathbf{q}}\exp\Big(-\beta\sum_{i=1}^{2n}E_i\Big)(\epsilon A(\mathbf{q}))^{2n}$$
$$\times\left(\frac{1}{E_1+E_2-4\delta}\right)\left(\frac{1}{E_2+E_3-4\delta}\right)\times\cdots\times\left(\frac{1}{E_{2n-1}+E_{2n}-4\delta}\right)$$
$$\left(\frac{1}{E_{2n}+E_1-4\delta}\right)\prod_{j=1}^{2n}(\chi(\mathbf{q},E_j))$$
$$-\frac{1}{4}\int\int\frac{dE_1dE_2}{(2\pi i)^2}\sum_{\mathbf{q}}\left(\frac{1}{E_1+E_2-4\delta}\right)^2\chi(\mathbf{q},E_1)\chi(\mathbf{q},E_2)(\epsilon A(\mathbf{q}))^2.$$

V. CONCLUDING REMARKS

In the present series of papers (cf. Refs. 1 and 2), we have written the grand potential and the magnetization of a strongly coupled Fermi fluid in terms of a reaction matrix expansion with self-energy effects due to the medium and to external fields included in the single particle propagators. The expressions we have obtained are exact. They are well behaved, even for particles with hard cores, and they never contain undefined energy denominators. Furthermore, the expressions we have obtained are in a form which can yield a quasiparticle picture of a Fermi fluid. The quasiparticle energies are simply given by the poles of single particle propagators. We can now begin to test many of the assumptions of the phenomenological theories of liquid He^3 by calculating phenomenological parameters directly for realistic He^3 potentials.

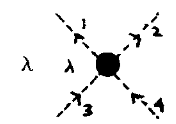
APPENDIX A

The A-matrix 0 diagrams and 0_M diagrams may be evaluated according to the following rules:

(A.i) Label the A vertices and D vertices from left to right from λ_1 to λ_Q, where Q is the number of vertices. (A vertices require one label but D vertices require two labels. One label of a D vertex is assigned according to this rule. The other is determined by the type of lines that leave the D vertex, and the labels of the vertices to which they attach. [cf. Rule (A.vi)].

(A.ii) Label the lines from 1 to n, where n is the number of lines, and associate with the jth line a momentum and spin $k_j=(\mathbf{k}_j, s_j)$ and an energy s_j. (There is one exception. If a line connects two vertices with the same label, λ, or connects a vertex to itself, it need not be assigned an energy s_j.)

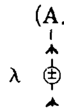
(A.iii) With each A vertex associate a factor



$$=A\begin{pmatrix}k_1k_2\\ k_3k_4\end{pmatrix}\equiv -C^2(\mathbf{k}_{12})_0\langle k_1k_2|A|k_3k_4\rangle_0^{15},$$

where the dotted lines can stand for either wavy or solid lines.

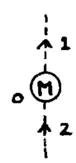
(A.iv) With each $\Delta H_{\pm}$ vertex, associate a factor



$$=\epsilon\frac{\mu}{2}s_1\delta_{s_1,s_2}H_r\,\delta(\mathbf{k}_1-\mathbf{k}_2\mp\mathbf{k}_0)$$

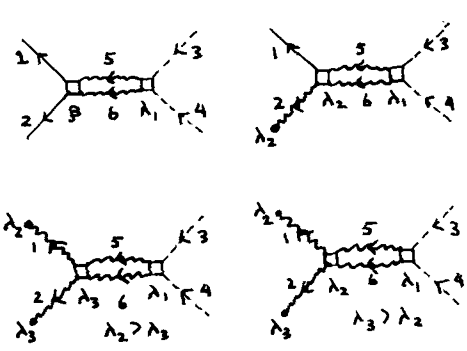
where the dotted lines may be either solid or wavy.

(A.v) With each M vertex associate a factor



$$=M\begin{pmatrix}k_1\\ k_2\end{pmatrix}=\mu\epsilon s_1\delta_{s_1s_2}\left(\frac{1}{2\pi}\right)^3\exp[-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}].$$

(A.vi) To each D vertex assign labels, λ_i, according to the following conventions:



In the above vertices, the right-most labels are assigned according to Rule (II. ii). The left-most labels are assigned according to the nature of the outgoing lines and the labels of the vertices to which they attach. For a D vertex with two outgoing solid lines, the left-most label is β. For D vertices with one outgoing wavy line the label of the left-most vertex is that of the vertex to which the wavy line attaches. For D vertices with two outgoing wavy lines which attach to vertices with temperatures λ_2 and λ_3, the temperature of the left-most vertex can be either λ_2 or λ_3. In general, both possibilities occur and lead to different expressions.

To each of the above D vertices assign a factor

$$D\begin{pmatrix} k_1k_2 \\ k_5k_6 \\ k_3k_4 \end{pmatrix} = -\mathbf{C}^2(\mathbf{k}_{56})_0\langle k_1k_2|A|k_5k_6\rangle_0^{(s)}\,{}_0\langle k_5k_6|A|k_3k_4\rangle_0^{(s)} \times P\left(\frac{1}{(\omega_5'+\omega_6'-\omega_1'-\omega_2')}\right),$$

where $\omega_1' = k_1^2/2m - g - \mu\, s_1 H_0$.

(A. vii) To each left directed wavy line (k_1, s_1) assign a factor $(s_1 - \omega_1' - \delta)^{-1}$. To each left directed solid line (k_1, s_1) assign a factor $(\epsilon\nu_1)(s_1 - \omega_1' - \delta)^{-1}$. To each right directed solid line (k_1, s_1) assign a factor $(\epsilon\nu_1) \times (s_1 + \omega_1' - \delta)^{-1}$. If a line (k_1, s_1) connects two vertices with the same label, assign no factor if it is a wavy line, and a factor $(\epsilon\nu_1)$ if it is a solid line.

(A. viii) To each line $(k_1 s_1)$ which attaches to a vertex with label $\lambda = \beta$, assign a factor $\exp(-\beta s_1)$.

(A. ix) To each A vertex or $\Delta H_\pm$ vertex with no lines entering or leaving to the left, assign a factor $(1/2\pi i)$, $\times \int ds_L \exp(-\beta s_L)(s_L - 4\delta)$ to the diagram unless the vertex has temperature label β. To each A vertex, D vertex, or $\Delta H_\pm$ vertex with no lines entering or leaving to the right assign a factor $(1/2\pi i)\int ds_R\,(s_R - 4\delta)$ to the diagram.

(A. x) In a given diagram, there will be either one or two vertices with a given label, λ [cf. Rule (II. vi)]. If only one vertex has label λ, then assign a delta function $(2\pi i)\delta(s \pm s_1 \pm s_2 \pm s_3 \pm s_4)$ to the diagram, where $s_1, \ldots, s_4$ are the energies of the lines that enter and leave the vertex. If two vertices have the label, λ, then assign a delta function $(2\pi i)\delta(s \pm s_1 \pm s_2 \pm \cdots \pm s_8)$ to the diagram, where $s_1, \ldots, s_8$ are the energies of the eight lines entering and leaving the two vertices. The energy, s, appears if the vertex satisfies the conditions of Rule (II. ix). It has values $s = +s_L$ or $s = -s_R$ depending on the type of vertex. The plus and minus signs in the delta functions are assigned as follows: An incoming line directed left is minus; an incoming line directed right is plus; an outgoing line directed left is plus; an outgoing line directed right is minus.

(A. xi) Multiply the entire expression by a factor $\epsilon^{P_B} S^{-1}$, where P_B is the number of permutations in the various matrix elements, and S is the symmetry number of the diagram.

(A. xii) Sum over all momenta and spins. Multiply by $(1/2\pi i)\int ds_j$ for each line of energy s_j and integrate. At the end of the calculation, take the limit $\delta \to 0$.

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A calculation of SU(4) Clebsch–Gordan coefficients*

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All the Clebsch–Gordan coefficients for SU(4) that would be required for particle physics, including those decomposed with respect to SU(3), are obtained by using the general formalism of Baird and Biedenharn.

INTRODUCTION

There has been a renewed interest in the unitary symmetry group SU(4), which was first considered in particle physics more than a decade ago.[1] The reason for this is the discovery[2] of the ψ-resonances at BNL and SLAC. The narrow width of the $\psi(3095)$ and the $\psi(3684)$ has led to the speculation that perhaps a new additive quantum number, called "charm,"[3] should be introduced. A verification of the SU(4) classification of the hadron spectrum awaits the discovery of resonances carrying nonzero charm number. At present, there is no compelling data to support any observational claim for the existence of charmed particles. In the following, we shall compute all the Clebsch–Gordan coefficients necessary to perform calculations within an SU(4) scheme involving scattering amplitudes and decay amplitudes.

The calculation of Clebsch–Gordan coefficients in SU(4) requires an evaluation of coefficients decomposed with respect to SU(3). This decomposition was undertaken earlier in the year[4] for two SU(4) expansions, namely $\underline{15}\otimes\underline{15}$ and $\underline{20}'\otimes\underline{15}$. In the present work we shall give coefficients for all the expansions likely to occur in particle physics. A list of the thirteen relevant relevant expansions[5] appears at the head of Table IV.

We summarize, in Sec. 2, a few important facts about SU(4), while the outline of the computational method for calculating the coefficients is carried out in Sec. 3. Section 4 contains the following Tables: the SU(3) subduction of all 26 SU(4) representations occurring in the expansions[6] (Table I), all the isoscalar factors required in the calculations (Table II) and their symmetry factors (Table III), the SU(3) singlet coefficients (Table IV) and the significant symmetry factors (Table V).

2. GENERAL PROPERTIES OF SU(4)

For a thorough coverage of the most important aspects of SU(n), the reader is referred to the papers of Biedenharn, and Baird and Biedenharn,[7] which we shall use extensively in our calculations of the SU(4) coefficients. We shall discuss some of the basic properties of SU(4) before computing coefficients.

SU(4) is the covering group corresponding to the rank-three Lie algebra A_3. The three conserved quantum numbers I_3, Y, and Z are related to the charge Q by the extended Gell-Mann–Nishijima relation

$$Q=I_3+\tfrac{1}{2}Y+aZ+bN, \tag{2.1}$$

where I_3, Y, and N are the third component of isotopic spin, hypercharge and baryon number, respectively.

The charm number C is defined by

$$C=aZ+bN. \tag{2.2}$$

Here, a and b are constants that depend on the choice of a specific model. In the remainder of this paper, only Z will be used, for it is model independent. The charm number of the various states can then be found from Eq. (2.2), using a and b determined by a specific choice of model for the quark charges.

The four quarks $q=(p,n,\lambda,c)$ are described by the fundamental representation $\underline{4}$ of SU(4) which has the SU(3) decomposition

$$\underline{4}=\underline{1}+\underline{3}. \tag{2.3}$$

For the choice of fractional charges of the four quarks in the Moffat model[8] we have $a=\frac{1}{3}$ and $b=-\frac{1}{4}$, while in the Glashow, Iliopoulos, and Maiani model[9] $a=-1$ and $b=\frac{3}{4}$.

The outer product of $\underline{4}$ and $\underline{4}^*$ gives

$$\underline{4}\otimes\underline{4}^*=\underline{1}+\underline{15}, \tag{2.4}$$

and the mesons are assigned to the adjoint representation $\underline{15}$. This representation has the SU(3) decomposition

$$\underline{15}=\underline{1}+\underline{3}+\underline{3}^*+\underline{8}. \tag{2.5}$$

The baryons fit into the product

$$\underline{4}\otimes\underline{4}\otimes\underline{4}=\underline{4}+2(\underline{20}')+\underline{20}. \tag{2.6}$$

The $J^p=\frac{1}{2}^+$ baryons are assigned to the representation $20'$. The $J^p=\frac{3}{2}^+$ isobars are placed in $\underline{20}$ which contains a decuplet with $C=0$. The $\underline{20}'$ and $\underline{20}$ have the SU(3) decompositions

$$\underline{20}'=\underline{3}+\underline{3}^*+\underline{6}+\underline{8}$$

and

$$\underline{20}=\underline{1}+\underline{3}+\underline{6}+\underline{10}.$$

Mass sum rules have been obtained for the mesons and baryons in SU(4) and SU(8),[10,11] including the familiar Gell-Mann–Okubo mass formula for the $C=0$ particles.

Let us consider some important computational aspects of the SU(4) coefficients. Each state in a representation can be described by a lexical Young tableau with partition $[\lambda]=(\lambda_1\geq\lambda_2\geq\lambda_3)$, where λ_i is the number of boxes in the ith row of the Young tableau.

Consider now the tableau with the form:

First row: m_{11} 1's, followed by $(m_{12}-m_{11})$ 2's, ..., $(m_{14}-m_{13})$ 4's.

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Second row: m_{22} 2's, followed by $(m_{23}-m_{22})$ 3's and $(m_{24}-m_{23})$ 4's.

Third row: m_{33} 3's, followed by $(m_{34}-m_{33})$ 4's.

The partition of this tableau is $[\lambda]=(m_{14}, m_{24}, m_{34})$. We denote the state associated with the tableau by

$$\begin{bmatrix} m_{14} & & m_{24} & & m_{34} & & m_{44} \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{bmatrix},$$

which Biedenharn called the Gel'fand pattern of the state. The m_{ij}'s are related to the eigenvalues of the state through the following equations:

$$\begin{aligned} I&=\tfrac{1}{2}(m_{12}-m_{22}),\\ I_z&=m_{11}-\tfrac{1}{2}(m_{12}+m_{22}),\\ Y&=m_{12}+m_{22}-\tfrac{2}{3}(m_{13}+m_{23}+m_{33}),\\ Z&=m_{13}+m_{23}+m_{33}-\tfrac{3}{4}(m_{14}+m_{24}+m_{34}+m_{44}). \end{aligned} \tag{2.7}$$

Here, m_{44} is always zero.

The requirement that the Young tableau be lexical leads to the condition that

$$m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1}. \tag{2.8}$$

The highest state in a given representation (denoted by m_{14}, m_{24}, and m_{34}) is uniquely defined by the triangular pattern, whose m_{ij}'s are as large as allowed by Eq. (2.8). In terms of m_{14}, m_{24}, and m_{34} (hereafter, called p, q, and r) the eigenvalues of the maximal state of an SU(4) representation read

$$\begin{aligned} I_H&=\tfrac{1}{2}(p-q),\\ Y_H&=\tfrac{1}{3}(p+q-2r),\\ Z_H&=\tfrac{1}{4}(p+q+r). \end{aligned} \tag{2.9}$$

This means that the highest state in a representation is the one that has maximal Z, then maximal Y, and then maximal I_Z. For instance, the maximal state in 15 has $(I_Z, Y, Z)=(\frac{1}{2}, \frac{1}{3}, 1)$. In a more precise, if more cumbersome, language, we could say that the maximal state is the one with highest I_Z in the SU(2) submultiplet having the highest Y, and belonging to the SU(3) submultiplet with the highest Z in the SU(4) representation.

The same considerations carry over to SU(3), where the maximal state in a representation is the one with highest hypercharge, and then highest I_Z. Of two SU(3) representations, the highest is defined to be the one that contains the SU(2) submultiplet with the highest hypercharge.

Thus, a higher dimensional representation in SU(3) is not necessarily higher in our sense (e.g., 6* is higher than 8, and 21 is higher than 24). Only in SU(2) do the two meanings coincide: The higher SU(2) multiplet is the higher dimensional one, corresponding to the higher isospin.

It is seen that these conventions differ from those adopted by de Swart[12] in his tabulation of SU(3) coefficients. Nevertheless, we feel that using the Gel'fand pattern, with Biedenharn's convention, is mathematically more consistent, especially as we proceed to higher dimensional SU(n) groups.

3. COMPUTATION OF THE SU(3) SINGLET COEFFICIENTS

To calculate the coefficients, we first need the matrix elements of the SU(4) generators. There are 15 of these, and we write the infinitesimal generators in terms of the matrices E_{ij} used by Weyl.[7] In the fundamental representation, E_{ij} is the matrix consisting of unity in the (ij)th position and zeros elsewhere. These generators obey the rule

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}, \tag{3.1}$$

where all indices run from 1 to 4.

Define the lowering operators:

$$\begin{aligned} &E_{21}=I_-, \quad E_{41}=K_-,\\ &E_{31}=V_-, \quad E_{42}=L_-,\\ &E_{32}=U_-, \quad E_{43}=M_-. \end{aligned} \tag{3.2}$$

E_{21} generates the isospin subgroup, E_{31} and E_{32} generate the V-spin and U-spin subgroups of SU(3). The other operators generate three SU(2) subgroups in SU(4).

The matrix elements, written in terms of the Gel'fand notation, are then given by Eqs. (60) and (61) of Baird and Biedenharn's Paper II. We reproduce Eq. (60) for the reader's convenience:

$$\begin{aligned} &E_{n,n-1}\left|\begin{bmatrix} m_{1,n} & m_{2,n} & \cdots & m_{nn} \\ m_{1,n-1} & \cdots & m_{i,n-1} & \cdots\ m_{n-1,n-1} \\ & \cdots & & \\ & m_{11} & & \end{bmatrix}\right\rangle \\ &=\sum_{i=1}^{n-1}\left\{\left[\frac{\prod_{j=1}^{n-2}(m_{j,n-2}-m_{i,n-1}-j+i)}{\prod_{j=1,\,j\neq i}^{n-1}(m_{j,n-1}-m_{i,n-1}-j+i)}\right]^{1/2}\left[(-)\frac{\prod_{j=1}^{n}(m_{j,n}-m_{i,n-1}-j+i+1)}{\prod_{j=1,\,j\neq i}^{n-1}(m_{j,n-1}-m_{i,n-1}-j+i+1)}\right]^{1/2}\right\} \\ &\times\left|\begin{bmatrix} m_{1,n} & m_{2,n} & \cdots & m_{nn} \\ m_{1,n-1} & \cdots\ m_{i,n-1}-1 & \cdots & m_{n-1,n-1} \\ & \cdots & & \\ & m_{11} & & \end{bmatrix}\right\rangle. \end{aligned} \tag{3.3}$$

This formula is very easily put on computer, allowing for a rapid calculation of all matrix elements in all representations, a task which would be otherwise rather forbidding.

Equation (61) in Baird and Biedenharn (II), gives the result for the general matrix element of a generator E_{nk} $(n>k)$. This may be obtained from the formula for

TABLE I. SU(3) subduction of SU(4) representations.

Rep. of SU(4)	4		6		10			15			20			
SU(3) submultiplets	3	1	$\bar{3}$	3	6	3	1	3	1+8	$\bar{3}$	10	6	3	1
Z	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	1	0	-1	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$-\frac{9}{4}$

Rep. of SU(4)	20′			20″			36				45			
SU(3) submultiplets	8	$\bar{3}+6$	3	$\bar{6}$	8	6	6	3+15	1+8	$\bar{3}$	15	8+10	$\bar{3}+6$	3
Z	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	1	0	-1	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{7}{4}$	1	0	-1	-2

Rep. of SU(4)	50				60				64			
SU(3) submultiplets	$\overline{10}$	$\overline{15}$	15	10	$\overline{15}$	$\bar{6}+15$	8+10	6	8	$\bar{3}+6+\overline{15}$	$3+\bar{6}+15$	8
Z	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{7}{4}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$

Rep. of SU(4)	70					84				
SU(3) submultiplets	10	6+24	3+15	1+8	$\bar{3}$	6	3+15	1+8+27	$\bar{3}+\overline{15}$	$\bar{6}$
Z	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	2	1	0	-1	-2

Rep. of SU(4)	84′					84″							105				
SU(3) submultiplets	24	15+15′	8+10	$\bar{3}+6$	3	28	21	15′	10	6	3	1	$\overline{15}'$	$\overline{24}$	27	24	15′
Z	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{7}{4}$	$-\frac{11}{4}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{9}{2}$	2	1	0	-1	-2

Rep. of SU(4)	120					
SU(3) submultiplets	15′	10+35	6+24	3+15	1+8	$\bar{3}$
Z	$\frac{7}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$-\frac{9}{4}$	$-\frac{13}{4}$

Rep. of SU(4)	126					140					
SU(3) submultiplets	27	$\overline{15}+24$	$\bar{6}+15+15'$	8+10	6	35	21+24	15+15′	8+10	$\bar{3}+6$	3
Z	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$

Rep. of SU(4)	140′					140″				
SU(3) submultiplets	$\overline{24}$	$\overline{10}+27$	$\overline{15}+24$	15+15′	10	15	8+10+27	$\bar{3}+6+\overline{15}+24$	$3+\bar{6}+15$	8
Z	$\frac{7}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$-\frac{9}{4}$	$\frac{7}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$-\frac{9}{4}$

Rep. of SU(4)	175					224					
SU(3) submultiplets	$\overline{15}$	$\bar{6}+15+\overline{24}$	$8+10+\overline{10}+27$	$6+\overline{15}+24$	15	42	27+35	$\overline{15}+21+24$	$\bar{6}+15+15'$	8+10	6
Z	2	1	0	-1	-2	$\frac{7}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$-\frac{9}{4}$	$-\frac{13}{4}$

Rep. of SU(4)	256					
SU(3) submultiplets	24	15+15′+42	8+10+27+35	$\bar{3}+6+\overline{15}+24$	$3+\bar{6}+15$	8
Z	2	1	0	-1	-2	-3

Rep. of SU(4)	300						
SU(3) submultiplets	10	6+24	3+15+42	1+8+27+64	$\bar{3}+\overline{15}+\overline{42}$	$\bar{6}+\overline{24}$	$\overline{10}$
Z	3	2	1	0	-1	-2	-3

$E_{n,n-1}$, through the use of the commutation relations (3.1).

Equation (3.3) also embodies the general sign convention for SU(n) generators: The matrix elements of $E_{n,n-1}$ are defined to be positive. This means that, in SU(4), I_-, U_-, and M_- are positive. The signs of the matrix ele-elements of the other generators then follow from the commutation relations.

The general SU(4) Clebsch–Gordan coefficient is written as

$$\mathrm{CG}=\left(\begin{array}{cc|c} \nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{array}\right)\times\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y \end{array}\right)\times C^{\lambda_1\lambda_2\lambda}_{I_{1Z}I_{2Z}I_Z}, \tag{3.4}$$

where

TABLES IIA.—IIP. Isoscalar factors$\begin{pmatrix}\mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}$.

Isoscalar factors are tabulated for the following expansions:

IIA: $3\otimes 3=\bar{3}+6$

IIB: $3\otimes\bar{3}=1+8$

IIC: $3\otimes 8=3+\bar{6}+15$

IID: $6\otimes 3=8+10$

IIE: $6\otimes\bar{3}=3+15$

IIF: $6\otimes 6=\bar{6}+15+15'$

IIG: $6\otimes\bar{6}=1+8+27$

IIH: $6\otimes 8=\bar{3}+6+\overline{15}+24$

II I: $8\otimes 8=1+8_D+8_F+10+\overline{10}+27$

IIJ: $10\otimes 3=15+15'$

IIK: $10\otimes\bar{3}=6+24$

IIL: $10\otimes 6=\overline{15}+21+24$

IIM: $10\otimes\bar{6}=3+15+42$

IIN: $10\otimes 8=8+10+27+35$

IIO: $10\otimes 10=\overline{10}+27+28+35$

IIP: $10\otimes\overline{10}=1+8+27+64$

$\begin{pmatrix}\nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z\end{pmatrix}$ is the SU(3) singlet factor,

$\begin{pmatrix}\mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda Y_2 & \lambda Y\end{pmatrix}$ is the SU(2) singlet (de Swart) factor,

$C^{\lambda_1\lambda_2\lambda}_{I_{1Z}I_{2Z}I_Z}$ is the SU(2) Clebsch—Gordan coefficient.

To complete our phase conventions, we will say that the highest Clebsch—Gordan coefficient is defined to be 1. This also makes the highest SU(3) singlet factor to be 1. From the considerations in Sec. II, the highest table in a given $\nu_1\otimes\nu_2$ SU(4) expansion is the one that has the highest Z, and then the highest μ, μ being an SU(3) subgroup. Within a table, the highest factor involves the highest μ_1, and then the highest μ_2. We emphasize that "highest" must be taken in the sense of Sec. II. If $\nu_1=\nu_2=8$, we define 8_D to be higher than 8_F.

In the same way, when computing Table II, which gives all relevant isoscalar factors, the highest table in a given $\mu_1\otimes\mu_2$ expansion is the one with the highest Y, then the highest λ. Within a table, the highest factor has the highest λ_1, then the highest λ_2. Here, of course, since λ_1, λ_2, and λ are SU(2) representations, highest also means highest dimensional.

Table III gives the phase factors ϵ_1 and ϵ_3 associated with the following symmetry properties of the isoscalar factors:

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}=\epsilon_1(-1)^{I-I_1-I_2}\begin{pmatrix}\mu_2 & \mu_1 & \mu \\ \lambda_2 Y_2 & \lambda_1 Y_1 & \lambda Y\end{pmatrix},$$

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}=\epsilon_3(-1)^{I-I_1-I_2}\times\begin{pmatrix}\bar{\mu}_1 & \bar{\mu}_2 & \bar{\mu} \\ \lambda_1,-Y_1 & \lambda_2,-Y_2 & \lambda,-Y\end{pmatrix}. \quad (3.5)$$

Table V gives the phase factors η_1 and η_3 associated with the symmetry properties of the SU(3) singlet factors:

$$\begin{pmatrix}\nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z\end{pmatrix}=\eta_1\epsilon_1\begin{pmatrix}\nu_2 & \nu_1 & \nu \\ \mu_2 Z_2 & \mu_1 Z_1 & \mu Z\end{pmatrix},$$

$$\begin{pmatrix}\nu_1 & \nu_2 & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z\end{pmatrix}=\eta_3\epsilon_3\begin{pmatrix}\bar{\nu}_1 & \bar{\nu}_2 & \bar{\nu} \\ \bar{\mu}_1,-Z_1 & \bar{\mu}_2,-Z_2 & \bar{\mu},-Z\end{pmatrix} \quad (3.6)$$

TABLE IIA. $\begin{pmatrix}3 & 3 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}$.

$3, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6
$2, \frac{1}{3}; 2, \frac{1}{3}$	1

$1, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{3}$
$2, \frac{1}{3}; 2, \frac{1}{3}$	1

$2, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{3}$	6
$1, -\frac{2}{3}; 2, \frac{1}{3}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$2, \frac{1}{3}; 1, -\frac{2}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$1, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6
$1, -\frac{2}{3}; 1, -\frac{2}{3}$	1

TABLE IIB. $\begin{pmatrix}3 & \bar{3} & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}$.

$2, 1$

$\lambda_1, Y_1; \lambda_2, Y_2$	8
$2, \frac{1}{3}; 1, \frac{2}{3}$	1

$3, 0$

$\lambda_1, Y_1; \lambda_2, Y_2$	8
$2, \frac{1}{3}; 2, -\frac{1}{3}$	1

$1, 0$

$\lambda_1, Y_1; \lambda_2, Y_2$	1	8
$1, -\frac{2}{3}; 1, \frac{2}{3}$	$1/\sqrt{3}$	$\sqrt{2/3}$
$2, \frac{1}{3}; 2, -\frac{1}{3}$	$\sqrt{2/3}$	$-1/\sqrt{3}$

$2, -1$

$\lambda_1, Y_1; \lambda_2, Y_2$	8
$1, -\frac{2}{3}; 2, -\frac{1}{3}$	1

TABLE IIC. $\begin{pmatrix}3 & 8 & \mu \\ \lambda_1 Y_1 & \lambda_2 Y_2 & \lambda Y\end{pmatrix}$.

$3, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$2, \frac{1}{3}; 2, 1$	1

$1, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{6}$
$2, \frac{1}{3}; 2, 1$	1

$4, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$2, \frac{1}{3}; 3, 0$	1

$2, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	$\bar{6}$	15
$1, -\frac{2}{3}; 2, 1$	$\sqrt{3}/2\sqrt{2}$	$-1/2$	$\sqrt{3}/2\sqrt{2}$
$2, \frac{1}{3}; 1, 0$	$-1/4$	$\sqrt{3}/2\sqrt{2}$	3/4
$2, \frac{1}{3}; 3, 0$	3/4	$\sqrt{3}/2\sqrt{2}$	$-1/4$

$3, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{6}$	15
$1, -\frac{2}{3}; 3, 0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$2, \frac{1}{3}; 2, -1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$1, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	15
$1, -\frac{2}{3}; 1, 0$	1/2	$\sqrt{3}/2$
$2, \frac{1}{3}; 2, -1$	$\sqrt{3}/2$	$-1/2$

$2, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$1, -\frac{2}{3}; 2, -1$	1

TABLE IID. $\left(\begin{smallmatrix}6 & 3\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ \lambda Y\end{smallmatrix}\right)$.

4, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	10
$3, \frac{2}{3}; 2, \frac{1}{3}$	1

2, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	8
$3, \frac{2}{3}; 2, \frac{1}{3}$	1

3, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8	10
$2, -\frac{1}{3}; 2, \frac{1}{3}$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$3, \frac{2}{3}; 1, -\frac{2}{3}$	$\sqrt{2/3}$	$1/\sqrt{3}$

1, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8
$2, -\frac{1}{3}; 2, \frac{1}{3}$	1

2, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	10
$1, -\frac{4}{3}; 2, \frac{1}{3}$	$-\sqrt{2/3}$	$1/\sqrt{3}$
$2, -\frac{1}{3}; 1, -\frac{2}{3}$	$1/\sqrt{3}$	$\sqrt{2/3}$

1, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	10
$1, -\frac{4}{3}; 1, -\frac{2}{3}$	1

TABLE IIF. $\left(\begin{smallmatrix}6 & 6\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ \lambda Y\end{smallmatrix}\right)$.

$5, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15′
$3, \frac{2}{3}; 3, \frac{2}{3}$	1

$3, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$3, \frac{2}{3}; 3, \frac{2}{3}$	1

$1, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{6}$
$3, \frac{2}{3}; 3, \frac{2}{3}$	1

$4, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	15′
$2, -\frac{1}{3}; 3, \frac{2}{3}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$3, \frac{2}{3}; 2, -\frac{1}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$2, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{6}$	15
$2, -\frac{1}{3}; 3, \frac{2}{3}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$3, \frac{2}{3}; 2, -\frac{1}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$3, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{6}$	15	15′
$1, -\frac{4}{3}; 3, \frac{2}{3}$	$1/\sqrt{3}$	$-1/\sqrt{2}$	$1/\sqrt{6}$
$2, -\frac{1}{3}; 2, -\frac{1}{3}$	$-1/\sqrt{3}$	0	$\sqrt{2/3}$
$3, \frac{2}{3}; 1, -\frac{4}{3}$	$1/\sqrt{3}$	$1/\sqrt{2}$	$1/\sqrt{6}$

$1, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$2, -\frac{1}{3}; 2, -\frac{1}{3}$	1

$2, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	15′
$1, -\frac{4}{3}; 2, -\frac{1}{3}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$2, -\frac{1}{3}; 1, -\frac{4}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$1, -\frac{8}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15′
$1, -\frac{4}{3}; 1, -\frac{4}{3}$	1

TABLE IIG. $\left(\begin{smallmatrix}6 & \bar{6}\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ \lambda Y\end{smallmatrix}\right)$.

3, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$3, \frac{2}{3}; 1, \frac{4}{3}$	1

4, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$3, \frac{2}{3}; 2, \frac{1}{3}$	1

2, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27
$2, -\frac{1}{3}; 1, \frac{4}{3}$	$\sqrt{2/5}$	$\sqrt{3/5}$
$3, \frac{2}{3}; 2, \frac{1}{3}$	$\sqrt{3/5}$	$-\sqrt{2/5}$

2, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$3, \frac{2}{3}; 3, -\frac{2}{3}$	1

3, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27
$2, -\frac{1}{3}; 2, \frac{1}{3}$	$1/\sqrt{5}$	$2/\sqrt{5}$
$3, \frac{2}{3}; 3, -\frac{2}{3}$	$2/\sqrt{5}$	$-1/\sqrt{5}$

1, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	1	8	27
$1, -\frac{4}{3}; 1, \frac{4}{3}$	$1/\sqrt{6}$	$2\sqrt{2/15}$	$\sqrt{3/10}$
$2, -\frac{1}{3}; 2, \frac{1}{3}$	$1/\sqrt{3}$	$1/\sqrt{15}$	$-\sqrt{3/5}$
$3, \frac{2}{3}; 3, -\frac{2}{3}$	$1/\sqrt{2}$	$-\sqrt{2/5}$	$1/\sqrt{10}$

4, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$2, -\frac{1}{3}; 3, -\frac{2}{3}$	1

2, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27
$1, -\frac{4}{3}; 2, \frac{1}{3}$	$\sqrt{2/5}$	$\sqrt{3/5}$
$2, -\frac{1}{3}; 3, -\frac{2}{3}$	$\sqrt{3/5}$	$-\sqrt{2/5}$

3, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$1, -\frac{4}{3}; 3, -\frac{2}{3}$	1

TABLE IIE. $\left(\begin{smallmatrix}6 & \bar{3}\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ \lambda Y\end{smallmatrix}\right)$.

$3, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$3, \frac{2}{3}; 1, \frac{2}{3}$	1

$4, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$3, \frac{2}{3}; 2, -\frac{1}{3}$	1

$2, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	15
$2, -\frac{1}{3}; 1, \frac{2}{3}$	$1/2$	$\sqrt{3}/2$
$3, \frac{2}{3}; 2, -\frac{1}{3}$	$\sqrt{3}/2$	$-1/2$

$3, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$2, -\frac{1}{3}; 2, -\frac{1}{3}$	1

$1, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	15
$1, -\frac{4}{3}; 1, \frac{2}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$2, -\frac{1}{3}; 2, -\frac{1}{3}$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$2, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$1, -\frac{4}{3}; 2, -\frac{1}{3}$	1

TABLE II H. $\begin{pmatrix} 6 & 8 \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{pmatrix} \bigg| \begin{matrix} \mu \\ \lambda Y \end{matrix}$.

$4, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$3, \frac{2}{3}; 2, 1$	1

$2, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$
$3, \frac{2}{3}; 2, 1$	1

$5, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$3, \frac{2}{3}; 3, 0$	1

$3, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6	$\overline{15}$	24
$2, -\frac{1}{3}; 2, 1$	$\sqrt{3/10}$	$-1/\sqrt{6}$	$2\sqrt{2/15}$
$3, \frac{2}{3}; 1, 0$	$-1/\sqrt{10}$	$1/\sqrt{2}$	$\sqrt{2/5}$
$3, \frac{2}{3}; 3, 0$	$\sqrt{3/5}$	$1/\sqrt{3}$	$-1/\sqrt{15}$

$1, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{3}$	$\overline{15}$
$2, -\frac{1}{3}; 2, 1$	$1/2$	$\sqrt{3}/2$
$3, \frac{2}{3}; 3, 0$	$\sqrt{3}/2$	$-1/2$

$4, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	24
$2, -\frac{1}{3}; 3, 0$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$3, \frac{2}{3}; 2, -1$	$\sqrt{2/3}$	$1/\sqrt{3}$

$2, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\bar{3}$	6	$\overline{15}$	24
$1, -\frac{4}{3}; 2, 1$	$-1/2$	$\sqrt{3/10}$	$-1/2$	$1/\sqrt{5}$
$2, -\frac{1}{3}; 1, 0$	$\sqrt{3}/4$	$1/2\sqrt{10}$	$\sqrt{3}/4$	$\sqrt{3/5}$
$2, -\frac{1}{3}; 3, 0$	$-\sqrt{3}/4$	$3/2\sqrt{10}$	$5/4\sqrt{3}$	$-1/\sqrt{15}$
$3, \frac{2}{3}; 2, -1$	$\sqrt{3}/2\sqrt{2}$	$3/2\sqrt{5}$	$-1/2\sqrt{6}$	$-\sqrt{2/15}$

$3, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	24
$1, -\frac{4}{3}; 3, 0$	$-\sqrt{2/3}$	$1/\sqrt{3}$
$2, -\frac{1}{3}; 2, -1$	$1/\sqrt{3}$	$\sqrt{2/3}$

$1, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6	24
$1, -\frac{4}{3}; 1, 0$	$\sqrt{2/5}$	$\sqrt{3/5}$
$2, -\frac{1}{3}; 2, -1$	$\sqrt{3/5}$	$-\sqrt{2/5}$

$2, -\frac{7}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$1, -\frac{4}{3}; 2, -1$	1

TABLE III I. $\begin{pmatrix} 8 & 8 \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{pmatrix} \bigg| \begin{matrix} \mu \\ \lambda Y \end{matrix}$.

$3, 2$

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$2, 1; 2, 1$	1

$1, 2$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$
$2, 1; 2, 1$	1

$4, 1$

$\lambda_1, Y_1; \lambda_2, Y_2$	10	27
$2, 1; 3, 0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$3, 0; 2, 1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$2, 1$

$\lambda_1, Y_1; \lambda_2, Y_2$	8_D	8_F	$\overline{10}$	27
$1, 0; 2, 1$	$-1/2\sqrt{5}$	$1/2$	$-1/2$	$3/2\sqrt{5}$
$2, 1; 1, 0$	$-1/2\sqrt{5}$	$-1/2$	$1/2$	$3/2\sqrt{5}$
$2, 1; 3, 0$	$-3/2\sqrt{5}$	$1/2$	$1/2$	$-1/2\sqrt{5}$
$3, 0; 2, 1$	$3/2\sqrt{5}$	$1/2$	$1/2$	$1/2\sqrt{5}$

$3, 0$

$\lambda_1, Y_1; \lambda_2, Y_2$	8_D	8_F	10	$\overline{10}$	27
$1, 0; 3, 0$	$1/\sqrt{5}$	0	$-1/2$	$-1/2$	$\sqrt{3/10}$
$2, -1; 2, 1$	$-\sqrt{3/10}$	$1/\sqrt{6}$	$1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{5}$
$2, 1; 2, -1$	$-\sqrt{3/10}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	$1/\sqrt{5}$
$3, 0; 1, 0$	$1/\sqrt{5}$	0	$1/2$	$1/2$	$\sqrt{3/10}$
$3, 0; 3, 0$	0	$\sqrt{2/3}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	0

$1, 0$

$\lambda_1, Y_1; \lambda_2, Y_2$	1	8_D	8_F	27
$1, 0; 1, 0$	$-1/2\sqrt{2}$	$-1/\sqrt{5}$	0	$3\sqrt{3}/2\sqrt{10}$
$2, -1; 2, 1$	$1/2$	$1/\sqrt{10}$	$1/\sqrt{2}$	$\sqrt{3}/2\sqrt{5}$
$2, 1; 2, -1$	$-1/2$	$-1/\sqrt{10}$	$1/\sqrt{2}$	$-\sqrt{3}/2\sqrt{5}$
$3, 0; 3, 0$	$\sqrt{3}/2\sqrt{2}$	$-\sqrt{3/5}$	0	$-1/2\sqrt{10}$

$5, 0$

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$3, 0; 3, 0$	1

$4, -1$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$	27
$2, -1; 3, 0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$3, 0; 2, -1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$2, -1$

$\lambda_1, Y_1; \lambda_2, Y_2$	8_D	8_F	10	27
$1, 0; 2, -1$	$-1/2\sqrt{5}$	$-1/2$	$-1/2$	$3/2\sqrt{5}$
$2, -1; 1, 0$	$-1/2\sqrt{5}$	$1/2$	$1/2$	$3/2\sqrt{5}$
$2, -1; 3, 0$	$3/2\sqrt{5}$	$1/2$	$-1/2$	$1/2\sqrt{5}$
$3, 0; 2, -1$	$-3/2\sqrt{5}$	$1/2$	$-1/2$	$-1/2\sqrt{5}$

$3, -2$

$\lambda_1, Y_1; \lambda_2, Y_2$	27
$2, -1; 2, -1$	1

$1, -2$

$\lambda_1, Y_1; \lambda_2, Y_2$	10
$2, -1; 2, -1$	-1

TABLE II J. $\begin{pmatrix} 10 & 3 \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{pmatrix} \begin{vmatrix} \mu \\ \lambda Y \end{vmatrix}$.

$5, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15′
$4, 1; 2, \frac{1}{3}$	1

$3, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$4, 1; 2, \frac{1}{3}$	1

$4, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	15′
$3, 0; 2, \frac{1}{3}$	$-1/2$	$\sqrt{3}/2$
$4, 1; 1, -\frac{2}{3}$	$\sqrt{3}/2$	$1/2$

$2, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$3, 0: 2, 1/3$	1

$3, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	15′
$2, -1; 2, \frac{1}{3}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$3, 0; 1, -\frac{2}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$1, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15
$2, -1; 2, \frac{1}{3}$	1

$2, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	15′
$1, -2; 2, \frac{1}{3}$	$-\sqrt{3}/2$	$1/2$
$2, -1; 1, -\frac{2}{3}$	$1/2$	$\sqrt{3}/2$

$1, -\frac{8}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15′
$1, -2; 1, -\frac{2}{3}$	1

TABLE II K. $\begin{pmatrix} 10 & \bar{3} \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{pmatrix} \begin{vmatrix} \mu \\ \lambda Y \end{vmatrix}$

$4, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$4, 1; 1, \frac{2}{3}$	1

$5, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$4, 1; 2, -\frac{1}{3}$	1

$3, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6	24
$3, 0; 1, \frac{2}{3}$	$1/\sqrt{5}$	$2/\sqrt{5}$
$4, 1; 2, -\frac{1}{3}$	$2/\sqrt{5}$	$-1/\sqrt{5}$

$4, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$3, 0; 2, -\frac{1}{3}$	1

$2, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6	24
$2, -1; 1, \frac{2}{3}$	$\sqrt{2/5}$	$\sqrt{3/5}$
$3, 0; 2, -\frac{1}{3}$	$\sqrt{3/5}$	$-\sqrt{2/5}$

$3, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$2, -1; 2, -\frac{1}{3}$	1

$1, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	6	24
$1, -2; 1, \frac{2}{3}$	$\sqrt{3/5}$	$\sqrt{2/5}$
$2, -1; 2, -\frac{1}{3}$	$\sqrt{2/5}$	$-\sqrt{3/5}$

$2, -\frac{7}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$1, -2; 2, -\frac{1}{3}$	1

TABLE II L. $\begin{pmatrix} 10 & 6 \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{pmatrix} \begin{vmatrix} \mu \\ \lambda Y \end{vmatrix}$.

$6, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	21
$4, 1; 3, \frac{2}{3}$	1

$4, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$4, 1; 3, \frac{2}{3}$	1

$2, \frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$
$4, 1; 3, \frac{2}{3}$	1

$5, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	21	24
$3, 0; 3, \frac{2}{3}$	$\sqrt{3/5}$	$-\sqrt{2/5}$
$4, 1; 2, -\frac{1}{3}$	$\sqrt{2/5}$	$\sqrt{3/5}$

$3, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	24
$3, 0; 3, \frac{2}{3}$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$4, 1; 2, -\frac{1}{3}$	$\sqrt{2/3}$	$1/\sqrt{3}$

$1, \frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$
$3, 0; 3, \frac{2}{3}$	1

$4, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	21	24
$2, -1; 3, \frac{2}{3}$	$1/\sqrt{6}$	$\sqrt{3/10}$	$-2\sqrt{2/15}$
$3, 0; 2, -\frac{1}{3}$	$-1/\sqrt{3}$	$\sqrt{3/5}$	$1/\sqrt{15}$
$4, 1; 1, -\frac{4}{3}$	$1/\sqrt{2}$	$1/\sqrt{10}$	$\sqrt{2/5}$

$2, -\frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	24
$2, -1; 3, \frac{2}{3}$	$-\sqrt{2/3}$	$1/\sqrt{3}$
$3, 0; 2, -\frac{1}{3}$	$1/\sqrt{3}$	$\sqrt{2/3}$

$3, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{15}$	21	24
$1, -2; 3, \frac{2}{3}$	$1/\sqrt{2}$	$1/\sqrt{10}$	$-\sqrt{2/5}$
$2, -1; 2, -\frac{1}{3}$	$-1/\sqrt{3}$	$\sqrt{3/5}$	$-1/\sqrt{15}$
$3, 0; 1, -\frac{4}{3}$	$1/\sqrt{6}$	$\sqrt{3/10}$	$2\sqrt{2/15}$

$1, -\frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	24
$2, -1; 2, -\frac{1}{3}$	1

$2, -\frac{7}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	21	24
$1, -2; 2, -\frac{1}{3}$	$\sqrt{2/5}$	$-\sqrt{3/5}$
$2, -1; 0, -\frac{4}{3}$	$\sqrt{3/5}$	$\sqrt{2/5}$

$1, -\frac{10}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	21
$1, -2; 1, -\frac{4}{3}$	1

TABLE IIM. $\left(\begin{matrix}10 & \overline{6}\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{matrix}\middle|\begin{matrix}\mu\\ \lambda Y\end{matrix}\right)$.

$4, \frac{7}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	42
$4, 1; 1, \frac{4}{3}$	1

$5, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	42
$4, 1; 2, \frac{1}{3}$	1

$3, \frac{4}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	42
$3, 0; 1, \frac{4}{3}$	$1/\sqrt{3}$	$\sqrt{2/3}$
$4, 1; 2, \frac{1}{3}$	$\sqrt{2/3}$	$-1/\sqrt{3}$

$6, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	42
$4, 1; 3, -\frac{2}{3}$	1

$4, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	42
$3, 0; 2, \frac{1}{3}$	$1/\sqrt{6}$	$\sqrt{5/6}$
$4, 1; 3, -\frac{2}{3}$	$\sqrt{5/6}$	$-1/\sqrt{6}$

$2, \frac{1}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	15	42
$2, -1; 1, \frac{4}{3}$	$1/\sqrt{10}$	$1/\sqrt{2}$	$\sqrt{2/5}$
$3, 0; 2, \frac{1}{3}$	$\sqrt{3/10}$	$1/\sqrt{6}$	$-2\sqrt{2/15}$
$4, 1; 3, -\frac{2}{3}$	$\sqrt{3/5}$	$-1/\sqrt{3}$	$1/\sqrt{15}$

$5, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	42
$3, 0; 3, -\frac{2}{3}$	1

$3, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	42
$3, 0; 3, -\frac{2}{3}$	$\sqrt{2/3}$	$-1/\sqrt{3}$
$2, -1; 2, \frac{1}{3}$	$1/\sqrt{3}$	$\sqrt{2/3}$

$1, -\frac{2}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	3	15	42
$1, -2; 1, \frac{4}{3}$	$\sqrt{3/10}$	$1/\sqrt{2}$	$1/\sqrt{5}$
$2, -1; 2, \frac{1}{3}$	$\sqrt{2/5}$	0	$-\sqrt{3/5}$
$3, 0; 3, -\frac{2}{3}$	$\sqrt{3/10}$	$-1/\sqrt{2}$	$1/\sqrt{5}$

$4, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	42
$2, -1; 3, -\frac{2}{3}$	1

$2, -\frac{5}{3}$

$\lambda_1, Y_1; \lambda_2, Y_2$	15	42
$1, -2; 2, \frac{1}{3}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$2, -1; 3, -\frac{2}{3}$	$1/\sqrt{2}$	$-1/\sqrt{2}$

TABLE IIN. $\left(\begin{matrix}10 & 8\\ \lambda_1 Y_1 & \lambda_2 Y_2\end{matrix}\middle|\begin{matrix}\mu\\ \lambda Y\end{matrix}\right)$.

5, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	35
4, 1; 2, 1	1

3, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	27
4, 1; 2, 1	1

6, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	35
4, 1; 3, 0	1

4, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	10	27	35
3, 0; 2, 1	$1/2$	$-1/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
4, 1; 1, 0	$-1/2\sqrt{2}$	$3/4$	$\sqrt{5}/4$
4, 1; 3, 0	$\sqrt{5}/2\sqrt{2}$	$\sqrt{5}/4$	$-1/4$

2, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27
3, 0; 2, 1	$1/\sqrt{5}$	$2/\sqrt{5}$
4, 1; 3, 0	$2/\sqrt{5}$	$-1/\sqrt{5}$

5, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	27	35
3, 0; 3, 0	$-1/2$	$\sqrt{3}/2$
4, 1; 2, −1	$\sqrt{3}/2$	$1/2$

3, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8	10	27	35
2, −1; 2, 1	$-\sqrt{2/15}$	$1/\sqrt{3}$	$-1/\sqrt{5}$	$1/\sqrt{3}$
3, 0; 1, 0	$1/\sqrt{5}$	0	$\sqrt{3/10}$	$1/\sqrt{2}$
3, 0; 3, 0	$-\sqrt{2/15}$	$1/\sqrt{3}$	$3/2\sqrt{5}$	$-1/2\sqrt{3}$
4, 1; 2, −1	$2\sqrt{2/15}$	$1/\sqrt{3}$	$-1/2\sqrt{5}$	$-1/2\sqrt{3}$

1, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27
2, −1; 2, 1	$\sqrt{2/5}$	$\sqrt{3/5}$
3, 0; 3, 0	$\sqrt{3/5}$	$-\sqrt{2/5}$

4, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	27	35
2, −1; 3, 0	$-1/\sqrt{2}$	$1/\sqrt{2}$
3, 0; 2, −1	$1/\sqrt{2}$	$1/\sqrt{2}$

2, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	10	27	35
1, −2; 2, 1	$-\sqrt{2/5}$	$1/2$	$-3/2\sqrt{10}$	$1/2\sqrt{2}$
2, −1; 1, 0	$1/\sqrt{5}$	$1/2\sqrt{2}$	$3/4\sqrt{5}$	$3/4$
2, −1; 3, 0	$-1/\sqrt{5}$	$1/2\sqrt{2}$	$7/4\sqrt{5}$	$-1/4$
3, 0; 2, −1	$1/\sqrt{5}$	$1/\sqrt{2}$	$-1/2\sqrt{5}$	$-1/2$

3, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	27	35
1, −2; 3, 0	$-\sqrt{3}/2$	$1/2$
2, −1; 2, −1	$1/2$	$\sqrt{3/4}$

1, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	10	35
1, −2; 1, 0	$1/\sqrt{2}$	$1/\sqrt{2}$
2, −1; 2, −1	$1/\sqrt{2}$	$-1/\sqrt{2}$

2, −3

$\lambda_1, Y_1; \lambda_2, Y_2$	35
1, −2; 2, −1	1

TABLE II O. $\left(\begin{smallmatrix} 10 & 10 \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{smallmatrix} \middle| \begin{smallmatrix} \mu \\ \lambda Y \end{smallmatrix}\right)$.

7, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	28
4, 1; 4, 1	1

5, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	35
4, 1; 4, 1	1

3, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	27
4, 1; 4, 1	1

1, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$
4, 1; 4, 1	1

6, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	28	35
3, 0; 4, 1	$1/\sqrt{2}$	$-1/\sqrt{2}$
4, 1; 3, 0	$1/\sqrt{2}$	$1/\sqrt{2}$

4, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	27	35
3, 0; 4, 1	$-1/\sqrt{2}$	$1/\sqrt{2}$
4, 1; 3, 0	$1/\sqrt{2}$	$1/\sqrt{2}$

2, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$	27
3, 0; 4, 1	$-1/\sqrt{2}$	$1/\sqrt{2}$
4, 1; 3, 0	$1/\sqrt{2}$	$1/\sqrt{2}$

5, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	27	28	35
2, −1; 4, 1	$\sqrt{3/10}$	$1/\sqrt{5}$	$-1/\sqrt{2}$
3, 0; 3, 0	$-\sqrt{2/5}$	$\sqrt{3/5}$	0
4, 1; 2, −1	$\sqrt{3/10}$	$1/\sqrt{5}$	$1/\sqrt{2}$

3, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$	27	35
2, −1; 4, 1	$1/\sqrt{3}$	$-1/\sqrt{2}$	$1/\sqrt{6}$
3, 0; 3, 0	$-1/\sqrt{3}$	0	$\sqrt{2/3}$
4, 1; 2, −1	$1/\sqrt{3}$	$1/\sqrt{2}$	$1/\sqrt{6}$

1, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	27
3, 0; 3, 0	1

4, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	$\overline{10}$	27	28	35
1, −2; 4, 1	$-1/2$	$3/2\sqrt{5}$	$1/2\sqrt{5}$	$-1/2$
2, −1; 3, 0	$1/2$	$-1/2\sqrt{5}$	$3/2\sqrt{5}$	$-1/2$
3, 0; 2, −1	$-1/2$	$-1/2\sqrt{5}$	$3/2\sqrt{5}$	$1/2$
4, 1; 1, −2	$1/2$	$3/2\sqrt{5}$	$1/2\sqrt{5}$	$1/2$

2, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	27	35
2, −1; 3, 0	$-1/\sqrt{2}$	$1/\sqrt{2}$
3, 0; 2, −1	$1/\sqrt{2}$	$1/\sqrt{2}$

3, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	27	28	35
1, −2; 3, 0	$\sqrt{3/10}$	$1/\sqrt{5}$	$-1/\sqrt{2}$
2, −1; 2, −1	$-\sqrt{2/5}$	$\sqrt{3/5}$	0
3, 0; 1, −2	$\sqrt{3/10}$	$1/\sqrt{5}$	$1/\sqrt{2}$

1, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	35
2, −1; 2, −1	1

2, −3

$\lambda_1, Y_1; \lambda_2, Y_2$	28	35
1, −2; 2, −1	$1/\sqrt{2}$	$-1/\sqrt{2}$
2, −1; 1, −2	$1/\sqrt{2}$	$1/\sqrt{2}$

1, −4

$\lambda_1, Y_1; \lambda_2, Y_2$	28
1, −2; 1, −2	1

TABLE II P. $\left(\begin{smallmatrix} 10 & \overline{10} \\ \lambda_1 Y_1 & \lambda_2 Y_2 \end{smallmatrix} \middle| \begin{smallmatrix} \mu \\ \lambda Y \end{smallmatrix}\right)$.

4, 3

$\lambda_1, Y_1; \lambda_2, Y_2$	64
4, 1; 1, 2	1

5, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	64
4, 1; 2, 1	1

3, 2

$\lambda_1, Y_1; \lambda_2, Y_2$	27	64
3, 0; 1, 2	$\sqrt{3/7}$	$2/\sqrt{7}$
4, 1; 2, 1	$2/\sqrt{7}$	$-\sqrt{3/7}$

6, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	64
4, 1; 3, 0	1

4, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	27	64
3, 0; 2, 1	$\sqrt{2/7}$	$\sqrt{5/7}$
4, 1; 3, 0	$\sqrt{5/7}$	$-\sqrt{2/7}$

2, 1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27	64
2, −1; 1, 2	$1/\sqrt{5}$	$3\sqrt{2/35}$	$\sqrt{2/7}$
3, 0; 2, 1	$\sqrt{2/5}$	$1/\sqrt{35}$	$-2/\sqrt{7}$
4, 1; 3, 0	$\sqrt{2/5}$	$-4/\sqrt{35}$	$1/\sqrt{7}$

7, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	64
4, 1; 4, −1	1

5, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	27	64
3, 0; 3, 0	$1/\sqrt{7}$	$\sqrt{6/7}$
4, 1; 4, −1	$\sqrt{6/7}$	$-1/\sqrt{7}$

3, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27	64
2, −1; 2, 1	$1/\sqrt{15}$	$4\sqrt{35}$	$\sqrt{10/21}$
3, 0; 3, 0	$2/\sqrt{15}$	$3/\sqrt{35}$	$-\sqrt{10/21}$
4, 1; 4, −1	$\sqrt{2/3}$	$-\sqrt{2/7}$	$1/\sqrt{21}$

1, 0

$\lambda_1, Y_1; \lambda_2, Y_2$	1	8	27	64
1, −2; 1, 2	$1/\sqrt{10}$	$\sqrt{2/5}$	$3\sqrt{3/10}$	$2/\sqrt{35}$
2, −1; 2, 1	$1/\sqrt{5}$	$1/\sqrt{5}$	$-\sqrt{3/35}$	$-3\sqrt{2/35}$
3, 0; 3, 0	$\sqrt{3/10}$	0	$-\sqrt{5/14}$	$2\sqrt{3/35}$
4, 1; 4, −1	$\sqrt{2/5}$	$-\sqrt{2/5}$	$\sqrt{6/35}$	$-1/\sqrt{35}$

6, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	64
3, 0; 4, −1	1

4, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	27	64
2, −1; 3, 0	$\sqrt{2/7}$	$\sqrt{5/7}$
3, 0; 4, −1	$\sqrt{5/7}$	$-\sqrt{2/7}$

2, −1

$\lambda_1, Y_1; \lambda_2, Y_2$	8	27	64
1, −2; 2, 1	$1/\sqrt{5}$	$3\sqrt{2/35}$	$\sqrt{2/7}$
2, −1; 3, 0	$\sqrt{2/5}$	$1/\sqrt{35}$	$-2/\sqrt{7}$
3, 0; 4, −1	$\sqrt{2/5}$	$-4/\sqrt{35}$	$1/\sqrt{7}$

5, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	64
2, −1; 4, −1	1

3, −2

$\lambda_1, Y_1; \lambda_2, Y_2$	27	64
1, −2; 3, 0	$\sqrt{3/7}$	$2/\sqrt{7}$
2, −1; 4, −1	$2/\sqrt{7}$	$-\sqrt{3/7}$

4, −3

$\lambda_1, Y_1; \lambda_2, Y_2$	64
1, −2; 4, −1	1

TABLE III. Phase factors involved in the symmetry properties of SU(3) singlet factors.

λ_1	λ_2	λ	ϵ_1	ϵ_3
3	3	$\bar{3}$	−1	1
		6	1	1
3	$\bar{3}$	1	−1	−1
		8	1	1
3	8	3	−1	−1
		$\bar{6}$	−1	1
		15	1	1
6	3	8	−1	1
		10	1	1
6	$\bar{3}$	3	−1	−1
		15	1	1
6	6	$\bar{6}$	1	1
		15	−1	−1
		15′	1	1
6	$\bar{6}$	1	1	1
		8	−1	−1
		27	1	1
6	8	$\bar{3}$	1	−1
		6	−1	−1
		$\overline{15}$	−1	1
		24	1	1
8	8	1	1	1
		8_D	1	1
		8_F	−1	−1
		10	−1	1
		$\overline{10}$	−1	1
		27	1	1
10	3	15	−1	1
		15′	1	1
10	$\bar{3}$	6	−1	−1
		24	1	1
10	6	$\overline{15}$	1	1
		21	1	1
		24	−1	1
10	$\bar{6}$	3	1	1
		15	−1	−1
		42	1	1
10	8	8	1	−1
		10	−1	−1
		27	−1	1
		35	1	1
10	10	$\overline{10}$	−1	1
		27	1	1
		28	1	1
		35	−1	1
10	$\overline{10}$	1	1	1
		8	1	1
		27	−1	−1
		64	1	1

TABLES IVA.—IVM. SU(3) singlet factors $\left(\begin{smallmatrix}\nu_1\\ \mu_1 Z_1\end{smallmatrix}\ \begin{smallmatrix}\nu_2\\ \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu\\ \mu Z\end{smallmatrix}\right)$. SU(3) singlet factors are tabulated for the following expansions:

IVA: $15\otimes 15 = 1 + 15_D + 15_F + 20'' + 45 + \overline{45} + 84$

IVB: $20\otimes 15 = 20 + 20' + 120 + 140''$

IVC: $20\otimes 20 = 50 + 84'' + 126 + 140$

IVD: $20\otimes \overline{20} = 1 + 15 + 84 + 300$

IVE: $20\otimes 20' = 64 + 70 + 126 + 140$

IVF: $20\otimes \overline{20}' = 15 + 45 + 84 + 256$

IVG: $20\otimes 20'' = \overline{36} + 140'' + 224$

IVH: $20'\otimes 15 = \bar{4} + 20 + 20'_1 + 20'_2 + \overline{36} + \overline{60} + 140''$

IVI: $20'\otimes 20' = 6 + 10 + \overline{10} + 50 + 64_D + 64_F + 70 + 126$

IVJ: $20'\otimes \overline{20}' = 1 + 15_1 + 15_2 + 20'' + 45 + \overline{45} + 84 + 175$

IVK: $20'\otimes 20'' = \bar{4} + 20' + \overline{36} + \overline{60} + 140' + 140''$

IVL: $20''\otimes 15 = 15 + 20'' + 45 + \overline{45} + 175$

IVM: $20''\otimes 20'' = 1 + 15 + 20'' + 84 + 105 + 175$

TABLE IVA. $\left(\begin{smallmatrix}15 & 15 \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

6,2

$\mu_1,Z_1;\mu_2,Z_2$	84
3,1;3,1	1

$\bar{3}$,2

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$
3,1;3,1	1

$\bar{6}$,1

$\mu_1,Z_1;\mu_2,Z_2$	20″	$\overline{45}$
3,1;8,0	$-1/\sqrt{2}$	$1/\sqrt{2}$
8,0;3,1	$1/\sqrt{2}$	$1/\sqrt{2}$

15,1

$\mu_1,Z_1;\mu_2,Z_2$	45	84
3,1;8,0	$-1/\sqrt{2}$	$1/\sqrt{2}$
8,0;3,1	$1/\sqrt{2}$	$1/\sqrt{2}$

3,1

$\mu_1Z_1;\mu_2,Z_2$	15_D	15_F	$\overline{45}$	84
1,0;3,1	$1/3\sqrt{2}$	$-1/\sqrt{6}$	$-1/\sqrt{3}$	2/3
3,1;1,0	$1/3\sqrt{2}$	$1/\sqrt{6}$	$1/\sqrt{3}$	2/3
3,1;8,0	$-2/3$	$1/\sqrt{3}$	$-1/\sqrt{6}$	$1/3\sqrt{2}$
8,0;3,1	2/3	$1/\sqrt{3}$	$-1/\sqrt{6}$	$-1/3\sqrt{2}$

27,0

$\mu_1,Z_1;\mu_2,Z_2$	84
8,0;8,0	1

$\overline{10}$,0

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$
8,0;8,0	1

10,0

$\mu_1,Z_1;\mu_2,Z_2$	45
8,0;8,0	1

8,0

$\mu_1,Z_1;\mu_2,Z_2$	15_D	15_F	20″	45	$\overline{45}$	84	
1,0;8,0	$-1/3\sqrt{2}$	0	$1/\sqrt{6}$	$-1/2$	$-1/2$	$\sqrt{5}/3\sqrt{2}$	
3,1;$\bar{3}$,−1	$1/\sqrt{6}$	$1/2\sqrt{2}$	$-1/2\sqrt{2}$	$-\sqrt{3}/4$	$\sqrt{3}/4$	$\sqrt{5}/2\sqrt{6}$	
$\bar{3}$,−1;3,1	$1/\sqrt{6}$	$-1/2\sqrt{2}$	$-1/2\sqrt{2}$	$\sqrt{3}/4$	$-\sqrt{3}/4$	$\sqrt{5}/2\sqrt{6}$	
8,0;1,0	$-1/3\sqrt{2}$	0	$1/\sqrt{6}$	1/2	1/2	$\sqrt{5}/3\sqrt{2}$	
8,0;8,0	0	$\sqrt{3}/2$	0	$1/2\sqrt{2}$	$-1/2\sqrt{2}$	0	*F*
8,0;8,0	$\sqrt{5}/3$	0	$\sqrt{5}/2\sqrt{3}$	0	0	$-1/6$	*D*

1,0

$\mu_1,Z_1;\mu_2,Z_2$	1	15_D	15_F	84
1,0;1,0	$-1/\sqrt{15}$	$\sqrt{2}/3$	0	$4\sqrt{2}/3\sqrt{5}$
3,1;$\bar{3}$,−1	$1/\sqrt{5}$	$-1/\sqrt{6}$	$1/\sqrt{2}$	$\sqrt{2/15}$
$\bar{3}$,−1;3,1	$-1/\sqrt{5}$	$1/\sqrt{6}$	$1/\sqrt{2}$	$-\sqrt{2/15}$
8,0;8,0	$2\sqrt{2/15}$	2/3	0	$-1/3\sqrt{5}$

$\overline{15}$,−1

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$	84
$\bar{3}$,1;8,0	$-1/\sqrt{2}$	$1/\sqrt{2}$
8,0;$\bar{3}$,1	$1/\sqrt{2}$	$1/\sqrt{2}$

6,−1

$\mu_1,Z_1;\mu_2,Z_2$	20″	45
$\bar{3}$,−1;8,0	$-1/\sqrt{2}$	$1/\sqrt{2}$
8,0;$\bar{3}$,−1	$1/\sqrt{2}$	$1/\sqrt{2}$

$\bar{3}$,−1

$\mu_1,Z_1;\mu_2,Z_2$	15_D	15_F	45	84
1,0;$\bar{3}$,−1	$1/3\sqrt{2}$	$1/\sqrt{6}$	$-1/\sqrt{3}$	2/3
$\bar{3}$,−1;1,0	$1/3\sqrt{2}$	$-1/\sqrt{6}$	$1/\sqrt{3}$	2/3
$\bar{3}$,−1;8,0	2/3	$1/\sqrt{3}$	$1/\sqrt{6}$	$-1/3\sqrt{2}$
8,0;$\bar{3}$,−1	$-2/3$	$1/\sqrt{3}$	$1/\sqrt{6}$	$1/3\sqrt{2}$

$\bar{6}$,−2

$\mu_1,Z_1;\mu_2,Z_2$	84
$\bar{3}$,−1;$\bar{3}$,−1	1

3,−2

$\mu_1,Z_1;\mu_2,Z_2$	45
$\bar{3}$,−1;$\bar{3}$,−1	1

TABLE IVB. $\left(\begin{smallmatrix}20 & 15\\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu\\ \mu Z\end{smallmatrix}\right)$.

$15', \frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120
$10,\frac{3}{4};3,1$	1

$15, \frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	140″
$10,\frac{3}{4};3,1$	1

$35, \frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120
$10,\frac{3}{4};8,0$	1

$27, \frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	140″
$10,\frac{3}{4};8,0$	1

$10, \frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	120	140″
$6,-\frac{1}{4};3,1$	$-2/\sqrt{21}$	$3/\sqrt{14}$	$-1/\sqrt{6}$
$10,\frac{3}{4};1,0$	$1/\sqrt{21}$	$\sqrt{2/7}$	$\sqrt{2/3}$
$10,\frac{3}{4};8,0$	$4/\sqrt{21}$	$1/\sqrt{14}$	$-1/\sqrt{6}$

$8, \frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20′	140″
$6,-\frac{1}{4};3,1$	$1/\sqrt{6}$	$\sqrt{5/6}$
$10,\frac{3}{4};8,0$	$\sqrt{5/6}$	$-1/\sqrt{6}$

$24, -\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120	140″
$6,-\frac{1}{4};8,0$	$\sqrt{3}/2$	$-1/2$
$10,\frac{3}{4};\bar{3},-1$	$1/2$	$\sqrt{3}/2$

$\overline{15}, -\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	140″
$6,-\frac{1}{4};8,0$	1

$6, -\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	20′	120	140″
$3,-\frac{5}{4};3,1$	$-4/3\sqrt{7}$	$-1/3$	$\sqrt{5/14}$	$-\sqrt{5}/3\sqrt{2}$
$6,-\frac{1}{4};1,0$	$-1/3\sqrt{21}$	$2/3\sqrt{3}$	$\sqrt{10/21}$	$\sqrt{10}/3\sqrt{3}$
$6,-\frac{1}{4};8,0$	$4\sqrt{5}/3\sqrt{21}$	$\sqrt{5}/3\sqrt{3}$	$\sqrt{2/21}$	$-2\sqrt{2}/3\sqrt{3}$
$10,\frac{3}{4};\bar{3},-1$	$2\sqrt{5}/3\sqrt{7}$	$-\sqrt{5}/3$	$1/\sqrt{14}$	$1/3\sqrt{2}$

$\bar{3}, -\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20′	140″
$3,-\frac{5}{4};3,1$	$1/\sqrt{3}$	$\sqrt{2/3}$
$6,-\frac{1}{4};8,0$	$-\sqrt{2/3}$	$1/\sqrt{3}$

$15, -\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120	140″
$3,-\frac{5}{4};8,0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$6,-\frac{1}{4};\bar{3},-1$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$\bar{6}, -\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	140″
$3,-\frac{5}{4};8,0$	1

$3, -\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	20′	120	140″
$1,-\frac{9}{4};3,1$	$2/\sqrt{21}$	$-1/\sqrt{3}$	$1/\sqrt{7}$	$-1/\sqrt{3}$
$3,-\frac{5}{4};1,0$	$5/3\sqrt{21}$	$2/3\sqrt{3}$	$2/\sqrt{7}$	$2/3\sqrt{3}$
$3,-\frac{5}{4};8,0$	$-4\sqrt{6}/9\sqrt{7}$	$2\sqrt{2}/3\sqrt{3}$	$1/\sqrt{14}$	$-5/3\sqrt{6}$
$6,-\frac{1}{4};\bar{3},-1$	$-4\sqrt{2}/3\sqrt{7}$	$-\sqrt{2}/3$	$\sqrt{3/14}$	$1/3\sqrt{2}$

$8, -\frac{9}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120	140″
$1,-\frac{9}{4};8,0$	$1/2$	$\sqrt{3}/2$
$3,-\frac{5}{4};\bar{3},-1$	$\sqrt{3}/2$	$-1/2$

$1, -\frac{9}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	120
$1,-\frac{9}{4};1,0$	$\sqrt{3/7}$	$2/\sqrt{7}$
$3,-\frac{5}{4};\bar{3},-1$	$-2/\sqrt{7}$	$\sqrt{3/7}$

$\bar{3}, -\frac{13}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	120
$1,-\frac{9}{4};\bar{3},-1$	1

TABLE IVC. $\left(\begin{smallmatrix}20 & 20 \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

$28, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″
$10, \frac{3}{4}; 10, \frac{3}{4}$	1

$35, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	140
$10, \frac{3}{4}; 10, \frac{3}{4}$	1

$27, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$10, \frac{3}{4}; 10, \frac{3}{4}$	1

$\overline{10}, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50
$10, \frac{3}{4}; 10, \frac{3}{4}$	1

$21, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″	140
$6, -\frac{1}{4}; 10, \frac{3}{4}$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$10, \frac{3}{4}; 6, -\frac{1}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$24, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126	140
$6, -\frac{1}{4}; 10, \frac{3}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$10, \frac{3}{4}; 6, -\frac{1}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$\overline{15}, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	126
$6, -\frac{1}{4}; 10, \frac{3}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$10, \frac{3}{4}; 6, -\frac{1}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$15', -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″	126	140
$3, -\frac{5}{4}; 10, \frac{3}{4}$	$1/\sqrt{5}$	$\sqrt{3/10}$	$-1/\sqrt{2}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$\sqrt{3/5}$	$-\sqrt{2/5}$	0
$10, \frac{3}{4}; 3, -\frac{5}{4}$	$1/\sqrt{5}$	$\sqrt{3/10}$	$1/\sqrt{2}$

$15, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	126	140
$3, -\frac{5}{4}; 10, \frac{3}{4}$	$1/\sqrt{3}$	$-1/\sqrt{2}$	$1/\sqrt{6}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{3}$	0	$\sqrt{2/3}$
$10, \frac{3}{4}; 3, -\frac{5}{4}$	$1/\sqrt{3}$	$1/\sqrt{2}$	$1/\sqrt{6}$

$\bar{6}, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	1

$10, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	84″	126	140
$1, -\frac{9}{4}; 10, \frac{3}{4}$	$-1/2$	$1/2\sqrt{5}$	$3/2\sqrt{5}$	$-1/2$
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$1/2$	$3/2\sqrt{5}$	$-1/2\sqrt{5}$	$-1/2$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$-1/2$	$3/2\sqrt{5}$	$-1/2\sqrt{5}$	$1/2$
$10, \frac{3}{4}; 1, -\frac{9}{4}$	$1/2$	$1/2\sqrt{5}$	$3/2\sqrt{5}$	$1/2$

$8, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126	140
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$6, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″	126	140
$1, -\frac{9}{4}; 6, -\frac{1}{4}$	$1/\sqrt{5}$	$\sqrt{3/10}$	$-1/\sqrt{2}$
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	$\sqrt{3/5}$	$-\sqrt{2/5}$	0
$6, -\frac{1}{4}; 1, -\frac{9}{4}$	$1/\sqrt{5}$	$\sqrt{3/10}$	$1/\sqrt{2}$

$\bar{3}, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	140
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	1

$3, -\frac{7}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″	140
$1, -\frac{9}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$3, -\frac{5}{4}; 1, -\frac{9}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$1, -\frac{9}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	84″
$1, -\frac{9}{4}; 1, -\frac{9}{4}$	1

TABLE IVD. $\left(\begin{smallmatrix}20 & \overline{20} \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle| \begin{smallmatrix}\nu\\ \mu Z\end{smallmatrix}\right)$.

$10,3$

$\mu_1,Z_1;\mu_2,Z_2$	300
$10,\frac{3}{4};1,\frac{9}{4}$	1

$24,2$

$\mu_1,Z_1;\mu_2,Z_2$	300
$10,\frac{3}{4};\bar{3},\frac{5}{4}$	1

$6,2$

$\mu_1,Z_1;\mu_2,Z_2$	84	300
$6,-\frac{1}{4};1,\frac{9}{4}$	$-\sqrt{3}/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
$10,\frac{3}{4};\bar{3},\frac{5}{4}$	$\sqrt{5}/2\sqrt{2}$	$\sqrt{3}/2\sqrt{2}$

$42,1$

$\mu_1,Z_1;\mu_2,Z_2$	300
$10,\frac{3}{4};\bar{6},-\frac{1}{4}$	1

$15,1$

$\mu_1,Z_1;\mu_2,Z_2$	84	300
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-1/2$	$\sqrt{3}/2$
$10,\frac{3}{4};\bar{6},\frac{1}{4}$	$\sqrt{3}/2$	$1/2$

$3,1$

$\mu_1,Z_1;\mu_2,Z_2$	15	84	300
$3,-\frac{5}{4};1,\frac{9}{4}$	$1/\sqrt{7}$	$-1/\sqrt{2}$	$\sqrt{5/14}$
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-2\sqrt{2/21}$	$1/2\sqrt{3}$	$\sqrt{15}/2\sqrt{7}$
$10,\frac{3}{4};\bar{6},\frac{1}{4}$	$\sqrt{10/21}$	$\sqrt{5}/2\sqrt{3}$	$\sqrt{3}/2\sqrt{7}$

$64,0$

$\mu_1,Z_1;\mu_2,Z_2$	300
$10,\frac{3}{4};\overline{10},-\frac{3}{4}$	1

$27,0$

$\mu_1,Z_1;\mu_2,Z_2$	84	300
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$\sqrt{7}/2\sqrt{2}$	$1/2\sqrt{2}$
$10,\frac{3}{4};\overline{10},-\frac{3}{4}$	$-1/2\sqrt{2}$	$\sqrt{7}/2\sqrt{2}$

$8,0$

$\mu_1,Z_1;\mu_2,Z_2$	15	84	300
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$1/\sqrt{21}$	$-\sqrt{5}/2\sqrt{3}$	$\sqrt{15}/2\sqrt{7}$
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$-\sqrt{5/21}$	$1/\sqrt{3}$	$\sqrt{3/7}$
$10,\frac{3}{4};\overline{10},-\frac{3}{4}$	$\sqrt{5/7}$	$1/2$	$1/2\sqrt{7}$

$1,0$

$\mu_1,Z_1;\mu_2,Z_2$	1	15	84	300
$1,-\frac{9}{4};1,\frac{9}{4}$	$-1/2\sqrt{5}$	$3/2\sqrt{7}$	$-3/2\sqrt{5}$	$\sqrt{5}/2\sqrt{7}$
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$\sqrt{3}/2\sqrt{5}$	$-5/2\sqrt{21}$	$-1/2\sqrt{15}$	$\sqrt{15}/2\sqrt{7}$
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$-\sqrt{3/10}$	$1/\sqrt{42}$	$7/2\sqrt{30}$	$\sqrt{15}/2\sqrt{14}$
$10,\frac{3}{4};\overline{10},-\frac{3}{4}$	$1/\sqrt{2}$	$\sqrt{5/14}$	$1/2\sqrt{2}$	$1/2\sqrt{14}$

$\overline{42},-1$

$\mu_1,Z_1;\mu_2,Z_2$	300
$6,-\frac{1}{4};\overline{10},-\frac{3}{4}$	1

$\overline{15},-1$

$\mu_1,Z_1;\mu_2,Z_2$	84	300
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$-1/2$	$\sqrt{3}/2$
$6,-\frac{1}{4};\overline{10},-\frac{3}{4}$	$\sqrt{3}/2$	$1/2$

$\bar{3},-1$

$\mu_1,Z_1;\mu_2,Z_2$	15	84	300
$1,-\frac{9}{4};\bar{3},\frac{5}{4}$	$1/\sqrt{7}$	$-1/\sqrt{2}$	$\sqrt{5/14}$
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$-2\sqrt{2/21}$	$1/2\sqrt{3}$	$\sqrt{15}/2\sqrt{7}$
$6,-\frac{1}{4};\overline{10},-\frac{3}{4}$	$\sqrt{10/21}$	$\sqrt{5}/2\sqrt{3}$	$\sqrt{3}/2\sqrt{7}$

$\overline{24},-2$

$\mu_1,Z_1;\mu_2,Z_2$	300
$3,-\frac{5}{4};\overline{10},-\frac{3}{4}$	1

$\bar{6},-2$

$\mu_1,Z_1;\mu_2,Z_2$	84	300
$1,-\frac{9}{4};\bar{6},\frac{1}{4}$	$-\sqrt{3}/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
$3,-\frac{5}{4};\overline{10},-\frac{3}{4}$	$\sqrt{5}/2\sqrt{2}$	$\sqrt{3}/2\sqrt{2}$

$\overline{10},-3$

$\mu_1,Z_1;\mu_2,Z_2$	300
$1,-\frac{9}{4};\overline{10},-\frac{3}{4}$	1

TABLE IVE. $\left(\begin{smallmatrix}20 & 20' \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

$35, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	140
$10, \frac{3}{4}; 8, \frac{3}{4}$	1

$27, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$10, \frac{3}{4}; 8, \frac{3}{4}$	1

$10, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	70
$10, \frac{3}{4}; 8, \frac{3}{4}$	1

$8, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64
$10, \frac{3}{4}; 8, \frac{3}{4}$	1

$21, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	140
$10, \frac{3}{4}; 6, -\frac{1}{4}$	1

$24, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	70	126	140
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$1/2$	$-1/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
$10, \frac{3}{4}; \bar{3}, -\frac{1}{4}$	$-1/2\sqrt{2}$	$3/4$	$\sqrt{5}/4$
$10, \frac{3}{4}; 6, -\frac{1}{4}$	$\sqrt{5}/2\sqrt{2}$	$\sqrt{5}/4$	$-1/4$

$\overline{15}, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	126
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$1/\sqrt{5}$	$2/\sqrt{5}$
$10, \frac{3}{4}; 6, -\frac{1}{4}$	$2/\sqrt{5}$	$-1/\sqrt{5}$

$6, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	70
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$10, \frac{3}{4}; \bar{3}, -\frac{1}{4}$	$\sqrt{2/3}$	$1/\sqrt{3}$

$\bar{3}, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64
$6, -\frac{1}{4}; 8, \frac{3}{4}$	1

$15', -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126	140
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$-1/2$	$\sqrt{3}/2$
$10, \frac{3}{4}; 3, -\frac{5}{4}$	$\sqrt{3}/2$	$1/2$

$15, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	70	126	140
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$-\sqrt{2/15}$	$1/\sqrt{3}$	$-1/\sqrt{5}$	$1/\sqrt{3}$
$6, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$1/\sqrt{5}$	0	$\sqrt{3/10}$	$1/\sqrt{2}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$-\sqrt{2/15}$	$1/\sqrt{3}$	$3/2\sqrt{5}$	$-1/2\sqrt{3}$
$10, \frac{3}{4}; \bar{3}, -\frac{5}{4}$	$2\sqrt{2/15}$	$1/\sqrt{3}$	$-1/2\sqrt{5}$	$-1/2\sqrt{3}$

$\bar{6}, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	126
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$\sqrt{2/5}$	$\sqrt{3/5}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$\sqrt{3/5}$	$-\sqrt{2/5}$

$3, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	70
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$-\sqrt{2/3}$	$1/\sqrt{3}$
$6, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$1/\sqrt{3}$	$\sqrt{2/3}$

$10, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126	140
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$8, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64	70	126	140
$1, -\frac{9}{4}; 8, \frac{3}{4}$	$-\sqrt{2}/5$	$1/2$	$-3/2\sqrt{10}$	$1/2\sqrt{2}$
$3, -\frac{5}{4}; \bar{3}, -\frac{1}{4}$	$1/\sqrt{5}$	$1/2\sqrt{2}$	$3/4\sqrt{5}$	$3/4$
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{5}$	$1/2\sqrt{2}$	$7/4\sqrt{5}$	$-1/4$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/\sqrt{5}$	$1/\sqrt{2}$	$-1/2\sqrt{5}$	$-1/2$

$6, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126	140
$1, -\frac{9}{4}; 6, -\frac{1}{4}$	$-\sqrt{3}/2$	$1/2$
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	$1/2$	$\sqrt{3}/2$

$\bar{3}, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	70	140
$1, -\frac{9}{4}; \bar{3}, -\frac{1}{4}$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$3, -\frac{7}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	140
$1, -\frac{9}{4}; 3, -\frac{5}{4}$	1

TABLE IVF. $\left(\begin{smallmatrix}20 & \overline{20}' \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

24,2

$\mu_1,Z_1;\mu_2,Z_2$	256
$10,\frac{3}{4};\bar{3},\frac{5}{4}$	1

6,2

$\mu_1,Z_1;\mu_2,Z_2$	84
$10,\frac{3}{4};\bar{3},\frac{5}{4}$	1

42,1

$\mu_1,Z_1;\mu_2,Z_2$	256
$10,\frac{3}{4};\bar{6},\frac{1}{4}$	1

15′,1

$\mu_1,Z_1;\mu_2,Z_2$	256
$10,\frac{3}{4};3,\frac{1}{4}$	1

15,1

$\mu_1,Z_1;\mu_2,Z_2$	45	84	256
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-\sqrt{3/14}$	$1/\sqrt{10}$	$2\sqrt{6/35}$
$10,\frac{3}{4};3,\frac{1}{4}$	$-1/\sqrt{7}$	$\sqrt{3/5}$	$-3/\sqrt{35}$
$10,\frac{3}{4};\bar{6},\frac{1}{4}$	$3/\sqrt{14}$	$\sqrt{3/10}$	$\sqrt{2/35}$

3,1

$\mu_1,Z_1;\mu_2,Z_2$	15	84
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-1/\sqrt{6}$	$\sqrt{5/6}$
$10,\frac{3}{4};\bar{6},\frac{1}{4}$	$\sqrt{5/6}$	$1/\sqrt{6}$

35,0

$\mu_1,Z_1;\mu_2,Z_2$	256
$10,\frac{3}{4};8,-\frac{3}{4}$	1

27,0

$\mu_1,Z_1;\mu_2,Z_2$	84	256
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$1/\sqrt{5}$	$2/\sqrt{5}$
$10,\frac{3}{4};8,-\frac{3}{4}$	$2/\sqrt{5}$	$-1/\sqrt{5}$

10,0

$\mu_1,Z_1;\mu_2,Z_2$	45	256
$6,-\frac{1}{4};3,\frac{1}{4}$	$-1/\sqrt{7}$	$\sqrt{6/7}$
$10,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{6/7}$	$1/\sqrt{7}$

8,0

$\mu_1,Z_1;\mu_2,Z_2$	15	45	84	256
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$-1/2\sqrt{3}$	$-3/2\sqrt{7}$	$1/\sqrt{6}$	$\sqrt{3/7}$
$6,-\frac{1}{4};3,\frac{1}{4}$	$-\sqrt{3}/4$	$-1/4\sqrt{7}$	$\sqrt{3}/2\sqrt{2}$	$-\sqrt{3/7}$
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$\sqrt{5}/4\sqrt{3}$	$3\sqrt{5}/4\sqrt{7}$	$7/2\sqrt{30}$	$\sqrt{3/35}$
$10,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{5}/2\sqrt{2}$	$-\sqrt{15}/2\sqrt{14}$	$1/2\sqrt{5}$	$-\sqrt{2/35}$

1,0

$\mu_1,Z_1;\mu_2,Z_2$	15	84
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$\sqrt{2/3}$	$1/\sqrt{3}$

24,−1

$\mu_1,Z_1;\mu_2,Z_2$	256
$6,-\frac{1}{4};8,-\frac{3}{4}$	1

$\overline{15}$,−1

$\mu_1,Z_1;\mu_2,Z_2$	84	256
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$\sqrt{2/5}$	$\sqrt{3/5}$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$\sqrt{3/5}$	$-\sqrt{2/5}$

6,−1

$\mu_1,Z_1;\mu_2,Z_2$	45	256
$3,-\frac{5}{4};3,\frac{1}{4}$	$-\sqrt{2/7}$	$\sqrt{5/7}$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$\sqrt{5/7}$	$\sqrt{2/7}$

$\bar{3}$,−1

$\mu_1,Z_1;\mu_2,Z_2$	15	45	84	256
$1,-\frac{9}{4};\bar{3},\frac{5}{4}$	$-1/2$	$-3/2\sqrt{7}$	$1/\sqrt{5}$	$2\sqrt{2/35}$
$3,-\frac{5}{4};3,\frac{1}{4}$	$-1/2$	$1/2\sqrt{7}$	$1/\sqrt{5}$	$-3\sqrt{2/35}$
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$1/\sqrt{6}$	$\sqrt{3/14}$	$2\sqrt{2/15}$	$\sqrt{3/35}$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$1/\sqrt{3}$	$-\sqrt{3/7}$	$1/\sqrt{15}$	$-\sqrt{6/35}$

15,−2

$\mu_1,Z_1;\mu_2 Z_2$	256
$3,-\frac{5}{4};8,-\frac{3}{4}$	1

$\bar{6}$,−2

$\mu_1,Z_1;\mu_2,Z_2$	84	256
$1,-\frac{9}{4};\bar{6},\frac{1}{4}$	$\sqrt{3/5}$	$\sqrt{2/5}$
$3,-\frac{5}{4};8,-\frac{3}{4}$	$\sqrt{2/5}$	$-\sqrt{3/5}$

3,−2

$\mu_1,Z_1;\mu_2,Z_2$	45	256
$1,-\frac{9}{4};3,\frac{1}{4}$	$-\sqrt{3/7}$	$2/\sqrt{7}$
$3,-\frac{5}{4};8,-\frac{3}{4}$	$2/\sqrt{7}$	$\sqrt{3/7}$

8,−3

$\mu_1,Z_1;\mu_2,Z_2$	256
$1,-\frac{9}{4};8,-\frac{3}{4}$	1

TABLE IVG. $\left(\begin{smallmatrix}20 & 20'' \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

$42, \frac{7}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$10, \frac{3}{4}; \bar{6}, 1$	1

$15, \frac{7}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''
$10, \frac{3}{4}; \bar{6}, 1$	1

$3, \frac{7}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$
$10, \frac{3}{4}; \bar{6}, 1$	1

$35, \frac{3}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$10, \frac{3}{4}; 8, 0$	1

$27, \frac{3}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''	224
$6, -\frac{1}{4}; \bar{6}, 1$	$1/\sqrt{3}$	$\sqrt{2/3}$
$10, \frac{3}{4}; 8, 0$	$\sqrt{2/3}$	$-1/\sqrt{3}$

$10, \frac{3}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''
$10, \frac{3}{4}; 8, 0$	1

$8, \frac{3}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$	140''
$6, -\frac{1}{4}; \bar{6}, 1$	$1/2$	$\sqrt{3}/2$
$10, \frac{3}{4}; 8, 0$	$\sqrt{3}/2$	$-1/2$

$1, \frac{3}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$
$6, -\frac{1}{4}; \bar{6}, 1$	1

$21, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$10, \frac{3}{4}; 6, -1$	1

$24, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''	224
$6, -\frac{1}{4}; 8, 0$	$1/\sqrt{6}$	$\sqrt{5/6}$
$10, \frac{3}{4}; 6, -1$	$\sqrt{5/6}$	$-1/\sqrt{6}$

$\overline{15}, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	36	140''	224
$3, -\frac{5}{4}; \bar{6}, 1$	$1/\sqrt{10}$	$1/\sqrt{2}$	$\sqrt{2/5}$
$6, -\frac{1}{4}; 8, 0$	$\sqrt{3/10}$	$1/\sqrt{6}$	$-2\sqrt{2/15}$
$10, \frac{3}{4}; 6, -1$	$\sqrt{3/5}$	$-1/\sqrt{3}$	$1/\sqrt{15}$

$6, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''
$6, -\frac{1}{4}; 8, 0$	1

$\bar{3}, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$	140''
$3, -\frac{5}{4}; \bar{6}, 1$	$1/\sqrt{2}$	$1/\sqrt{2}$
$6, -\frac{1}{4}; 8, 0$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$15', -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$6, -\frac{1}{4}; 6, -1$	1

$15, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''	224
$3, -\frac{5}{4}; 8, 0$	$1/\sqrt{3}$	$\sqrt{2/3}$
$6, -\frac{1}{4}; 6, -1$	$\sqrt{2/3}$	$-1/\sqrt{3}$

$\bar{6}, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$	140''	224
$1, -\frac{9}{4}; \bar{6}, 1$	$\sqrt{3/10}$	$1/\sqrt{2}$	$1/\sqrt{5}$
$3, -\frac{5}{4}; 8, 0$	$\sqrt{2/5}$	0	$-\sqrt{3/5}$
$6, -\frac{1}{4}; 6, 1$	$\sqrt{3/10}$	$-1/\sqrt{2}$	$1/\sqrt{5}$

$3, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''
$3, -\frac{5}{4}; 8, 0$	1

$10, -\frac{9}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$3, -\frac{5}{4}; 6, -1$	1

$8, -\frac{9}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	140''	224
$1, -\frac{9}{4}; 8, 0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$3, -\frac{5}{4}; 6, -1$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$6, -\frac{13}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	224
$1, -\frac{9}{4}; 6, -1$	1

TABLE IVH. $\left(\begin{smallmatrix}20' & 15\\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu\\ \mu Z\end{smallmatrix}\right)$.

$15,\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$8,\frac{3}{4};3,1$	1

$\bar{6},\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{60}$
$8,\frac{3}{4};3,1$	1

$3,\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{36}$
$8,\frac{3}{4};3,1$	1

$27,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$8,\frac{3}{4};8,0$	1

$\overline{10},\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{60}$
$8,\frac{3}{4};8,0$	1

$10,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	$140''$
$6,-\frac{1}{4};3,1$	$-1/\sqrt{3}$	$\sqrt{2/3}$
$8,\frac{3}{4};8,0$	$\sqrt{2/3}$	$1/\sqrt{3}$

$8,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$20'_1$	$20'_2$	$\overline{36}$	$\overline{60}$	$140''$	
$\bar{3},-\frac{1}{4};3,1$	$3\sqrt{3}/4\sqrt{26}$	$-\sqrt{2/13}$	$1/4\sqrt{2}$	$-3/4\sqrt{2}$	$3\sqrt{5}/4\sqrt{6}$	
$6,-\frac{1}{4};3,1$	$-17/4\sqrt{78}$	$-\sqrt{2/13}$	$3/4\sqrt{2}$	$3/4\sqrt{2}$	$\sqrt{5}/4\sqrt{6}$	
$8,\frac{3}{4};1,0$	$1/2\sqrt{39}$	$1/\sqrt{13}$	$-1/2$	$1/2$	$\sqrt{5}/2\sqrt{3}$	
$8,\frac{3}{4};8,0$	$-5/4\sqrt{78}$	$2\sqrt{2/13}$	$3/4\sqrt{2}$	$-1/4\sqrt{2}$	$\sqrt{5}/4\sqrt{6}$	F
$8,\frac{3}{4};8,0$	$\sqrt{65}/4\sqrt{6}$	0	$\sqrt{5}/4\sqrt{2}$	$\sqrt{5}/4\sqrt{2}$	$-1/4\sqrt{6}$	D

$1,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\bar{4}$	$\overline{36}$
$\bar{3},-\frac{1}{4};3,1$	$-2/\sqrt{5}$	$1/\sqrt{5}$
$8,\frac{3}{4};8,0$	$1/\sqrt{5}$	$2/\sqrt{5}$

$24,-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$6,-\frac{1}{4};8,0$	1

$\overline{15},-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{36}$	$\overline{60}$	$140''$
$\bar{3},-\frac{1}{4};8,0$	$1/4$	$-\sqrt{3}/2\sqrt{2}$	$3/4$
$6,-\frac{1}{4};8,0$	$3/4$	$\sqrt{3}/2\sqrt{2}$	$1/4$
$8,\frac{3}{4};\bar{3},-1$	$-\sqrt{3}/2\sqrt{2}$	$1/2$	$\sqrt{3}/2\sqrt{2}$

$6,-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	$20'_1$	$20'_2$	$\overline{60}$	$140''$
$3,-\frac{5}{4};3,1$	$-1/3$	$11/6\sqrt{13}$	$-2/\sqrt{39}$	$-1/2$	$\sqrt{5}/3\sqrt{2}$
$\bar{3},-1;8,0$	$1/\sqrt{3}$	$-\sqrt{13}/4\sqrt{3}$	0	$-\sqrt{3}/4$	$\sqrt{5}/2\sqrt{6}$
$6,-\frac{1}{4};1,0$	$-2/3\sqrt{3}$	$-7/3\sqrt{39}$	$-1/3\sqrt{13}$	$1/\sqrt{3}$	$\sqrt{10}/3\sqrt{3}$
$6,-\frac{1}{4};8,0$	$-\sqrt{5}/3\sqrt{3}$	$-5\sqrt{5}/12\sqrt{39}$	$4\sqrt{5}/3\sqrt{13}$	$-\sqrt{5}/4\sqrt{3}$	$1/6\sqrt{6}$
$8,\frac{3}{4};\bar{3},-1$	$\sqrt{2}/3$	$17/6\sqrt{26}$	$2\sqrt{2/39}$	$1/2\sqrt{2}$	$\sqrt{5}/6$

$\bar{3},-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\bar{4}$	$20'_1$	$20'_2$	$\overline{36}$	$140''$
$3,-\frac{5}{4};3,1$	$1/\sqrt{5}$	$-1/\sqrt{39}$	$-2/\sqrt{13}$	$\sqrt{3/10}$	$1/\sqrt{6}$
$\bar{3},-\frac{1}{4};1,0$	$-2/3\sqrt{5}$	$2/\sqrt{39}$	$-1/3\sqrt{13}$	$-\sqrt{2/15}$	$\sqrt{2/3}$
$\bar{3},-\frac{1}{4};8,0$	$-\sqrt{2}/3\sqrt{5}$	$7/2\sqrt{78}$	$4\sqrt{2}/3\sqrt{13}$	$11/4\sqrt{15}$	$1/4\sqrt{3}$
$6,-\frac{1}{4};8,0$	$-\sqrt{2/5}$	$-\sqrt{13}/2\sqrt{6}$	0	$\sqrt{3}/4\sqrt{5}$	$1/4\sqrt{3}$
$8,\frac{3}{4};\bar{3},-1$	$2/\sqrt{15}$	$-3/2\sqrt{13}$	$4/\sqrt{39}$	$-1/2\sqrt{10}$	$1/2\sqrt{2}$

$15,-\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{60}$	$140''$
$3,-\frac{5}{4};8,0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$6,-\frac{1}{4};\bar{3},-1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$\bar{6},-\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{36}$	$140''$
$3,-\frac{5}{4};8,0$	$\sqrt{3}/2$	$1/2$
$\bar{3},-\frac{1}{4};\bar{3},-1$	$-1/2$	$\sqrt{3}/2$

$3,-\frac{5}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20	$20'_1$	$20'_2$	$140''$
$3,-\frac{5}{4};1,0$	$-2/3\sqrt{3}$	$4/3\sqrt{39}$	$-5/3\sqrt{13}$	$4/3\sqrt{3}$
$3,-\frac{5}{4};8,0$	$-2\sqrt{2}/3\sqrt{3}$	$17/3\sqrt{78}$	$4\sqrt{2}/3\sqrt{13}$	$-1/3\sqrt{6}$
$\bar{3},-\frac{1}{4};\bar{3},-1$	$1/\sqrt{3}$	$1/\sqrt{39}$	$2/\sqrt{13}$	$1/\sqrt{3}$
$6,-\frac{1}{4};\bar{3},-1$	$-\sqrt{2}/3$	$-11/3\sqrt{26}$	$2\sqrt{2/39}$	$1/3\sqrt{2}$

$8,-\frac{9}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$3,-\frac{5}{4};\bar{3},-1$	1

$1,-\frac{9}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	20
$3,-\frac{5}{4};\bar{3},-1$	-1

TABLE IV I. $\begin{pmatrix} 20' & 20' & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{pmatrix}$.

$27, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$8, \frac{3}{4}; 8, \frac{3}{4}$	1

$\overline{10}, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50
$8, \frac{3}{4}; 8, \frac{3}{4}$	1

$10, \frac{3}{2}$

$\mu_1 Z_1; \mu_2, Z_2$	70
$8, \frac{3}{4}; 8, \frac{3}{4}$	1

$8, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64_D	64_F	
$8, \frac{3}{4}; 8, \frac{3}{4}$	0	1	F
$8, \frac{3}{4}; 8, \frac{3}{4}$	1	0	D

$1, \frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{10}$
$8, \frac{3}{4}; 8, \frac{3}{4}$	1

$24, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	70	126
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$8, \frac{3}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$

$\overline{15}, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	64_D	64_F	126
$\bar{3}, -\frac{1}{4}; 8, \frac{3}{4}$	$-1/2$	$-1/2\sqrt{5}$	$1/2$	$3/2\sqrt{5}$
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$1/2$	$3/2\sqrt{5}$	$1/2$	$1/2\sqrt{5}$
$8, \frac{3}{4}; \bar{3}, -\frac{1}{4}$	$1/2$	$-1/2\sqrt{5}$	$-1/2$	$3/2\sqrt{5}$
$8, \frac{3}{4}; 6, -\frac{1}{4}$	$1/2$	$-3/2\sqrt{5}$	$1/2$	$-1/2\sqrt{5}$

$6, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	10	64_D	64_F	70
$\bar{3}, -\frac{1}{4}; 8, \frac{3}{4}$	$1/2\sqrt{3}$	$-\sqrt{5}/2\sqrt{3}$	$1/2\sqrt{3}$	$\sqrt{5}/2\sqrt{3}$
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$-\sqrt{5}/2\sqrt{3}$	$-1/2\sqrt{3}$	$\sqrt{5}/2\sqrt{3}$	$-1/2\sqrt{3}$
$8, \frac{3}{4}; \bar{3}, -\frac{1}{4}$	$-1/2\sqrt{3}$	$\sqrt{5}/2\sqrt{3}$	$1/2\sqrt{3}$	$\sqrt{5}/2\sqrt{3}$
$8, \frac{3}{4}; 6, -\frac{1}{4}$	$\sqrt{5}/2\sqrt{3}$	$1/2\sqrt{3}$	$\sqrt{5}/2\sqrt{3}$	$-1/2\sqrt{3}$

$\bar{3}, \frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	6	$\overline{10}$	64_D	64_F
$\bar{3}, -\frac{1}{4}; 8, \frac{3}{4}$	$1/2\sqrt{5}$	$1/2$	$1/2$	$3/2\sqrt{5}$
$6, -\frac{1}{4}; 8, \frac{3}{4}$	$-3/2\sqrt{5}$	$1/2$	$-1/2$	$1/2\sqrt{5}$
$8, \frac{3}{4}; \bar{3}, -\frac{1}{4}$	$1/2\sqrt{5}$	$-1/2$	$-1/2$	$3/2\sqrt{5}$
$8, \frac{3}{4}; 6, -\frac{1}{4}$	$3/2\sqrt{5}$	$1/2$	$-1/2$	$-1/2\sqrt{5}$

$15', -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	1

$15, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	64_D	64_F	70	126
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$-1/\sqrt{6}$	$\sqrt{3/10}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	$1/\sqrt{5}$
$\bar{3}, -\frac{1}{4}; 6, -\frac{1}{4}$	$-1/2$	$-1/\sqrt{5}$	0	$-1/2$	$\sqrt{3/10}$
$6, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$1/2$	$-1/\sqrt{5}$	0	$1/2$	$\sqrt{3/10}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$1/\sqrt{6}$	0	$-\sqrt{2/3}$	$-1/\sqrt{6}$	0
$8, \frac{3}{4}; 3, -\frac{5}{4}$	$1/\sqrt{6}$	$\sqrt{3/10}$	$1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{5}$

$\bar{6}, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{10}$	64_D	64_F	126
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$1/2$	$1/\sqrt{10}$	$1/\sqrt{2}$	$\sqrt{3}/2\sqrt{5}$
$\bar{3}, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$-1/2\sqrt{2}$	$-1/\sqrt{5}$	0	$3\sqrt{3}/2\sqrt{10}$
$6, -\frac{1}{4}; 6, -\frac{1}{4}$	$\sqrt{3}/2\sqrt{2}$	$-\sqrt{3/5}$	0	$-1/2\sqrt{10}$
$8, \frac{3}{4}; 3, -\frac{5}{4}$	$-1/2$	$-1/\sqrt{10}$	$1/\sqrt{2}$	$-\sqrt{3}/2\sqrt{5}$

$3, -\frac{1}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	6	10	64_D	64_F	70
$3, -\frac{5}{4}; 8, \frac{3}{4}$	$\sqrt{3/10}$	$-1/\sqrt{3}$	$1/\sqrt{6}$	$1/\sqrt{30}$	$-1/\sqrt{6}$
$\bar{3}, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$1/\sqrt{10}$	0	0	$\sqrt{2/5}$	$1/\sqrt{2}$
$\bar{3}, -\frac{1}{4}; 6, -\frac{1}{4}$	$-\sqrt{3}/2\sqrt{5}$	$1/\sqrt{6}$	$1/\sqrt{3}$	$2/\sqrt{15}$	$-1/2\sqrt{3}$
$6, -\frac{1}{4}; \bar{3}, -\frac{1}{4}$	$-\sqrt{3}/2\sqrt{5}$	$-1/\sqrt{6}$	$-1/\sqrt{3}$	$2/\sqrt{15}$	$-1/2\sqrt{3}$
$8, \frac{3}{4}; 3, -\frac{5}{4}$	$\sqrt{3/10}$	$1/\sqrt{3}$	$-1/\sqrt{6}$	$1/\sqrt{30}$	$-1/\sqrt{6}$

$10, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	50	126
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$

$8, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	64_D	64_F	70	126
$3, -\frac{5}{4}; \bar{3}, -\frac{1}{4}$	$1/2\sqrt{5}$	$-1/2$	$1/2$	$3/2\sqrt{5}$
$3, -\frac{5}{4}; 6, -\frac{1}{4}$	$-3/2\sqrt{5}$	$-1/2$	$-1/2$	$1/2\sqrt{5}$
$\bar{3}, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/2\sqrt{5}$	$1/2$	$-1/2$	$3/2\sqrt{5}$
$6, -\frac{1}{4}; 3, -\frac{5}{4}$	$3/2\sqrt{5}$	$-1/2$	$-1/2$	$-1/2\sqrt{5}$

$1, -\frac{3}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	10	70
$3, -\frac{5}{4}; \bar{3}, -\frac{1}{4}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$\bar{3}, -\frac{1}{4}; 3, -\frac{5}{4}$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$6, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	126
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	1

$\bar{3}, -\frac{5}{2}$

$\mu_1, Z_1; \mu_2, Z_2$	70
$3, -\frac{5}{4}; 3, -\frac{5}{4}$	-1

TABLE IVJ. $\left(\begin{smallmatrix} 20'' & \overline{20}'' & \nu \\ \mu_1 Z_1 & \mu_2 Z_2 & \mu Z \end{smallmatrix}\right)$.

$\overline{15},2$

$\mu_1,Z_1;\mu_2,Z_2$	175
$8,\frac{3}{4};\bar{3},\frac{5}{4}$	1

$6,2$

$\mu_1,Z_1;\mu_2,Z_2$	84
$8,\frac{3}{4};\bar{3},\frac{5}{4}$	1

$\bar{3},2$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$
$8,\frac{3}{4};\bar{3},\frac{5}{4}$	1

$\overline{24},1$

$\mu_1,Z_1;\mu_2,Z_2$	175
$8,\frac{3}{4};\bar{6},\frac{1}{4}$	1

$15,1$

$\mu_1,Z_1;\mu_2,Z_2$	45	84	175
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-\sqrt{3}/2\sqrt{2}$	$1/2$	$\sqrt{3}/2\sqrt{2}$
$8,\frac{3}{4};3,\frac{1}{4}$	$1/4$	$-\sqrt{3}/2\sqrt{2}$	$3/4$
$8,\frac{3}{4};\bar{6},\frac{1}{4}$	$3/4$	$\sqrt{3}/2\sqrt{2}$	$1/4$

$\bar{6},1$

$\mu_1,Z_1;\mu_2,Z_2$	$20''$	$\overline{45}$	175
$\bar{3},-\frac{1}{4};\bar{3},\frac{5}{4}$	$-1/2$	$1/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
$8,\frac{3}{4};3,\frac{1}{4}$	$-1/2\sqrt{2}$	$3/4$	$-\sqrt{5}/4$
$8,\frac{3}{4};\bar{6},\frac{1}{4}$	$\sqrt{5}/2\sqrt{2}$	$\sqrt{5}/4$	$1/4$

$3,1$

$\mu_1,Z_1;\mu_2,Z_2$	15_1	15_2	$\overline{45}$	84
$\bar{3},-\frac{1}{4};\bar{3},\frac{5}{4}$	0	$-1/2$	$1/2$	$1/\sqrt{2}$
$6,-\frac{1}{4};\bar{3},\frac{5}{4}$	$-1/\sqrt{2}$	$1/2\sqrt{6}$	$\sqrt{3}/2\sqrt{2}$	$-1/2\sqrt{3}$
$8,\frac{3}{4};3,\frac{1}{4}$	$1/2$	$-\sqrt{3}/4$	$\sqrt{3}/4$	$-\sqrt{3}/2\sqrt{2}$
$8,\frac{3}{4};\bar{6},\frac{1}{4}$	$1/2$	$5/4\sqrt{3}$	$\sqrt{3}/4$	$1/2\sqrt{6}$

$27,0$

$\mu_1,Z_1;\mu_2,Z_2$	84	175
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$8,\frac{3}{4};8,-\frac{3}{4}$	$-1/\sqrt{2}$	$1/\sqrt{2}$

$\overline{10},0$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$	175
$\bar{3},-\frac{1}{4};\bar{6},\frac{1}{4}$	$1/2$	$\sqrt{3}/2$
$8,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{3}/2$	$-1/2$

$10,0$

μ_1,Z_1,μ_2,Z_2	45	175
$6,-\frac{1}{4};3,\frac{1}{4}$	$-1/2$	$\sqrt{3}/2$
$8,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{3}/2$	$1/2$

$8,0$

$\mu_1,Z_1;\mu_2,Z_2$	15_1	15_2	$20''$	45	$\overline{45}$	84	175	
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$-1/4$	$-5/8\sqrt{3}$	$-\sqrt{3}/4$	$-3/8$	$3/8$	$\sqrt{5}/4\sqrt{3}$	$\sqrt{15}/8$	
$\bar{3},-\frac{1}{4};3,\frac{1}{4}$	$1/8$	$9/16\sqrt{3}$	$-\sqrt{3}/8$	$3/16$	$-3/16$	$-\sqrt{15}/8$	$3\sqrt{15}/16$	
$\bar{3},-\frac{1}{4};\bar{6},\frac{1}{4}$	$3/8$	$\sqrt{3}/16$	$\sqrt{3}/8$	$9/16$	$7/16$	$\sqrt{15}/8$	$\sqrt{15}/16$	
$6,-\frac{1}{4};3,\frac{1}{4}$	$-3/8$	$-\sqrt{3}/16$	$-\sqrt{3}/8$	$7/16$	$9/16$	$-\sqrt{15}/8$	$-\sqrt{15}/16$	
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$-\sqrt{5}/8$	$7\sqrt{5}/16\sqrt{3}$	$\sqrt{15}/8$	$-3\sqrt{5}/16$	$3\sqrt{5}/16$	$-1/8\sqrt{3}$	$\sqrt{3}/16$	
$8,\frac{3}{4};8,-\frac{3}{4}$	$3/4\sqrt{6}$	$-7/8\sqrt{2}$	$3/4\sqrt{2}$	$-\sqrt{3}/8\sqrt{2}$	$\sqrt{3}/8\sqrt{2}$	$-\sqrt{5}/4\sqrt{2}$	$\sqrt{5}/8\sqrt{2}$	*F*
$8,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{15}/4\sqrt{2}$	$\sqrt{5}/8\sqrt{2}$	$-\sqrt{5}/4\sqrt{2}$	$-\sqrt{15}/8\sqrt{2}$	$\sqrt{15}/8\sqrt{2}$	$-1/4\sqrt{2}$	$-3/8\sqrt{2}$	*D*

$1,0$

$\mu_1,Z_1;\mu_2,Z_2$	1	15_1	15_2	84
$3,-\frac{5}{4};\bar{3},\frac{5}{4}$	$3/2\sqrt{15}$	$-1/2$	$1/\sqrt{3}$	$-2/\sqrt{15}$
$\bar{3},-\frac{1}{4};3,\frac{1}{4}$	$-\sqrt{3}/2\sqrt{5}$	$1/2$	0	$-\sqrt{3/5}$
$6,-\frac{1}{4};\bar{6},\frac{1}{4}$	$-\sqrt{3/10}$	$-1/\sqrt{2}$	$-1/\sqrt{6}$	$-1/\sqrt{30}$
$8,\frac{3}{4};8,-\frac{3}{4}$	$\sqrt{2/5}$	0	$-1/\sqrt{2}$	$-1/\sqrt{10}$

$24,-1$

$\mu_1,Z_1;\mu_2,Z_2$	175
$6,-\frac{1}{4};8,-\frac{3}{4}$	1

$\overline{15},-1$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$	84	175
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$\sqrt{3}/2\sqrt{2}$	$1/2$	$\sqrt{3}/2\sqrt{2}$
$\bar{3},-\frac{1}{4};8,-\frac{3}{4}$	$-1/4$	$-\sqrt{3}/2\sqrt{2}$	$3/4$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$3/4$	$-\sqrt{3}/2\sqrt{2}$	$-1/4$

$6,-1$

$\mu_1,Z_1;\mu_2,Z_2$	$20''$	45	175
$3,-\frac{5}{4};3,\frac{1}{4}$	$-1/2$	$-1/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$
$\bar{3},-\frac{1}{4};8,-\frac{3}{4}$	$1/2\sqrt{2}$	$3/4$	$\sqrt{5}/4$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$\sqrt{5}/2\sqrt{2}$	$-\sqrt{5}/4$	$1/4$

$\bar{3},-1$

$\mu_1,Z_1;\mu_2,Z_2$	15_1	15_2	45	84
$3,-\frac{5}{4};3,\frac{1}{4}$	0	$1/2$	$1/2$	$-1/\sqrt{2}$
$3,-\frac{5}{4};\bar{6},\frac{1}{4}$	$-1/\sqrt{2}$	$1/2\sqrt{6}$	$-\sqrt{3}/2\sqrt{2}$	$-1/2\sqrt{3}$
$\bar{3},-\frac{1}{4};8,\frac{3}{4}$	$1/2$	$-\sqrt{3}/4$	$-\sqrt{3}/4$	$-\sqrt{3}/2\sqrt{2}$
$6,-\frac{1}{4};8,-\frac{3}{4}$	$-1/2$	$-5/4\sqrt{3}$	$\sqrt{3}/4$	$-1/2\sqrt{6}$

$15,-2$

$\mu_1,Z_1;\mu_2,Z_2$	175
$3,-\frac{5}{4};8,-\frac{3}{4}$	1

$\bar{6},-2$

$\mu_1,Z_1;\mu_2,Z_2$	84
$3,-\frac{5}{4};8,-\frac{3}{4}$	-1

$3,-2$

$\mu_1,Z_1;\mu_2,Z_2$	45
$3,-\frac{5}{4};8,-\frac{3}{4}$	-1

TABLE IVK. $\left(\begin{smallmatrix}20' & 20'' \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

$\overline{24},\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140'$
$8,\frac{3}{4};\bar{6},1$	1

$15,\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$8,\frac{3}{4};\bar{6},1$	1

$\bar{6},\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{60}$
$8,\frac{3}{4};\bar{6},1$	1

$3,\frac{7}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{36}$
$8,\frac{3}{4};\bar{6},1$	1

$27,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140'$	$140''$
$6,-\frac{1}{4};\bar{6},1$	$1/\sqrt{3}$	$\sqrt{2/3}$
$8,\frac{3}{4};8,0$	$\sqrt{2/3}$	$-1/\sqrt{3}$

$\overline{10},\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{60}$	$140'$
$\bar{3},-\frac{1}{4};\bar{6},1$	$\sqrt{2/5}$	$\sqrt{3/5}$
$8,\frac{3}{4};8,0$	$\sqrt{3/5}$	$-\sqrt{2/5}$

$10,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140''$
$8,\frac{3}{4};8,0$	1

$8,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$20'$	$\overline{36}$	$\overline{60}$	$140''$	
$\bar{3},-\frac{1}{4};\bar{6},1$	$\sqrt{3}/4\sqrt{2}$	$3/4\sqrt{2}$	$\sqrt{5}/4\sqrt{2}$	$3\sqrt{5}/4\sqrt{6}$	
$6,-\frac{1}{4};\bar{6},1$	$-3\sqrt{5}/4\sqrt{6}$	$\sqrt{5}/4\sqrt{2}$	$3/4\sqrt{2}$	$-\sqrt{3}/4\sqrt{2}$	
$8,\frac{3}{4};8,0$	$3/4\sqrt{2}$	$-\sqrt{3}/4\sqrt{2}$	$3\sqrt{5}/4\sqrt{6}$	$-\sqrt{5}/4\sqrt{2}$	*F*
$8,\frac{3}{4};8,0$	$\sqrt{5}/4\sqrt{2}$	$3\sqrt{5}/4\sqrt{6}$	$-\sqrt{3}/4\sqrt{2}$	$-3/4\sqrt{2}$	*D*

$1,\frac{3}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\bar{4}$	$\overline{36}$
$6,-\frac{1}{4};\bar{6},1$	$-\sqrt{3/5}$	$-\sqrt{2/5}$
$8,\frac{3}{4};8,0$	$\sqrt{2/5}$	$-\sqrt{3/5}$

$24,-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$140'$	$140''$
$6,-\frac{1}{4};8,0$	$\sqrt{2/3}$	$1/\sqrt{3}$
$8,\frac{3}{4};6,-1$	$1/\sqrt{3}$	$-\sqrt{2/3}$

$\overline{15},-\frac{1}{4}$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{36}$	$\overline{60}$	$140'$	$140''$
$3,-\frac{5}{4};\bar{6},1$	$1/2$	$\sqrt{3/10}$	$1/\sqrt{5}$	$1/2$
$\bar{3},-\frac{1}{4};8,0$	$-\sqrt{3}/4$	$1/2\sqrt{10}$	$\sqrt{3/5}$	$-\sqrt{3}/4$
$6,-\frac{1}{4};8,0$	$\sqrt{3}/4$	$3/2\sqrt{10}$	$-1/\sqrt{15}$	$-5/4\sqrt{3}$
$8,\frac{3}{4};6,-1$	$-\sqrt{3}/2\sqrt{2}$	$3/2\sqrt{5}$	$-\sqrt{2/15}$	$1/2\sqrt{6}$

TABLE IVK. (*Continued*).

$6, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$20'$	$\overline{60}$	$140''$
$\bar{3}, -\frac{1}{4}; 8, 0$	$-1/4$	$-\sqrt{5}/4$	$-\sqrt{5}/2\sqrt{2}$
$6, -\frac{1}{4}; 8, 0$	$-\sqrt{5}/4$	$3/4$	$-1/2\sqrt{2}$
$8, \frac{3}{4}; 6, -1$	$\sqrt{5}/2\sqrt{2}$	$1/2\sqrt{2}$	$-1/2$

$\bar{3}, -\frac{1}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\bar{4}$	$20'$	$\overline{36}$	$140''$
$3, -\frac{5}{4}; \bar{6}, 1$	$-1/\sqrt{5}$	$-1/\sqrt{2}$	$-1/2\sqrt{5}$	$-1/2$
$\bar{3}, -\frac{1}{4}; 8, 0$	$1/\sqrt{5}$	$1/2\sqrt{2}$	$-3/4\sqrt{5}$	$-3/4$
$6, -\frac{1}{4}; 8, 0$	$-1/\sqrt{5}$	$1/2\sqrt{2}$	$-7/4\sqrt{5}$	$1/4$
$8, \frac{3}{4}; 6, -1$	$\sqrt{2/5}$	$-1/2$	$-3/2\sqrt{10}$	$1/2\sqrt{2}$

$15', -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$140'$
$6, -\frac{1}{4}; 6, -1$	1

$15, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{60}$	$140'$	$140''$
$3, -\frac{5}{4}; 8, 0$	$\sqrt{3/10}$	$2\sqrt{2/15}$	$1/\sqrt{6}$
$\bar{3}, -\frac{1}{4}; 6, -1$	$-1/\sqrt{10}$	$\sqrt{2/5}$	$-1/\sqrt{2}$
$6, -\frac{1}{4}; 6, -1$	$\sqrt{3/5}$	$-1/\sqrt{15}$	$-1/\sqrt{3}$

$\bar{6}, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{36}$	$140''$
$3, -\frac{5}{4}; 8, 0$	$-1/2$	$-\sqrt{3}/2$
$6, -\frac{1}{4}; 6, -1$	$-\sqrt{3}/2$	$1/2$

$3, -\frac{5}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$20'$	$140''$
$3, -\frac{5}{4}; 8, 0$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$\bar{3}, -\frac{1}{4}; 6, -1$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$10, -\frac{9}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$140'$
$3, -\frac{5}{4}; 6, -1$	1

$8, -\frac{9}{4}$

$\mu_1, Z_1; \mu_2, Z_2$	$140''$
$3, -\frac{5}{4}; 6, -1$	-1

TABLE IVL. $\left(\begin{smallmatrix}20''' & 15 \\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu \\ \mu Z\end{smallmatrix}\right)$.

$\overline{15}, 2$

$\mu_1, Z_1; \mu_2, Z_2$	175
$\bar{6}, 1; 3, 1$	1

$\bar{3}, 2$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{45}$
$\bar{6}, 1; 3, 1$	1

$\overline{24}, 1$

$\mu_1, Z_1; \mu_2, Z_2$	175
$\bar{6}, 1; 8, 0$	1

$15, 1$

$\mu_1, Z_1; \mu_2, Z_2$	45	175
$\bar{6}, 1; 8, 0$	$-\sqrt{3}/2$	$1/2$
$8, 0; 3, 1$	$1/2$	$\sqrt{3}/2$

$\bar{6}, 1$

$\mu_1, Z_1; \mu_2, Z_2$	$20''$	45	175
$\bar{6}, 1; 1, 0$	$-1/3$	$-1/\sqrt{3}$	$\sqrt{5}/3$
$\bar{6}, 1; 8, 0$	$-\sqrt{5}/3$	$\sqrt{5}/2\sqrt{3}$	$1/6$
$8, 0; 3, 1$	$1/\sqrt{3}$	$1/2$	$\sqrt{5}/2\sqrt{3}$

$3, 1$

$\mu_1, Z_1; \mu_2, Z_2$	15	$\overline{45}$
$\bar{6}, 1; 8, 0$	$-\sqrt{3}/2$	$1/2$
$8, 0; 3, 1$	$1/2$	$\sqrt{3}/2$

$27, 0$

$\mu_1, Z_1; \mu_2, Z_2$	175
$8, 0; 8, 0$	1

$\overline{10}, 0$

$\mu_1, Z_1; \mu_2, Z_2$	$\overline{45}$	175
$\bar{6}, 1; \bar{3}, -1$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$8, 0; 8, 0$	$1/\sqrt{2}$	$1/\sqrt{2}$

$10, 0$

$\mu_1, Z_1; \mu_2, Z_2$	45	175
$6, -1; 3, 1$	$1/\sqrt{2}$	$1/\sqrt{2}$
$8, 0; 8, 0$	$-1/\sqrt{2}$	$1/\sqrt{2}$

8,0

$\mu_1,Z_1;\mu_2,Z_2$	15	$20''$	45	$\overline{45}$	175	
$6,-1;3,1$	$3/4\sqrt{2}$	$-1/2$	$1/4\sqrt{2}$	$3/4\sqrt{2}$	$\sqrt{5}/4\sqrt{2}$	
$\bar{6},1;\bar{3},-1$	$3/4\sqrt{2}$	$1/2$	$-3/4\sqrt{2}$	$-1/4\sqrt{2}$	$\sqrt{5}/4\sqrt{2}$	
$8,0;1,0$	$-1/2\sqrt{2}$	0	$1/2\sqrt{2}$	$-1/2\sqrt{2}$	$\sqrt{5}/2\sqrt{2}$	
$8,0;8,0$	0	$1/\sqrt{2}$	$1/2$	$1/2$	0	*F*
$8,0;8,0$	$-\sqrt{5}/4$	0	$-\sqrt{5}/4$	$\sqrt{5}/4$	$1/4$	*D*

1,0

$\mu_1,Z_1;\mu_2,Z_2$	15
$8,0;8,0$	1

24,−1

$\mu_1,Z_1;\mu_2,Z_2$	175
$6,-1;8,0$	1

$\overline{15},-1$

$\mu_1,Z_1;\mu_2,Z_2$	$\overline{45}$	175
$6,-1;8,0$	$\sqrt{3}/2$	$1/2$
$8,0;\bar{3},-1$	$-1/2$	$\sqrt{3}/2$

6, −1

$\mu_1,Z_1;\mu_2,Z_2$	$20''$	45	175
$6,-1;1,0$	$1/3$	$1/\sqrt{3}$	$\sqrt{5}/3$
$6,-1;8,0$	$-\sqrt{5}/3$	$\sqrt{5}/2\sqrt{3}$	$-1/6$
$8,0;\bar{3},-1$	$-1/\sqrt{3}$	$-1/2$	$\sqrt{5}/2\sqrt{3}$

$\bar{3},-1$

$\mu_1,Z_1;\mu_2,Z_2$	15	45
$6,-1;8,0$	$\sqrt{3}/2$	$1/2$
$8,0;\bar{3},-1$	$-1/2$	$\sqrt{3}/2$

15,−2

$\mu_1,Z_1;\mu_2,Z_2$	175
$6,-1;\bar{3},-1$	1

3,−2

$\mu_1,Z_1;\mu_2,Z_2$	45
$6,-1;\bar{3},-1$	1

TABLE IVM. $\left(\begin{smallmatrix}20'' & 20''\\ \mu_1 Z_1 & \mu_2 Z_2\end{smallmatrix}\middle|\begin{smallmatrix}\nu\\ \mu Z\end{smallmatrix}\right)$.

$\overline{15}',2$

$\mu_1,Z_1;\mu_2,Z_2$	105
$\bar{6},1;\bar{6},1$	1

$\overline{15},2$

$\mu_1,Z_1;\mu_2,Z_2$	175
$\bar{6},1;\bar{6},1$	1

6,2

$\mu_1,Z_1;\mu_2,Z_2$	84
$\bar{6},1;\bar{6},1$	1

$\overline{24},1$

$\mu_1,Z_1;\mu_2,Z_2$	105	175
$8,0;\bar{6},1$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$\bar{6},1;8,0$	$1/\sqrt{2}$	$1/\sqrt{2}$

15,1

$\mu_1,Z_1;\mu_2,Z_2$	84	175
$8,0;\bar{6},1$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$\bar{6},1;8,0$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$\bar{6},1$

$\mu_1,Z_1;\mu_2,Z_2$	$20''$	175
$8,0;\bar{6},1$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$\bar{6},1;8,0$	$1/\sqrt{2}$	$1/\sqrt{2}$

3,1

$\mu_1,Z_1;\mu_2,Z_2$	15	84
$8,0;\bar{6},1$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$\bar{6},1;8,0$	$1/\sqrt{2}$	$-1/\sqrt{2}$

TABLE IVM. (*Continued*).

27,0

$\mu_1,Z_1;\mu_2,Z_2$	84	105	175
$6,-1;\bar{6},1$	$-1/\sqrt{3}$	$1/\sqrt{6}$	$-1/\sqrt{2}$
$8,0;8,0$	$1/\sqrt{3}$	$\sqrt{2/3}$	0
$\bar{6},1;6,-1$	$-1/\sqrt{3}$	$1/\sqrt{6}$	$1/\sqrt{2}$

$\overline{10},0$

$\mu_1,Z_1;\mu_2,Z_2$	175
$8,0;8,0$	1

10,0

$\mu_1,Z_1;\mu_2,Z_2$	175
$8,0;8,0$	1

8,0

$\mu_1,Z_1;\mu_2,Z_2$	15	$20''$	84	175	
$6,-1;\bar{6},1$	$\sqrt{5}/4$	$-\sqrt{3}/2\sqrt{2}$	$1/2\sqrt{2}$	$\sqrt{3}/4$	
$8,0;8,0$	$-\sqrt{3}/2\sqrt{2}$	0	0	$\sqrt{5}/2\sqrt{2}$	*F*
$8,0;8,0$	0	$1/2$	$\sqrt{3}/2$	0	*D*
$\bar{6},1;6,-1$	$\sqrt{5}/4$	$\sqrt{3}/2\sqrt{2}$	$-1/2\sqrt{2}$	$\sqrt{3}/4$	

1,0

$\mu_1,Z_1;\mu_2,Z_2$	1	15	84
$6,-1;\bar{6},1$	$\sqrt{3/10}$	$1/\sqrt{2}$	$-1/\sqrt{5}$
$8,0;8,0$	$-\sqrt{2/5}$	0	$-\sqrt{3/5}$
$\bar{6},1;6,-1$	$\sqrt{3/10}$	$-1/\sqrt{2}$	$-1/\sqrt{5}$

24,−1

$\mu_1,Z_1;\mu_2,Z_2$	105	175
$6,-1;8,0$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$8,0;6,-1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$\overline{15},-1$

$\mu_1,Z_1;\mu_2,Z_2$	84	175
$6,-1;8,0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$8,0;6,-1$	$-1/\sqrt{2}$	$1/\sqrt{2}$

6,−1

$\mu_1,Z_1;\mu_2,Z_2$	$20''$	175
$6,-1;8,0$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$8,0;6,-1$	$1/\sqrt{2}$	$1/\sqrt{2}$

$\bar{3},-1$

$\mu_1,Z_1;\mu_2,Z_2$	15	84
$6,-1;8,0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$8,0;6,-1$	$-1/\sqrt{2}$	$1/\sqrt{2}$

$15',-2$

$\mu_1,Z_1;\mu_2,Z_2$	105
$6,-1;6,-1$	1

15,−2

$\mu_1,Z_1;\mu_2,Z_2$	175
$6,-1;6,-1$	1

$\bar{6},-2$

$\mu_1,Z_1;\mu_2,Z_2$	84
$6,-1;6,-1$	1

TABLE V. Phase factors involved in the symmetry properties of SU(3) singlet factors.

μ_1	μ_2	μ	η_1	η_2
15	15	1	1	1
		15_D	1	1
		15_F	−1	−1
		$20''$	1	1
		45	−1	1
		$\overline{45}$	−1	1
		84	1	1
20	15	20	−1	1
		$20'$	1	−1
		120	1	1
		$140''$	−1	−1

TABLE V. (*Continued*)

20	20	50	−1	1
		$84''$	1	1
		126	1	1
		140	−1	1
20	$\overline{20}$	1	−1	−1
		15	−1	−1
		84	−1	−1
		300	1	1
20	$20'$	64	1	1
		70	−1	1
		126	−1	1
		140	1	1
20	$\overline{20}'$	15	1	−1
		45	−1	−1
		84	−1	1
		256	1	1
20	$20''$	$\overline{36}$	1	1
		$140''$	−1	1
		224	1	1
$20'$	15	$\bar{4}$	1	−1
		20	−1	1
		$20'_1$	1	1
		$20'_2$	−1	−1
		$\overline{36}$	−1	1
		$\overline{60}$	−1	1
		$140''$	1	1
$20'$	$20'$	6	−1	−1
		10	1	1
		$\overline{10}$	1	−1
		50	−1	1
		64_D	1	1
		64_F	−1	−1
		70	−1	−1
		126	1	1
$20'$	$\overline{20}'$	1	1	1
		15_1	1	1
		15_2	1	1
		$20''$	−1	−1
		45	−1	1
		$\overline{45}$	−1	1
		84	−1	−1
		175	1	1
$20'$	$20''$	$\bar{4}$	−1	1
		$20'$	1	1
		$\overline{36}$	1	−1
		$\overline{60}$	−1	1
		$140'$	1	1
		$140''$	−1	−1
$20''$	15	15	−1	1
		$20''$	1	−1
		45	1	−1
		$\overline{45}$	−1	−1
		175	1	1
$20''$	$20''$	1	1	1
		15	−1	−1
		$20''$	1	1
		84	1	1
		105	1	1
		175	−1	−1

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[1]For a comprehensive review of SU(4) and SU(8) schemes of hadron physics, see: J.W. Moffat, "SU(4) and the New Hadrons," lectures at the McGill Summer School of Physics, June 1975 [to appear in Proceedings of the McGill Summer School of Physics, June 1975 (to be published)].

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[6]In Ref. 5, the SU(3) content of SU(4) representations up to $\underline{84}'$ are given. Note that the representation $\underline{60}$ contains a $\underline{15}$ and a $\underline{6}^*$, not a $\underline{15}^*$ and a $\underline{6}$, at $Z=\frac{1}{4}$.

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Green's functions for a face centered orthorhombic lattice*

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Diagonal and off-diagonal matrix elements of the Green's functions for a *face centered orthorhombic* lattice are presented in terms of integrals of complete elliptic integrals of the first and third kind. These Green's functions are also applicable to structures like that of the benzene crystal (space group D_{2h}^{15}, interchange symmetry D_2).

I. INTRODUCTION

Lattice Green's functions proved to be a powerful tool in the determination of impurity states in crystals. Considerable difficulties have been encountered in numerical calculations even for simple types of Green's function matrix elements.[1] Analytical expressions simplify the calculation of the Green's function matrix element, however these analytical expressions are only available for the simplest type crystal energy dispersion relations. Extensive use has been made of the complete elliptic integral of the first kind for the derivation of the diagonal matrix elements of the Green's function of square and rectangular lattices.[2,3] Green's function diagonal and off-diagonal matrix elements were given in terms of complete elliptic integrals of the first, second, and third kind by Hoshen and Jortner[4] for square lattices for the energy dispersion relation $2p\cos(x)+4q\cos(x/2)\cos(y/2)$, where p and q are intermolecular interaction parameters. Diagonal matrix elements of Green's functions for the three-dimensional cubic lattice can also be expressed in terms of integrals of complete elliptic integrals of the first kind.[5] Expressions for the Green's functions of products of complete elliptic integrals are available for fcc and bcc lattices.[3] Horiguchi, Yamazuki, and Morita[6] derived Green's function expressions for orthorhombic lattices in terms of complete elliptic integrals of the first kind. In Sec. II of this paper diagonal and off-diagonal matrix elements of the Green's function for face centered orthorhombic lattices will be presented. These matrix elements will be given in terms of integrals of complete elliptic integrals of the first and third kind. The expression derived in Sec. II will be applied in Sec. III for a numerical calculation of the Green's function matrix elements for some dispersion relations.

It should be noted that the lattice Green's function for the face centered orthorhombic Green's functions derived in this paper can be applied to benzene crystals belonging to the D_{2h}^{15} space group which contains four molecules per unit cell. The application of these Green's functions for isotopic impurity clusters in the benzene crystal will be given elsewhere.[7]

II. DERIVATIONS OF THE GREEN'S FUNCTIONS MATRIX ELEMENTS

In this section, expressions will be derived for the diagonal and three off-diagonal matrix elements of the Green's function for face centered orthorhombic crystals. The off-diagonal matrix elements correspond to the three nearest face centered neighbor molecules. The energy dispersion relation for this system is

$$h(x,y,z)=4A\cos x\cos y+4B\cos y\cos z+4C\cos z\cos x, \tag{1}$$

where

$$-\pi<x\le\pi,\quad -\pi<y\le\pi,\quad -\pi<z\le\pi.$$

A, B, C are the interaction parameters between a molecule at the origin and the three face centered molecules, respectively.

The four Green's function matrix elements are given by

$$g_0(E)=\frac{1}{4\pi^3}\int_{-\pi/2}^{\pi/2}dy\int_{-\pi}^{\pi}dz\int_{-\pi}^{\pi}\frac{dx}{E-h(x,y,z)}, \tag{2}$$

$$g_1(E)=\frac{1}{4\pi^3}\int_{-\pi/2}^{\pi/2}dy\exp(iy)\int_{-\pi}^{\pi}dz\int_{-\pi}^{\pi}\frac{\exp(ix)\,dx}{E-h(x,y,z)}, \tag{3}$$

$$G_2(E)=\frac{1}{4\pi^3}\int_{-\pi/2}^{\pi/2}dy\exp(iy)\int_{-\pi}^{\pi}dz\exp(iz)\int_{-\pi}^{\pi}\frac{dx}{E-h(x,y,z)}, \tag{4}$$

$$g_3(E)=\frac{1}{4\pi^3}\int_{-\pi/2}^{\pi/2}dy\int_{-\pi}^{\pi}dz\exp(iz)\int_{-\pi}^{\pi}\frac{dx\exp(ix)}{E-h(x,y,z)}. \tag{5}$$

It should be noted that the integration limits over the y variable can be changed. Thus the following expression would hold for Eqs. (2)–(5):

$$2\int_{-\pi/2}^{\pi/2}dy\int_{-\pi}^{\pi}dz\int_{-\pi}^{\pi}dx\,F(x,y,z)=\int_{-\pi}^{\pi}dy\int_{-\pi}^{\pi}dz\int_{-\pi}^{\pi}dx\,F(x,y,z), \tag{6}$$

where $F(x,y,z)$ represents the integrands in Eq. (2)–(5).

Equation (2)–(5) can be recast in the following form:

$$g_0(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\int_0^{\pi}dz\,I_0(y,z), \tag{7}$$

$$g_1(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}dz\,I_1(y,z), \tag{8}$$

$$g_2(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}dz\cos z\,I_0(y,z), \tag{9}$$

$$g_3(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\int_0^{\pi}dz\cos z\,I_1(y,z), \tag{9'}$$

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TABLE I. Setting signs for A, B, and C.

Given B, C*		Setting signs for: A	B	C
$B>0$,	$C>0$	A	B	C
$B>0$,	$C<0$	$-A$	B	$-C$
$B<0$,	$C>0$	$-A$	$-B$	C
$B<0$,	$C<0$	A	$-B$	$-C$

*The sign of A can be either positive or negative.

where

$$I_n(y,z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx\exp(inx)}{E-4(A\cos x\cos y+B\cos y\cos z+C\cos x\cos z)}, \tag{10}$$

and $n=0,1$.

The integral I_n can be easily evaluated by a complex contour integration. When the density of state function for the energy dispersion Eq. (1) is nonzero, $I_n(y,z)$ should be treated as a special case. In this case E is substituted by $E-i\epsilon$, where ϵ is a small positive number. ϵ is set to zero when the limit of $I_n(y,z)$ is taken.

Substituting $u=\exp(ix)$ in Eq. (10) and integrating over the unit circle in the complex u plane we obtain

$$I_n(y,z)=\lim_{\epsilon\to+0}\frac{i}{\pi}\oint\frac{u^n\,du}{xu^2-2(E-i\epsilon-\mu)u+\chi}, \tag{11}$$

where

$$\chi=4(A\cos y+C\cos z),$$

$$\mu=4B\cos y\cos z.$$

$I_n(y,z)$ has a real value for

$$(E-\mu)^2\geqslant\chi^2 \tag{12}$$

and is given by

$$I_n(y,z)=\xi u^m/\Delta', \tag{13}$$

where

$\xi=1$ and $u_m=u^-$ for $E>\mu$,

$\xi=-1$ and $u_m=u^+$ for $E<\mu$.

Δ' is given by

$$\Delta'=\lim_{\epsilon\to+0}[(E-i\epsilon-\mu)^2-\chi)]^{1/2}. \tag{14}$$

$u^\pm$ are represented by:

$$u^\pm=\lim_{\epsilon\to+0}\frac{E-i\epsilon-\mu\pm\Delta'}{\chi}. \tag{15}$$

It should be noted that for the case represented by Eq. (12) it is immaterial whether the limits are taken before or after the integration of Eq. (11), since we deal with two poles, neither one of which is located on the unit circle for $\epsilon\to+0$.

The situation is different for

$$(E-\mu)^2<\chi^2. \tag{16}$$

The two roots $u^\pm$ lie on the unit circle for which $\epsilon\to+0$, so that the limit is determined after the residues of Eq. (11) are calculated. The limiting process is described in Appendix A. The complex integral $I_n(y,z)$ is given for this case by:

$$I_n(y,z)=\xi u^+/\Delta', \tag{17}$$

where $\xi=-1$, and u^+ and Δ' are given by Eqs. (15) and (14), respectively. Hence for the complex I_n, ξ is independent of y and z.

At this point it becomes necessary to assume certain relationships between the interaction parameters, A, B, and C. Without loss of generality we can assume $|C|>|B|>|A|$. This can be done because the dispersion relation (1) is symmetrical with respect to x, y, and z. In addition, we may assume that $C>0$ and $B>0$. When B or C (or both) are negative they can be set positive according to Table I. This setting leaves the Green's functions invariant.

Equations (7)–(10) can be recast in the form

$$g_0(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\int_0^{\pi}\frac{\xi\,dz}{\Delta'}, \tag{18}$$

$$\begin{aligned} g_1(E)&=\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}\frac{\xi(E-\mu)\,dz}{\chi\Delta'} -\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}\frac{dz}{4(A\cos y+C\cos z)}\\ &=\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}\frac{\xi(E-4B\cos y\cos z)\,dz}{4(A\cos y+C\cos z)\Delta'}, \end{aligned} \tag{19}$$

$$g_2(E)=\frac{2}{\pi^2}\int_0^{\pi/2}dy\cos y\int_0^{\pi}\frac{\xi\cos z\,dz}{\Delta'}, \tag{20}$$

$$\begin{aligned} g_3(E)&=\frac{2}{\pi^2}\int_0^{\pi/2}dy\int_0^{\pi}\frac{\xi\cos z(E-\mu)\,dz}{\chi\Delta'} -\frac{2}{\pi^2}\int_0^{\pi/2}dy\int_0^{\pi}\frac{\cos z\,dz}{4(A\cos y+C\cos z)}\\ &=\frac{E}{4C}g_0(E)-\frac{A}{C}g_1(E)-\frac{B}{C}g_2(E)-\frac{1}{4C}. \end{aligned} \tag{21}$$

The Green's functions matrix elements, Eqs. (18)–(21) are real for the inequality Eq. (12) and complex for the inequality Eq. (16). $g_3(E)$ is given in terms of $g_0(E)$, $g_1(E)$, and $g_2(E)$. Thus we shall limit the discussion only to those three Green's functions matrix elements.

Substituting $t=\cos z$ in Eqs. (18)–(21), the $g_i(E)$ functions, $i=0,1,2$, can be expressed in the form

$$g_i(E)=\int_0^{\pi/2}H_i(y)\,dy, \tag{22}$$

where

$$H_i(y)=\int_{-1}^{1}u_i(t)\,dt \tag{23}$$

and the $u_i(t)$ are given by

$$u_0(t)=\frac{2\xi}{\pi^2\Delta}, \tag{24}$$

$$u_1(t)=\frac{2B(\cos^2 y)\xi(p_1-t)}{\pi^2C(p-t)\Delta}, \tag{25}$$

$$u_2(t)=\frac{2(\cos y)\xi t}{\pi^2\Delta}, \tag{26}$$

TABLE II. Boundaries for $\mathcal{R}$.

Boundary \ Condition	$A<0$, $C>B-A$ Case (i)	$A<0$, $C<B-A$ Case (ii)	$A>0$, $C>B+A$ Case (iii)	$A>0$, $C<A+B$ Case (iv)
A_1	$4(A-B-C)$	$4(A-B-C)$	$4(A-B-C)$	$4(A-B-C)$
A_2	$-4C$	$-4C$	$-4C$	$-4C$
A_3	$4(B-C-A)$	0	$4(B-A-C)$	$4(B-A-C)$
A_4	0	$4(B-C-A)$	$-4AB/C$	$-4AB/C$
A_5	$-4AB/C$	$-4AB/C$	0	$4(C-A-B)$
A_6	$4(C-A-B)$	$4(C-A-B)$	$4(C-A-B)$	0
A_7	$4C$	$4C$	$4C$	$4C$
A_8	$4(A+B+C)$	$4(C+A+B)$	$4(C+A+B)$	$4(C+B+A)$

where

$$p=-A\cos y/C, \quad p_1=E/4B\cos y,$$

$$\Delta=[16(C^2-B^2\cos^2 y)(t-1)(t+1)(t-\gamma)(t-\delta)]^{1/2}, \quad (27)$$

and γ and δ are

$$\gamma=\frac{E+4A\cos y}{4B\cos y-4C}, \quad (28)$$

$$\delta=\frac{E-4A\cos y}{4B\cos y+4C}. \quad (29)$$

The following relationships hold for γ and δ: When $E>r$ then $\delta>\gamma$, when $E<r$ then $\gamma>\delta$, where

$$r=-4AB\cos^2 y/C. \quad (30)$$

By utilizing Eqs. (28)—(30), it can be shown that for the real part of $g_i(E)$ ξ is independent of z, and depends only on y and is given by

$$\xi=\mathrm{sgn}(E-r). \quad (31)$$

The integrals over y of Eq. (22) take different forms in each of the nine $\mathcal{R}_i$ energy regions. The $\mathcal{R}_i$ regions for $i=2,3,\ldots,8$ are defined by $A_i>E>A_{i-1}$, $\mathcal{R}_1$ is defined by $E<A_1$ and $\mathcal{R}_9$ is defined by $E>A_8$. The A_i are the boundaries of these regions. There are four cases for these boundaries, and they are specified in Table II.

The $g_i(E)$ can be represented in term of the integrals $V_i(y_1,y_2)$, where the V_i are defined by:

$$V_i^{(l)}(y_s,y_j)=\int_{y_s}^{y_j} H_i^{(l)}(y)\,dy \quad (32)$$

where

$$0<y_s<y_j\leq\pi/2.$$

The index l denotes six energy regions $\mathcal{S}_l$, specified in Table III, for which $H_i(y)$ assumes a different form for each of the six regions. Hence Eqs. (22) are given as follows:

For the $\mathcal{R}_1$ Cases: (i), (ii), (iii), (iv),

$$g_i(E)=V_i^{(6)}(0,\pi/2). \quad (33)$$

For the $\mathcal{R}_2$ Cases: (i), (ii), (iii), (iv),

$$g_i(E)=V_i^{(5)}(0,y_1)+V_i^{(6)}(y_1,\pi/2). \quad (34)$$

For the $\mathcal{R}_3$ Cases: (i), (ii), (iii), (iv),

$$g_i(E)=V_i^{(5)}(0,y_2)+V_i^{(4)}(y_2,\pi/2). \quad (35)$$

For the $\mathcal{R}_4$ Cases: (i), (iii), (iv),

$$g_i(E)=V_i^{(4)}(0,\pi/2). \quad (36a)$$

For the $\mathcal{R}_4$ Case: (ii),

$$g_i(E)=V_i^{(5)}(0,y_2)+V_i^{(4)}(y_2,y_3)+V_i^{(3)}(y_3,\pi/2). \quad (36b)$$

For the $\mathcal{R}_5$ Cases: (iii) and (iv),

$$g_i(E)=V_i^{(3)}(0,y_3)+V_i^{(4)}(y_3,\pi/2). \quad (37a)$$

For the $\mathcal{R}_5$ Cases: (i) and (ii),

$$g_i(E)=V^{(4)}(0,y_3)+V^{(3)}(y_3,\pi/2). \quad (37b)$$

For the $\mathcal{R}_6$ Cases: (i), (ii), and (iii),

$$g_i(E)=V_i^{(3)}(0,\pi/2). \quad (38a)$$

For the $\mathcal{R}_6$ Case: (iv),

$$g_i(E)=V_i^{(2)}(0,y_4)+V_i^{(3)}(y_4,y_3)+V_i^{(4)}(y_3,\pi/2). \quad (38b)$$

For the $\mathcal{R}_7$ Cases: (i), (ii), (iii), and (iv),

$$g_i(E)=V_i^{(2)}(0,y_4)+V_i^{(3)}(y_4,\pi/2). \quad (39)$$

For the $\mathcal{R}_8$ Cases: (i), (ii), (iii), (iv),

$$g_i(E)=V_i^{(2)}(0,y_5)+V_i^{(1)}(y_5,\pi/2). \quad (40)$$

For the $\mathcal{R}_9$ Cases: (i), (ii), (iii), (iv),

$$g_i(E)=V_i^{(1)}(0,\pi/2). \quad (41)$$

The y_1, y_2, y_3, y_4, and y_5 are

$$y_1=\arccos\frac{E+4C}{4A-4B}, \quad (42a)$$

$$y_2=\arccos\frac{E+4C}{4B-4A}, \quad (42b)$$

TABLE III. The $\mathcal{S}_i$ regions.

$\mathcal{S}_l$ regions	Definition of energy regions*	Relationships for γ and δ
$\mathcal{S}_1$	$E>a$	$\delta>1;\ -1>\gamma$
$\mathcal{S}_2$	$a>E>b$	$1>\delta>-1>\gamma$
$\mathcal{S}_3$	$b>E>r$	$1>\delta>\gamma>-1$
$\mathcal{S}_4$	$r>E>c$	$1>\gamma>\delta>-1$
$\mathcal{S}_5$	$c>E>d$	$\gamma>1>\delta>-1$
$\mathcal{S}_6$	$E<d$	$\gamma>1;\ -1>\delta$

*$a=4(A+B)\cos y+4C$; $b=-4(A+B)\cos y+4C$; $c=4(B-A)\cos y-4C$; $d=4(A-B)\cos y-4C$; $r=-4AB\cos^2 y/C$; $a>b>r>c>d$.

TABLE IV. The Y parameters of Eqs. (57) and (58).

$\mathcal{S}_l$ regions	Y Real part	Y Imaginary part Range 1	Range 2
1	1	-1	-1
2	1	$-\delta$	-1
3	$-\gamma$	$-\gamma^{\dagger}$	$-\delta^{\dagger}$
4	$-\delta$	$-\delta^{\ddagger}$	$-\gamma^{\ddagger}$
5	$-\delta$	$-\delta$	-1
6	1	-1	-1

†,‡see the footnotes to Table V.

$$y_3 = \arccos\left(\frac{-EC}{4AB}\right)^{1/2}, \tag{42c}$$

$$y_3 = \arccos\frac{4C-E}{4A+4B}, \tag{42d}$$

$$y_5 = \arccos\frac{E-4C}{4A+4B}. \tag{42e}$$

It should be noted that $V_i^{(1)}(y_k, y_j)$ and $V_i^{(6)}(y_k, y_j)$ are real and have no imaginary component.

In order to carry the integration of Eq. (23) for $H_i(y)$ [or rather $H_i^{(l)}(y)$], we have to separate the real and imaginary parts of $H_i(y)$. This can be done by defining $u_i''(t)$ functions, where:

$$u_0''(t) = \frac{2}{\pi^2\Delta''}, \tag{43}$$

$$u_1''(t) = \frac{-2B\cos^2 y(p_1-t)}{\pi^2 C(p-t)\Delta''}, \tag{44}$$

$$u_2''(t) = \frac{2t\cos y}{\pi^2\Delta''}, \tag{45}$$

where Δ'' is given by

$$\Delta'' = [16(C^2 - B^2\cos^2 y)(t-1)(t+1)(t-\gamma)(\delta-t)]^{1/2} = i\Delta \tag{46}$$

and Δ was given by Eq. (27).

The following relation holds between $u_i''(t)$, Eqs. (43)–(45) and $u_i(t)$, Eqs. (24)–(26), for the imaginary part of the Green's functions:

$$u_i''(t) = -iu_i(t). \tag{47}$$

Let us define two auxiliary functions $W(t_1, t_2)$ and $W''(t_1, t_2)$:

$$W_i(t_1, t_2) = \int_{t_1}^{t_2} u_i(t)\,dt \tag{48}$$

and

$$W_i''(t_1, t_2) = \int_{t_1}^{t_2} u_i''(t)\,dt, \tag{49}$$

where $t_2 > t_1$, $t_1 = -1, \gamma, \delta$, and $t_2 = \gamma, \delta, 1$. The $H_i^{(l)}(y)$ are given for each of the $\mathcal{S}_l$ regions in the form

$$H_i^{(1)}(y) = W_i(-1, 1), \tag{50}$$

$$H_i^{(2)}(y) = W_i(-1, \delta) + iW''(\delta, 1), \tag{51}$$

$$H_i^{(3)}(y) = W_i(\gamma, \delta) + i[W_i''(-1, \gamma) + W_i''(\delta, 1)], \tag{52}$$

$$H_i^{(4)}(y) = W_i(\delta, \gamma) + i[W_i''(-1, \delta) + W_i''(\gamma, 1)], \tag{53}$$

$$H_i^{(5)}(y) = W_i(\delta, 1) + iW_i''(-1, \delta), \tag{54}$$

$$H_i^{(6)}(y) = W_i(-1, 1). \tag{55}$$

The W_i and W_i'' functions can be expressed in terms of complete elliptic integrals of the first and third kind,[8] and are given by

$$W_0'(t_1, t_2) = \xi_l fK(k), \tag{56}$$

$$W_1'(t_1, t_2) = -\xi_l f\frac{B\cos^2 y(p_1+Y)}{C(p+Y)}T(k, \alpha_4^2, \alpha_3^2), \tag{57}$$

$$W_2'(t_1, t_2) = -\xi_l f\cos y\, YT(k, \alpha_1^2, \alpha_2^2), \tag{58}$$

where W_i' represents either W_i or W_i''.

For $W_i'' = W_i'$, $\xi_l = 1$; and for $W_i = W_i'$, $\xi_l = 1$ in the regions $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, but $\xi_l = -1$ for the regions $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{S}_6$ (see Table III). The parameter Y in Eqs. (57) and (58) is given in Table IV. $K(k)$ in Eq. (56) denotes a complete elliptic integral of the first kind with a modulus k. The T functions in Eqs. (57) and (58) are given in the form[8]

$$T(k, \alpha^2, \beta^2) = (1/\alpha^2)[(\alpha^2 - \beta^2)\Pi(k, \alpha^2) + \beta^2 K(k)]. \tag{59}$$

TABLE V. Parameters for the imaginary parts of the Green's functions.

$\mathcal{S}_l$ region	k'^2	Range	α_1^2	α_2^2	α_3^2	α_4^2
2	$\frac{V^2-\epsilon_m}{8q}$		$\frac{(1-\delta)}{2}$	$\frac{(\delta-1)}{2\delta}$	$\frac{(1-\delta)(p_1+1)}{2(p_1+\delta)}$	$\frac{(1-\delta)(p_1+1)}{2(p_1+\delta)}$
3	$\frac{\epsilon_m-V}{\epsilon_p-W}$	† 1	$\frac{\gamma+1}{\delta+1}$	$\frac{\delta(\gamma+1)}{\gamma(\delta+1)}$	$\frac{(p_1-\delta)(\gamma+1)}{(p_1-\gamma)(\delta+1)}$	$\frac{(p-\delta)(\delta+1)}{(p-\gamma)(\gamma+1)}$
		2	$\frac{1-\delta}{1-\gamma}$	$\frac{\gamma(1-\delta)}{\delta(1-\gamma)}$	$\frac{(p_1-\gamma)(1-\delta)}{(p_1-\delta)(1-\gamma)}$	$\frac{(p-\gamma)(1-\delta)}{(p-\delta)(1-\gamma)}$
4	$\frac{\epsilon_p-W}{\epsilon_m-V}$	‡ 1	$\frac{\delta+1}{\gamma+1}$	$\frac{\gamma(\delta+1)}{\delta(\gamma+1)}$	$\frac{(p_1-\gamma)(\delta+1)}{(p_1-\delta)(\gamma+1)}$	$\frac{(p-\gamma)(\delta+1)}{(p-\delta)(\gamma+1)}$
		2	$\frac{1-\gamma}{1-\delta}$	$\frac{\delta(1-\gamma)}{\gamma(1-\delta)}$	$\frac{(p_1-\delta)(1-\gamma)}{(p_1-\gamma)(1-\delta)}$	$\frac{(p-\delta)(1-\gamma)}{(p-\gamma)(1-\delta)}$
5	$\frac{\epsilon_p-W}{8q}$		$\frac{\delta+1}{2}$	$\frac{\delta+1}{2\delta}$	$\frac{(\delta+1)(p_1-1)}{2(p_1-\delta)}$	$\frac{(\delta+1)(p-1)}{2(p-\delta)}$

†(1) parameters for $W_i''(-1,\gamma)$; (2) parameters for $W_i''(\delta,1)$ (see Eq. 52).

‡(1) parameters for $W_i''(-1,\delta)$; (2) parameters for $W_i''(\gamma,1)$ (see Eq. 53). V, W, ϵ_m, ϵ_p, q are defined in the footnote to Table VI.

TABLE VI. Parameters for the real part of the Green's functions.

S_l region	f	k^2	α_1^2	α_2^2	α_3^2	α_4^2
1	$\frac{4}{\pi^2(\epsilon_p - W)^{1/2}}$	$\frac{8q}{\epsilon_p - W}$	$\frac{2}{1-\gamma}$	$\frac{2}{\gamma-1}$	$\frac{2(p_1-\gamma)}{(1-\gamma)(p_1+1)}$	$\frac{2(p-\gamma)}{(1-\gamma)(p+1)}$
2	$\frac{1}{\pi^2}\left(\frac{2}{q}\right)^{1/2}$	$\frac{\epsilon_p - W}{8q}$	$\frac{\delta+1}{\delta-\gamma}$	$\frac{\gamma(\delta+1)}{\gamma-\delta}$	$\frac{(\delta+1)(p_1-\gamma)}{(\delta-\gamma)(p_1+1)}$	$\frac{(\delta+1)(p-\gamma)}{(\delta-\gamma)(p+1)}$
3	$\frac{4}{\pi^2(\epsilon_p - W)^{1/2}}$	$\frac{8q}{\epsilon_p - W}$	$\frac{\delta-\gamma}{\delta+1}$	$\frac{\gamma-\delta}{\gamma(\delta+1)}$	$\frac{(\delta-\gamma)(p_1+1)}{(\delta+1)(p_1-\gamma)}$	$\frac{(\delta-\gamma)(p+1)}{(\delta+1)(p-\gamma)}$
4	$\frac{4}{\pi^2(\epsilon_m - V)^{1/2}}$	$\frac{8q}{V-\epsilon_m}$	$\frac{\gamma-\delta}{\gamma+1}$	$\frac{\delta-\gamma}{\delta(\gamma+1)}$	$\frac{(\gamma-\delta)(p_1+1)}{(\gamma+1)(p_1-\delta)}$	$\frac{(\gamma-\delta)(p+1)}{(\gamma+1)(p-\delta)}$
5	$\frac{1}{\pi^2}\left(\frac{-2}{q}\right)^{1/2}$	$\frac{V-\epsilon_m}{8q}$	$\frac{1-\delta}{2}$	$\frac{(\delta-1)}{2}$	$\frac{(1-\delta)(p_1+1)}{2(p_1-\delta)}$	$\frac{(1-\delta)(p+1)}{2(p-\delta)}$
6	$\frac{4}{\pi^2(\epsilon_m - V)^{1/2}}$	$\frac{8q}{V-\epsilon_m}$	$\frac{2}{1-\delta}$	$\frac{2\delta}{\delta-1}$	$\frac{2(p_1-\delta)}{(1-\delta)(p_1+1)}$	$\frac{2(p-\delta)}{(1-\delta)(p+1)}$

$V=16(A+B)^2\cos^2 y$, $W=16(A-B)^2\cos^2 y$, $q=8AB\cos^2 y+2CE$, $\epsilon_m=(E-4C)^2$, $\epsilon_p=(E+4C)^2$. $\epsilon_m=(E-4C)^2$, $\epsilon_p=(E+4C)^2$.

The modulus k, and the parameters α_1^2, α_2^2, α_3^2, and α_4^2 and f are given for W_i in Table VI and for W_i'' in Table V. Π denotes a complete elliptic integral of the third kind. It should be noted that the modulus k of W_i'' is the complementary modulus k' of W_i.

The following relationships hold for the parameters of $\Pi(k,\alpha^2)$ for W_1 and W_2 (real case):

$$0<\alpha_1^2<k^2, \tag{60}$$

$$\alpha_4^2>1. \tag{61}$$

This is known as the hyperbolic case for Π, where Π can be given in terms of the Jacobian zeta function Z.

The relationships for the parameters of $\Pi(k,\alpha^2)$ for W_i'' and W_2'' are (imaginary case)

$$k^2<\alpha_1^2<1, \tag{62}$$

$$\alpha_4^2<0. \tag{63}$$

This is known as the circular case for Π, where Π can be given in terms of the Heuman lambda function Λ_0.

III. NUMERICAL CALCULATIONS

The Green's function matrix elements g_0, g_1, and g_2 can be evaluated utilizing the integration formulas Eqs. (33)–(41). $g_3(E)$ can be simply determined from Eq. (21), after g_0, g_1, and g_2 are evaluated. The integrations Eqs. (33)–(41) as defined in Eq. (32) can be determined numerically. There are no available analytical expressions for these integrals. The integrand of Eq. (32), $H_i^{(l)}$ includes both a real and an imaginary component for $l=2,3,4,5$ (see Table III). The $H_i^{(l)}$ functions are defined by Eqs. (50)–(55) in terms of the W_i and W_i'' functions. The W_i and W'' functions are given in terms of complete elliptic integrals of the first and third kind [see Eqs. (56)–(59)]. The various parameters of Eqs. (56)–(59) are displayed in Tables IV, V, and VI. The complete elliptic integrals of the third kind can be given in terms of Z or Λ_0 functions.[8] The Z and Λ_0 functions can be represented in terms of both complete elliptic integrals of the first and second kind, and incomplete elliptic integrals of the first and second kind. All these complete and incomplete elliptic integrals can be calculated utilizing standard computer programs.[9]

The integrations in Eqs. (33)–(41) are somewhat complicated due to the existence of a logarithmic singularity in $K(k)$ for $k=1$. However, the singularity will be located at one of the integration limits of Eq. (32), when they occur. To avoid the singularities, a Gauss quadrature was applied for the numerical integrations of Eqs. (33)–(41). It was determined that the singularities do not have a significant effect on the calculations. This effect was explored by removing the singularity from the integrand of Eq. (32) for W_0'' and W_0. An example of such a process is given in Appendix B. It was found that the singularities could be safely ignored, since removal of the singularities changed the final results by 0.1% at most.

Results for the computation of the $g_0(E)$, $g_1(E)$, $g_2(E)$, and $g_3(E)$ are given in Figs. 1 and 2 for the real and imaginary parts of the Green's function. The A, B, and C parameters in Fig. 1 are taken from Kopelman and Laufer,[10] corresponding to case (iii) whereas the parameters in Fig. 2 are arbitrary, and correspond to case (iv).

A specific example for determining a Green's function matrix element is given in Appendix C. The complete program for calculating the Green's function matrix elements for the various A, B, and C interaction parameters has been coded in FORTRAN, and is available upon request from the authors of this paper.

APPENDIX A

Utilizing a binomial expansion of the radical Δ' of Eq. (14), and returning the term linear in ϵ we obtain for the complex case of Eq. (15)

$$u^{\pm}=\frac{1}{\chi}\left\{E-i\epsilon-\mu\pm i(\chi^2-(E-\mu)^2)^{1/2}\left(1+\frac{i\epsilon(E-\mu)}{\chi^2-(E-\mu)^2}\right)\right\}$$
$$=\frac{1}{\chi}\left\{[E-\mu\pm i(\chi^2-(E-\mu)^2)^{1/2}]\left(1\pm\frac{\epsilon}{(\chi^2-(E-\mu)^2)^{1/2}}\right)\right\}. \tag{A1}$$

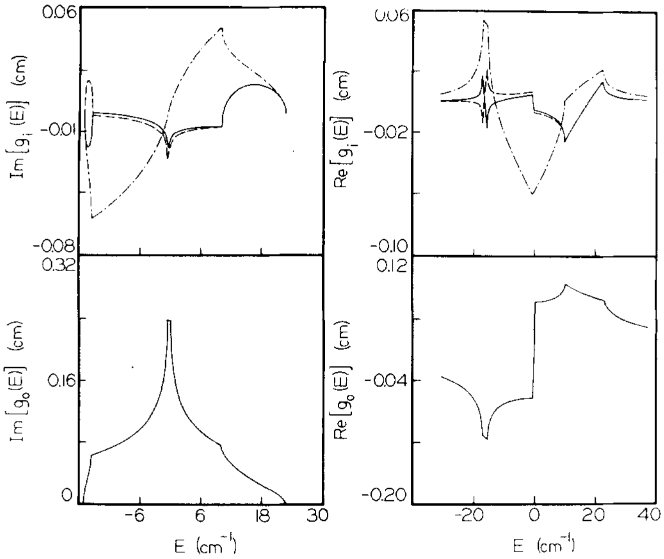

FIG. 1. Real and imaginary parts of the Green's functions matrix element $g_i(E)$. Bottom figures denote diagonal element $g_0(E)$. Top figures denote off-diagonal elements $g_i(E)$: --- $g_1(E)$, —$g_2(E)$, — — —$g_3(E)$ for interaction parameters $A=0.7\ \mathrm{cm}^{-1}$, $B=0.9\ \mathrm{cm}^{-1}$, $C=4.1\ \mathrm{cm}^{-1}$ (see Eq. 1).

To determine whether u^+ or u^- lies within the unit circle the absolute value of the poles is taken:

$$|u^{\pm}| = \left|\frac{E-\mu \pm i\chi^2-(E-\mu)^2}{\chi}\right| \left|1 \mp \frac{\epsilon}{(\chi^2-(E-\mu))^{1/2}}\right|$$

$$=1\cdot\left|1 \mp \frac{\epsilon}{(\chi^2-(E-\mu)^2)^{1/2}}\right|. \tag{A2}$$

Since ϵ is a positive number, $|u^+|<0$ and $|u^-|>0$. Hence $|u^+|$ lies within the unit circle and contributes to the residue.

It should be noted that the choice of ϵ to be positive is required by the physical situation. The density of states function $\rho(E)$ given by[1]

$$\rho(E)=\frac{1}{\pi}\,\mathrm{Im}\,g_0(E) \tag{A3}$$

must be positive. If ϵ is taken to be negative, u^- would lie within the unit circle, and the $\rho(E)$ would be negative.

APPENDIX B

The effect of the singularities of the integral given by Eq. (32), can be best illustrated by treating an example of such an integral. Let us look at the imaginary part of Eq. (37a) for $i=0$,

$$\mathrm{Im}\,g_0(E)=\mathrm{Im}\,V_0^{(3)}(0,y_3)+\mathrm{Im}\,V_0^{(4)}(y_3,\pi/2)$$

$$=\int_0^{y_3} f_3K(k_3)\,dy+\int_{y_3}^{\pi/2} f_4K(k_4)\,dy, \tag{B1}$$

f_3 and f_4 denote the f parameters for regions $\mathcal{S}_3$ and $\mathcal{S}_4$, respectively. The values of f_3 and f_4 are given in Table V. For $y=y_3$ we shall define f':

$$f'=f_3=f_4. \tag{B2}$$

The moduli k_3 and k_4 are given by Table VI and for $y=y_3$ both approach the value of one.

Utilizing the limit[1]

$$\lim_{k\to 1} K(k)=\ln\frac{4}{k'}, \tag{B3}$$

we obtain for Eq. (B1),

$$\mathrm{Im}\,g_0(E)=\int_0^{y_3}[f_3K(k_3)-f'\ln(-q)]\,dy$$
$$+\int_{y_3}^{\pi/2}[f_4K(k_4)-f'\ln q]\,dy+f'\int_0^{\pi/2}\ln|q|\,dy, \tag{B4}$$

where q is given in Table VI.

For the limits

$$\lim_{y\to y_3} k_3=\lim_{y\to y_3} k_4=1, \tag{B5}$$

the following expressions are obtained:

$$\lim_{y\to y_3}[f_3K(k_3)-f'\ln(-q)]=\lim_{y\to y_3}[f_4K(k_4)-f'\ln q]=0. \tag{B6}$$

By utilizing Eq. (B6), we may observe that the singularities have been removed from the first two integrals of Eq. (B4). The singularity exists only for the integral $J_1=\int_0^{\pi/2}\ln|q|\,dy$. However, this integral has an analytical expression

$$=(\pi/2)k|2CE|+\int_0^{\pi/2}\ln|1+p_2\cos^2 y|, \tag{B7}$$

where p_2 is given by

$$p_2=4AB/CE.$$

The integral

$$J_2=\int_0^{\pi/2}\ln|1+p_2\cos^2 y|\,dy \tag{B8}$$

is given by

$$J_2=\pi\ln\left(\frac{1+(1+p_2)^{1/2}}{2}\right). \tag{B9}$$

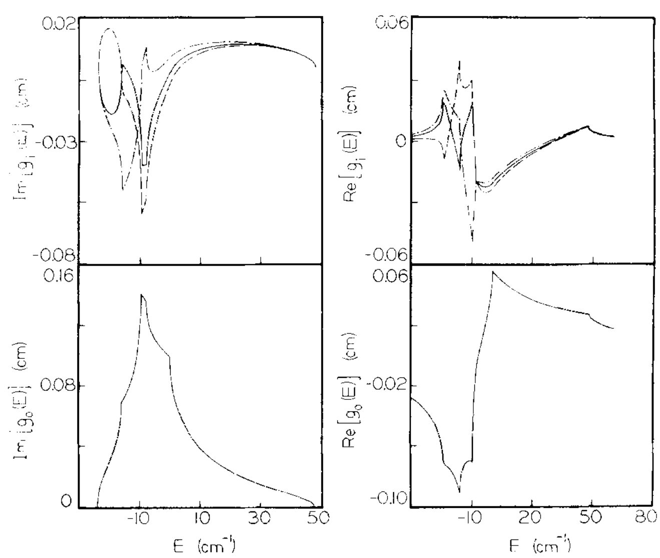

FIG. 2. Real and imaginary parts of the Green's function matrix elements $g_i(E)$ [for notations see footnotes to Fig. 1] for interaction parameters $A=2\ \mathrm{cm}^{-1}$, $B=3\ \mathrm{cm}^{-1}$, and $C=4\ \mathrm{cm}^{-1}$.

APPENDIX C

The representation of the Green's function matrix elements $q_i(E)$ for the crystal structure and interactions, under consideration, are numerous and complex. These representations change with the relative magnitude of the interaction parameters A, B, and C, as well as the interactions signs, and energy value E. A specific example for determining the appropriate representation, using the prescription given above would be illustrative.

We shall assume $A=2$, $B=3$, $C=4$, and that we are looking for the imaginary part of $q_2(E)$, where $E=-10$ (see Fig. 2). In order to elucidate the form of $\operatorname{Im} g_2(E)$, we shall envoke the following steps:

(a) Utilizing Table I, we observe that the signs of A, B, and C remain unchanged, because $B>0$, and $C>0$.

(b) Inspecting Table II, we find that Case (iv) is applicable for our parameter set, and that $A_3 < E < A_4$, implying $E \in \mathcal{R}_4$.

(c) The Green's function matrix element $\operatorname{Im} g_2(E)$, which corresponds to Case (iv) and region $\mathcal{R}_4$, is given by Eq. (36a). Hence,

$$\operatorname{Im} g_2(E) = \operatorname{Im} V_2^{(4)}(0, \pi/2) = \int_0^{\pi/2} \operatorname{Im} H_2^{(4)}(y)\, dy. \qquad \text{(C1)}$$

(d) The superscript (4) in $H_2^{(4)}$ of Eq. (C1) denotes energy region $\mathcal{S}_4$, defined in Table III. $\operatorname{Im} H_2^{(4)}(E)$ is expressed by the auxiliary function W_2'' given by Eq. (53). Taking the imaginary part of Eq. (53), we obtain for Eq. (C1):

$$\operatorname{Im} g_2(E) = \int_0^{\pi/2} W_2''(-1, \delta)\, dy + \int_0^{\pi/2} W_2''(\gamma, 1)\, dy, \qquad \text{(C2)}$$

where γ and δ are given by Eqs. (28) and (29), respectively.

(e) The W_2'' functions are given by Eq. (58) for which $\xi_l = 1$. Thus, the W_2'' functions given by Eq. (C2) can be represented in the form:

$$W_2''(t_1, t_2) = f Y \alpha_1^{-2} [(\alpha_1^2 - \alpha_2^2) \Pi(k', \alpha_1^2) + \alpha_1^2 K(k')] \cos y. \qquad \text{(C3)}$$

The parameters f and $(k')^2$, for the region $\mathcal{S}_4$, are displayed in Tables VI and V, respectively. Similarly, the parameters α_1^2 and α_2^2 of Eq. (C3), for range (1) of region $\mathcal{S}_4$, corresponding to the function $W''(-1, \delta)$, and for range (2) to the function $W''(1, \gamma)$, are given in Table V. The Y parameters for ranges (1) and (2) of region $\mathcal{S}_4$ are given in Table IV.

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Automorphisms of the Lie algebra of polynomials under Poisson bracket

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By constructing the one-parameter group of automorphisms generated by a typical derivation and generalizing certain special cases arising, we find all the automorphisms of the Lie algebra of polynomials under Poisson bracket. We introduce the notion of quasi-Hamiltonian equations, and investigate the effect of transformations $(q,p)\to(\alpha(q),\alpha(p))$ (α an arbitrary automorphism) on such equations. By considering linear quasi-Hamiltonian equations with constant coefficients we obtain a conserved quantity for an *arbitrary* (2×2) linear system with constant coefficients.

1. INTRODUCTION

Let F denote the collection of all real polynomials in the $2n$ real variables $(q,p)=(q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n)$, and define the *Poisson bracket* of elements $f,g\in F$ to be

$$\{f,g\}_{(q,p)}=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial q_j}\cdot\frac{\partial g}{\partial p_j}-\frac{\partial f}{\partial p_j}\cdot\frac{\partial g}{\partial q_j}\right).$$

This defines a Lie bracket on F and we shall also denote the corresponding Lie algebra by F.

The restriction to polynomials in this work is for convenience. Many of the results will, with suitable continuity hypotheses, extend to a wider class of functions, but are then more tedious to state.

An *automorphism of* F is a bijective linear map $\alpha: F\to F$ which satisfies

$$\{\alpha(f),\alpha(g)\}_{(q,p)}=\alpha(\{f,g\}_{(q,p)})\quad \text{all } f,g\in F. \tag{1}$$

Let aut$F$ denote the set of all automorphisms of F; it can be verified that aut$F$ is a group under composition. The automorphism property (1) does not imply that the transformation

$$(q,p)\to(Q,P)=(\alpha(q),\alpha(p)) \tag{2}$$

(using an obvious notation) is canonical. The situation is described by

Lemma 1: Let $\alpha\in\mathrm{aut}F$. Then $\alpha(1)$ is a nonzero space constant [we use *space constant* to mean independent of (q,p), and *constant* to mean independent of (q,p) and all parameters] but the transformation (2) is canonical if and only if $\alpha(1)=1$.

Proof: Since $\alpha\in\mathrm{aut}F$ is bijective, it has an inverse. Denote it by α^{-1}. From (1) and linearity we find (here and in the sequel i and j run from 1 to n)

$$\frac{\partial}{\partial q_i}\alpha(1)=\{\alpha(1),p_i\}_{(q,p)}=\alpha(\{1,\alpha^{-1}(p_i)\}_{(q,p)})=\alpha(0)=0,$$

$$\frac{\partial}{\partial p_i}\alpha(1)=\{q_i,\alpha(1)\}_{(q,p)}=\alpha(\{\alpha^{-1}(q_i),1\}_{(q,p)})=\alpha(0)=0,$$

implying $\alpha(1)$ is a space constant; $\alpha(1)\neq 0$ now follows from the fact that $\alpha(0)=0$ and α is bijective.

Now (1) and linearity tell us also that

$$\{\alpha(q_i),\alpha(q_j)\}_{(q,p)}=\alpha(\{q_i,q_j\}_{(q,p)})=\alpha(0)=0, \tag{3a}$$

$$\{\alpha(p_i),\alpha(p_j)\}_{(q,p)}=\alpha(\{p_i,p_j\}_{(q,p)})=\alpha(0)=0, \tag{3b}$$

$$\{\alpha(q_i),\alpha(p_j)\}_{(q,p)}=\alpha(\{q_i,p_j\}_{(q,p)})=\alpha(\delta_{ij})=\alpha(1)\cdot\delta_{ij} \tag{3c}$$

and these reduce to the canonical Poisson bracket relations, which are the conditions for (2) to be canonical, if and only if $\alpha(1)=1$. This proves Lemma I.

Remark: We show later (see Lemma 5) that $\alpha\in\mathrm{aut}F$ is uniquely determined by its effect on q_i,p_j.

It is convenient at this point to state the following result:

Lemma 2: Let $\alpha\in\mathrm{aut}F$. Then for all $f,g\in F$

$$\{f,g\}_{(q,p)}=\alpha(1)\cdot\{f,g\}_{(Q,P)}, \tag{4}$$

where, in the right-hand side, f and g are expressed in (Q,P) terms.

Proof: See the Appendix.

Now if $\alpha\in\mathrm{aut}F$ is *time-independent*, then transformation (2), even if not canonical, will nevertheless preserve the Hamiltonian form of equations. [The effect of (2) on (5a) and (5b) in the case when α is time-dependent is considered in Sec. 5B.] For suppose we have (here and in the sequel a dot denotes differentiation with respect to t)

$$\dot{q}_i=\frac{\partial h(q,p,t)}{\partial p_i}, \tag{5a}$$

$$\dot{p}_i=-\frac{\partial h(q,p,t)}{\partial q_i} \tag{5b}$$

for some (possibly time-dependent) $h(q,p,t)\in F$, and suppose $\alpha\in\mathrm{aut}F$ is time-independent. Then

$$\frac{\partial Q_i}{\partial t}=\frac{\partial P_i}{\partial t}=0,$$

and thus

$$\begin{aligned}\dot{Q}_i&=\{Q_i,h\}_{(q,p)}\\&=\alpha(1)\cdot\{Q_i,h\}_{(Q,P)},\ \text{using (4)}\\&=\alpha(1)\cdot\frac{\partial k(Q,P,t)}{\partial P_i}=\frac{\partial}{\partial P_i}[\alpha(1)\cdot k(Q,P,t)]\end{aligned}$$

and
$$\begin{aligned}\dot{P}_i&=\{P_i,h\}_{(q,p)}=\alpha(1)\cdot\{P_i,h\}_{(Q,P)}\\&=\alpha(1)\cdot\left[-\frac{\partial k(Q,P,t)}{\partial Q_i}\right]=-\frac{\partial}{\partial Q_i}[\alpha(1)\cdot k(Q,P,t)],\end{aligned}$$

where $k(Q,P,t)$ is $h(q,p,t)$ expressed in (Q,P) terms,

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and we have used the fact that $\alpha(1)$ is a space constant. We have shown that Eqs. (5a) and (5b), when expressed in (Q,P) terms, are again of Hamiltonian form, the new Hamiltonian being $\alpha(1)\cdot h(q,p,t)$ expressed in (Q,P) terms.

When $\alpha(1)=1$ this reduces (cf., Lemma 1) to the familiar result for time-independent canonical transformations For this reason, as well as their algebraic interest, automorphisms merit study. A useful source of automorphisms is the following (cf., Helgason[1] and Sagle and Walde[2]). Let D be a derivation of F, and for each real t define the map α_t by

$$\alpha_t f(q,p)=\exp(tD)f(q,p), \quad \text{all } f\in F, \tag{6}$$

where we interpret $\alpha_t f(q,p)$ to mean $(\alpha_t(f))(q,p)$; that is, we write $\alpha_t f$ or $\alpha_t(f)$, whichever is convenient. Then, if D does not depend explicitly on t, the set $\{\alpha_t\}$ forms, under composition, a one-parameter group of automorphisms of F, that is: each $\alpha_t\in \operatorname{aut} F$; α_0 is the identity map on F; $\alpha_{t_1}\cdot\alpha_{t_2}=\alpha_{t_1+t_2}$, all real t_1,t_2. It is called the *one-parameter group of automorphisms generated by D*. Furthermore, all one-parameter groups of automorphisms arise in this way. Now we have shown (Wollenberg[3]) that every derivation D of F has the form

$$D(f)=c\left(f-\sum_{j=1}^{n} p_j\frac{\partial f}{\partial p_j}\right)+\{f,H\}_{(q,p)} \quad \text{all } f\in F \tag{7}$$

for some space constant c and some $H\in F$. [By redefining H in (7) we could bring (7) to a form symmetrical in the q's and p's, but it is more convenient as it stands.]

In Sec. 3, we construct the one-parameter group of automorphisms generated by a typical derivation (7), and use certain special cases of these to motivate other examples of automorphisms. Section 2 contains results used in Sec. 3. In Sec. 4, we prove that the examples of Sec. 3 exhaust all possibilities. In Sec. 5, we introduce the notion of quasi-Hamiltonian equations, and investigate the effect of arbitrary (i.e., possibly time-dependent) transformations (2) on such equations. We consider linear quasi-Hamiltonian equations with constant coefficients, and obtain a conserved quantity for such equations. This yields, as a special case, a conserved quantity for an *arbitrary* (2×2) linear system with constant coefficients.

2. SOME RESULTS ON VECTOR FIELDS

The results in this section are used in Sec. 3.

By a *vector field* we mean an operator X of form

$$X=\sum_{j=1}^{n}\left(a_j(q,p)\circ\frac{\partial}{\partial q_j}+b_j(q,p)\circ\frac{\partial}{\partial p_j}\right), \tag{8}$$

where each $a_j,\ b_j\in F$.

It is easy to verify that for all $f,g\in F$

$$X(f\circ g)=(Xf)\circ g+f\circ(Xg). \tag{9}$$

This in turn implies (see Helgason,[1] Sagle and Walde[2]) that for all real t

$$\exp(tX)(f\cdot g)=[\exp(tX)f]\cdot[\exp(tX)g] \quad \text{all } f,g\in F, \tag{10}$$

i.e., $\exp(tX)$ preserves multiplication. (We assume $\exp(tX)f$ is defined for all t,f. That is, we omit problems of global integrability.)

From now on in this section, and for most of the rest of the paper, we will assume, for convenience, that $n=1$. The results extend immediately in an obvious way to the case of general n. Let $q=q_1,\ p=p_1$, $q_t=\exp(tX)q,\ p_t=\exp(tX)p$. Repeated application of (10), together with linearity, tells us that

$$\exp(tX)f(q,p)=f[\exp(tX)q,\exp(tX)p]=f(q_t,p_t) \quad \text{all } f\in F. \tag{11}$$

Lemma 3: Let X be a vector field. Then X is form-invariant under the transformations $(q,p)\to[\exp(tX)q,\exp(tX)p]$.

Proof: It can be verified that X, when expressed in (q_t,p_t) terms, takes the form

$$X=\hat a(q_t,p_t)\circ\frac{\partial}{\partial q_t}+\hat b(q_t,p_t)\circ\frac{\partial}{\partial p_t}.$$

In particular,

$$\begin{aligned}\hat a(q_t,p_t)&=Xq_t=X\exp(tX)q\\&=\exp(tX)\circ Xq \quad \text{since } X \text{ and } \exp(tX) \text{ commute}\\&=\exp(tX)a(q,p)\\&=a[\exp(tX)q,\ \exp(tX)p] \quad \text{using (11)}\\&=a(q_t,p_t).\end{aligned}$$

Similarly, $\hat b(q_t,p_t)=b(q_t,p_t)$, and thus

$$X=a(q_t,p_t)\cdot\frac{\partial}{\partial q_t}+b(q_t,p_t)\cdot\frac{\partial}{\partial p_t}. \tag{12}$$

Comparing (8) (with $n=1$) and (12) we obtain the result. This proves Lemma 3.

Remark: This result is no doubt well known, but the author cannot recall having seen an explicit proof before.

From (12), we obtain

$$\dot q_t=\frac{d}{dt}[\exp(tX)q]=X\exp(tX)q=Xq_t=a(q_t,p_t), \tag{13a}$$

$$\dot p_t=\frac{d}{dt}[\exp(tX)p]=X\exp(tX)p=Xp_t=b(q_t,p_t). \tag{13b}$$

We have also

$$q_0=q \quad \text{and} \quad p_0=p. \tag{13c}$$

Thus, by use of (11), the problem of finding the effect of $\exp(tX)$ is reduced to the (by no means trivial) problem of solving (13a)–(13c). It is worth emphasizing that (10)–(13a) and (13b) rely on the vector field property (9).

3. CONSTRUCTION OF AUTOMORPHISMS

A. One-parameter groups

We now construct the one-parameter group $\{\alpha_t^{(c,H)}\}$ of automorphisms generated by a typical derivation (7) in which c,H do not depend explicitly on t. Write (7) as

$$Df=cf+Xf=(c1+X)f, \tag{14}$$

where 1 is the identity operator on F and X is the vector field

$$\frac{\partial H(q,p)}{\partial p}\cdot\frac{\partial}{\partial q}-\left[cp+\frac{\partial H(q,p)}{\partial q}\right]\cdot\frac{\partial}{\partial p}.$$

Thus for all $f\in F$

$$\alpha_t^{(c,H)}f(q,p)=\exp(tD)f(q,p)$$

$$=\exp[t(c1+X)]f(q,p)$$

$=\exp(ct)\cdot\exp(tX)f(q,p)$ since 1 and X commute

$$=\exp(ct)f(q_t,p_t)\quad \text{cf. (11)}, \tag{15}$$

where [cf. (13a)–(13c)] $q_t=\exp(tX)q$ and $p_t=\exp(tX)p$ are obtained by solving

$$\dot{q}_t=\frac{\partial H(q_t,p_t)}{\partial p_t}, \tag{16a}$$

$$\dot{p}_t=-cp_t-\frac{\partial H(q_t,p_t)}{\partial q_t}, \tag{16b}$$

$$q_0=q \text{ and } p_0=p. \tag{16c}$$

Since we can write (16b) as

$$\frac{d}{dt}[p_t\exp(ct)]=-\frac{\partial}{\partial q_t}[\exp(ct)H(q_t,p_t)] \tag{16b$'$}$$

and (16a) as

$$\dot{q}_t\exp(-ct)\frac{\partial}{\partial p_t}[\exp(ct)H(q_t,p_t)]$$

$$=\frac{\partial}{\partial[p_t\exp(ct)]}[\exp(ct)H(q_t,p_t)]. \tag{16a$'$}$$

choose new variables $(\bar{q}_t,\bar{p}_t)=[q_t,p_t\exp(ct)]$ and let

$\bar{H}_c(\bar{q}_t,\bar{p}_t,t)=\exp(ct)H(q_t,p_t)$ expressed in $(\bar{q}_t,\bar{p}_t)$ terms

$$=\exp(ct)H[\bar{q}_t,\bar{p}_t\exp(-ct)]. \tag{17}$$

Then (16a)$'$ and (16b)$'$ take the Hamiltonian form

$$\dot{\bar{q}}_t=\frac{\partial\bar{H}_c(\bar{q}_t,\bar{p}_t,t)}{\partial\bar{p}_t}, \tag{18a}$$

$$\dot{\bar{p}}_t=-\frac{\partial\bar{H}_c(\bar{q}_t,\bar{p}_t,t)}{\partial q_t} \tag{18b}$$

and (16c) gives

$$\bar{q}_0=q \text{ and } \bar{p}_0=p. \tag{18c}$$

Hence from (15)

$$\alpha_t^{(c,H)}f(q,p)=\exp(ct)f(q_t,p_t)=\exp(ct)f[\bar{q}_t,\bar{p}_t\exp(-ct)]$$

$$\text{all } f\in F, \tag{15$'$}$$

where $\bar{q}_t,\bar{p}_t$ are obtained by solving (18a), (18b), (18c). In particular $\bar{H}_c(\bar{q}_t,\bar{p}_t,t)=\alpha_t^{(c,H)}H(q,p)$, a relation which does not seem either obvious or useful. An alternative expression is given below, see (20$'$).

Thus finding $\alpha_t^{(c,H)}$ is reduced to the problem of solving (18a)–(18c). The Hamiltonian form of (18a) and (18b) tells us that the transformations $(q,p)\to(\bar{q}_t,\bar{p}_t)$ are canonical although, because of possible t dependence of $\bar{H}_c(\bar{q}_t,\bar{p}_t,t)$, they do not necessarily form a one-parameter group.

B. Examples and generalizations

(a) When $H=0$ the corresponding automorphisms are given [cf. (15$'$)] by

$$\alpha_t^{(c,0)}f(q,p)=\exp(ct)f[\bar{q}_t,\bar{p}_t\exp(-ct)]\quad\text{all } f\in F,$$

where $\bar{q}_t,\bar{p}_t$ are obtained by solving (18a)–(18c) with $\bar{H}_c(\bar{q}_t,\bar{p}_t,t)=0$: i.e., where $\bar{q}_t=\bar{q}_0=q$ and $\bar{p}_t=\bar{p}_0=p$. Thus,

$$\alpha_t^{(c,0)}f(q,p)=\exp(ct)f[q,p\exp(-ct)]\quad\text{all } f\in F. \tag{19}$$

For example, from (17),

$$\bar{H}_c(q,p,t)=\exp(ct)H[q,p\exp(-ct)]=\alpha_t^{(c,0)}H(q,p), \tag{20}$$

i.e.,

$$\bar{H}_c=\alpha_t^{(c,0)}H. \tag{20$'$}$$

[We do not make use of (20$'$).]

(b) When $c=0$ the corresponding automorphisms are given by

$$\alpha_t^{(0,H)}f(q,p)=f(\bar{q}_t,\bar{p}_t)\quad\text{all } f\in F, \tag{21}$$

where $\bar{q}_t,\bar{p}_t$ are obtained by solving (18a)–(18c) with $\bar{H}_c(\bar{q}_t,\bar{p}_t,t)=H(\bar{q}_t,\bar{p}_t)$. In particular, it can be verified that

$$\alpha_t^{(c,H)}=\alpha_t^{(0,\bar{H}_c)}\cdot\alpha_t^{(c,0)}.$$

For example if $H(q,p)=\frac{1}{2}p^2$, then

$$\bar{H}_c(\bar{q}_t,\bar{p}_t,t)=\tfrac{1}{2}\exp(ct)[\bar{p}_t\exp(-ct)]^2=\tfrac{1}{2}\bar{p}_t^2\exp(-ct).$$

Thus (we omit the details),

$$\alpha_t^{(0,\bar{H}_c)}f(q,p)=f\{q+c^{-1}p[1-\exp(-ct),p\},\quad c\neq 0,$$

$$=f(q+tp,p),\qquad c=0,$$

and

$$\alpha_t^{(c,H)}f(q,p)$$

$$=\exp(ct)f\{q+c^{-1}p[1-\exp(-ct)],p\exp(-ct)\},\quad c\neq 0,$$

$$=f(q+tp,p),\quad c=0.$$

(c) The automorphisms (19) and (21) can be generalized. For it can be verified that if λ is an arbitrary nonzero space constant and $\gamma:(q,p)\to(\bar{q},\bar{p})$ is an arbitrary canonical transformation, then the maps α_λ and α_γ, defined by

$$\alpha_\lambda f(q,p)=\lambda f(q,p/\lambda)\quad\text{all } f\in F, \tag{22}$$

$$\alpha_\gamma f(q,p)=f(\bar{q},\bar{p})\quad\text{all } f\in F, \tag{23}$$

are automorphisms.

Maps of form (22) have been used by Souriau.[4]

The automorphisms (23) are well known.

Composing (23) and (22) we obtain the automorphism

$$\alpha_{\lambda,\gamma}f(q,p)=\alpha_\gamma\cdot\alpha_\lambda f(q,p)=\lambda f(\bar{q},\bar{p}/\lambda)\quad\text{all } f\in F. \tag{24}$$

4. ALL THE AUTOMORPHISMS

In fact, *every* automorphism of F takes the form $\alpha_{\lambda,\gamma}$ for some λ,γ.

The proof (formulated for the case of general n) is in stages. First note that (Wollenberg,[3] Lemma 2) the smallest Lie subalgebra of F containing the set $S=\{q_i,q_i^2,q_iq_j,q_i^3,p_i^2\}$ is F itself.

Lemma 4: Let $\alpha \in \operatorname{aut} F$. If $\alpha(q_i)=q_i$ and $\alpha(p_j)=p_j$, then $\alpha(f)=f$ all $f\in F$.

Proof: From the automorphism condition (1)

$$\frac{\partial}{\partial p_i}\alpha(f)=\{q_i,\alpha(f)\}_{(q,p)}=\{\alpha(q_i),\alpha(f)\}_{(q,p)}$$
$$=\alpha(\{q_i,f\}_{(q,p)})$$
$$=\alpha\left(\frac{\partial f}{\partial p_i}\right)\quad \text{all } f\in F. \tag{25a}$$

Similarly,

$$\frac{\partial}{\partial q_j}\alpha(f)=\alpha\left(\frac{\partial f}{\partial q_j}\right)\quad \text{all } f\in F. \tag{25b}$$

Thus,

$$\frac{\partial}{\partial p_j}\alpha(q_i^2)=0,$$
$$\frac{\partial}{\partial q_j}\alpha(q_i^2)=2\delta_{ij}q_i,$$
$$\therefore\quad \alpha(q_i^2)=q_i^2+d_i,$$

where the d_i are constants. Hence, from (1)

$$\alpha(\{q_i^2,f\}_{(q,p)})=\{\alpha(q_i^2),\alpha(f)\}_{(q,p)}=\{q_i^2,\alpha(f)\}_{(q,p)},$$

i.e.,

$$\alpha\left(2q_i\frac{\partial f}{\partial p_i}\right)=2q_i\cdot\frac{\partial\alpha(f)}{\partial p_i}$$
$$=2q_i\alpha\left(\frac{\partial f}{\partial p_i}\right)\quad [\text{using (25a)}]\ \text{all } f\in F. \tag{26}$$

Substituting $f=q_ip_i,\ q_i^2p_i,\ q_jp_i$, respectively, into (26) gives

$$\alpha(q_i\cdot q_i)=q_i\cdot\alpha(q_i)=q_i^2,$$
$$\alpha(q_i\cdot q_i^2)=q_i\cdot\alpha(q_i^2)=q_i^3,$$
$$\alpha(q_i\cdot q_j)=q_i\cdot\alpha(q_j)=q_iq_j.$$

Similarly, we can show

$$\alpha(p_i^2)=p_i^2.$$

Now if $\alpha(f)=f$ and $\alpha(g)=g$, then from (1)

$$\alpha(\{f,g\}_{(q,p)})=\{f,g\}_{(q,p)}$$

and by linearity, for all constants c_1, c_2

$$\alpha(c_1f+c_2g)=c_1\alpha(f)+c_2\alpha(g)=c_1f+c_2g.$$

Thus $\alpha(f)=f$ for all f belonging to some Lie subalgebra of F containing S, that is, for all $f\in F$. This proves Lemma 4.

Remark: It can be proved, from (7) or directly from the derivation property, that if D is a derivation and $D(q_i)=D(p_j)=0$, then $D(f)=0$ all $f\in F$. That is, a derivation is uniquely determined by its effect on q_i, p_j. However, we have not been able to show that this result implies (or is implied by) Lemma 4. Note that the proof of Lemma 4 is similar to that of a previous result of the author (see Wollenberg,[3] Lemma 3) which led to (7).

Lemma 5: Let $\alpha\in\operatorname{aut} F$. Then α is uniquely determined by $\alpha(q_i)$ and $\alpha(p_j)$.

Proof: Suppose the automorphisms α_1 and α_2 satisfy

$$\alpha_1(q_i)=\alpha_2(q_i),$$
$$\alpha_1(p_j)=\alpha_2(p_j).$$

Then $\alpha=\alpha_1\circ\alpha_2^{-1}$ is an automorphism which satisfies $\alpha(q_i)=q_i$, $\alpha(p_j)=p_j$. Hence, by Lemma 4, α is the identity map on F. Thus $\alpha_1=\alpha_2$. This proves Lemma 5.

Remark: The situation is very different if only $\alpha(1)$ is specified. For let λ be an arbitrary nonzero space constant, then *all* the automorphisms (24) satisfy $\alpha(1)=\lambda$. In particular, $\alpha(1)$ is constant does not imply that α is independent of all parameters (e.g., time). We can now prove the main result of this section.

Theorem 1: Every automorphism of F takes the form $\alpha_{\lambda,\gamma}$ for some λ,γ.

Proof: Let $\alpha\in\operatorname{aut} F$, and let $\lambda=\alpha(1)$. Then since, by Lemma 1, λ is a nonzero space constant, conditions (3a)–(3c) can be written

$$\left\{\frac{\alpha(q_i)}{\lambda},\frac{\alpha(q_j)}{\lambda}\right\}_{(q,p)}=\{\alpha(p_i),\alpha(p_j)\}_{(q,p)}=0,$$
$$\left\{\frac{\alpha(q_i)}{\lambda},\alpha(p_j)\right\}_{(q,p)}=\delta_{ij}$$

which tells us that the transformation

$$\gamma:(q,p)\to(\bar q,\bar p)=\left(\frac{\alpha(q)}{\lambda},\alpha(p)\right) \tag{27}$$

is canonical. Hence [cf. (24)], the map

$$\alpha_{\lambda,\gamma}f(q,p)=\lambda f(\bar q,\bar p/\lambda)=\alpha(1)f\left(\frac{\alpha(q)}{\alpha(1)},\frac{\alpha(p)}{\alpha(1)}\right)\quad \text{all } f\in F$$

is an automorphism. Since clearly $\alpha_{\lambda,\gamma}(q_i)=\alpha(q_i)$ and $\alpha_{\lambda,\gamma}(p_j)=p_j$, Theorem 1 now follows from Lemma 5.

5. QUASI-HAMILTONIAN EQUATIONS

A. Motivation and definition

We return to consideration of the one-parameter group $\{\alpha_t^{(c,H)}\}$ and examine the form (15′) takes when expressed in $(Q_t,P_t)=(\alpha_t(q),\alpha_t(p))$ terms and obtain the equations satisfied by Q_t,P_t. For convenience we replace $\alpha_t^{(c,H)}$ by α_t.

Since $Q_t=\alpha_t(q)=\bar q_t\exp(ct)$ and $P_t=\alpha_t(p)=\bar p_t$, clearly (15′) becomes

$$\alpha_tf(q,p)=\exp(ct)f[Q_t\exp(-ct),P_t\exp(-ct)]\quad \text{all } f\in F. \tag{15''}$$

The equations satisfied by Q_t, P_t can of course be obtained by expressing (18a), (18b) in (Q_t,P_t) terms. A more interesting method is the following, which makes fuller use of the automorphism property:

$$\dot Q_t=\frac{d}{dt}[\exp(tD)q]=D\exp(tD)q$$
$$=\exp(tD)Dq\quad \text{since } D \text{ and } \exp(tD) \text{ commute}$$
$$=\alpha_t[cq+\{q,H\}_{(q,p)}]$$
$$=c\alpha_t(q)+\{\alpha_t(q),\alpha_t(H)\}_{(q,p)}\quad \text{using linearity and (1)}$$

$$= cQ_t + \boldsymbol{\alpha}_t(1)\cdot\{Q_t, \boldsymbol{\alpha}_t(H)\}_{(Q_t,P_t)} \quad \text{using (4)}$$

$$= cQ_t + \frac{\partial H_c(Q_t,P_t,t)}{\partial P_t}. \tag{28a}$$

Similarly,

$$\dot{P}_t = \frac{d}{dt}[\exp(tD)p] = \exp(tD)Dp = \boldsymbol{\alpha}_t[\{p,H\}_{(q,p)}]$$

$$= \{\boldsymbol{\alpha}_t(p), \boldsymbol{\alpha}_t(H)\}_{(q,p)}$$

$$= \boldsymbol{\alpha}_t(1)\cdot\{P_t, \boldsymbol{\alpha}_t(H)\}_{(Q_t,P_t)}$$

$$= -\frac{\partial H_c(Q_t,P_t t)}{\partial Q_t}, \tag{28b}$$

where

$$H_c(Q_t,P_t,t) = \boldsymbol{\alpha}_t(1)\circ\boldsymbol{\alpha}_t(H) \quad \text{expressed in } (Q_t,P_t) \text{ terms}$$

$$= \exp(2ct)H[Q_t\exp(-ct),\ P_t\exp(-ct)]$$

$$\text{from } (15'')$$

and we have used the fact that $\boldsymbol{\alpha}_t(1)$ is a space constant.

Also, from (18c),

$$Q_0 = q \quad \text{and} \quad P_0 = p. \tag{28c}$$

Note that by this method we derive (28a) and (28b) without any reference to (15″) or (18a) and (18b), although we do use (15″) to get a " practical" form for $H_c(Q_t,P_t,t)$. Equations (28a) and (28b) might thus be regarded as more fundamental than (18a) and (18b). However, (15′) and (18a)–(18c) are more convenient than (15″) and (28a)–(28c).

Now since, as is well known, Eqs. (18a) and (18b) are derivable from the variational principle

$$\delta\int_{t_1}^{t_2}[\bar{p}_t\dot{\bar{q}}_t - \bar{H}_c(\bar{q}_t,\bar{p}_t,t)]\,dt = 0 \quad \text{all } t_2 > t_1, \tag{29}$$

Eqs. (28a) and (28b), which are (18a) and (18b) expressed in (Q_t,P_t) terms, are also derivable from a variational principle. To find it we just express the integrand in (29) in (Q_t,P_t) terms. We have

$$\bar{H}_c(\bar{q}_t,\bar{p}_t,t) = \exp(ct)H[\bar{q}_t,\bar{p}_t\exp(-ct)]$$

$$= \exp(ct)H[Q_t\exp(-ct),\ P_t\exp(-ct)]$$

$$= \exp(-ct)H_c(Q_t,P_t,t).$$

Thus,

$$[\bar{p}_t\mathring{\bar{q}}_t - \bar{H}_c(\bar{q}_t,\bar{p}_t,t)$$

$$= P_t\frac{d}{dt}[Q_t\exp(-ct)] - \exp(-ct)H_c(Q_t,P_t,t)$$

$$= \exp(-ct)\{P_t\mathring{Q}_t - [H_c(Q_t,P_t,t) + cP_tQ_t]\}.$$

Thus, (28a) and (28b) are derivable from

$$\delta\int_{t_1}^{t_2}\exp(-ct)\{P_t\dot{Q}_t - [H_c(Q_t,P_t,t) + cP_tQ_t]\}\,dt$$
$$= 0 \quad \text{all } t_2 > t_1. \tag{30}$$

A similar computation shows that (16a) and (16b) are derivable from

$$\delta\int_{t_1}^{t_2}\exp(ct)[p_t\mathring{q}_t - H(q_t,p_t)]\,dt = 0 \quad \text{all } t_2 > t_1. \tag{31}$$

These considerations motivate the following (formulated for the case of general n).

Definition: The equations

$$\dot{q}_i = M_i(q,p,t), \tag{32a}$$

$$\dot{p}_i = N_i(q,p,t) \tag{32b}$$

are called *quasi-Hamiltonian* if they are derivable from a variational principle of the form

$$\delta\int_{t_1}^{t_2}\mu(t)[(\sum_{i=1}^{n}p_i\dot{q}_i) - \mathcal{H}(q,p,t)]\,dt = 0 \quad \text{all } t_2 > t_1, \tag{33}$$

where $\mu(t)$ is a nonvanishing function of t alone and $\mathcal{H} \in \boldsymbol{F}$.

In such a case, (32a) and (32b) take the form

$$\dot{q}_i = \frac{\partial\mathcal{H}(q,p,t)}{\partial p_i}, \tag{33a}$$

$$\dot{p}_i = -\frac{\dot{\mu}}{\mu}p_i - \frac{\partial\mathcal{H}(q,p,t)}{\partial q_i}, \tag{33b}$$

these being the Euler–Lagrange equations arising from (33). Thus (16a) and (16b) and (28a) and (28b) are quasi-Hamiltonian. We get the usual Hamiltonian form if and only if $\mu(t)$ is constant, but we can always reduce quasi-Hamiltonian equations to Hamiltonian form, for by writing the integrand in (33) as

$$(\sum_{i=1}^{n}(\mu p_i)\dot{q}_i) - \mu\mathcal{H}(q,p,t)$$

we see that the change to new variables.

$$(\bar{\bar{q}},\bar{\bar{p}}) = (q,\mu p) \tag{34a}$$

will reduce (33a) and (33b) to Hamiltonian form with Hamiltonian $\bar{\bar{\mathcal{H}}}(\bar{\bar{q}},\bar{\bar{p}},t)$ given by

$$\begin{aligned}\bar{\bar{\mathcal{H}}}(\bar{\bar{q}},\bar{\bar{p}},t) &= \mu\mathcal{H}(q,p,t) \quad \text{expressed in } (\bar{\bar{q}},\bar{\bar{p}}) \text{ terms}\\ &= \mu\mathcal{H}(\bar{\bar{q}},\bar{\bar{p}}/\mu,t).\end{aligned} \tag{34b}$$

(It is not the only possibility. See Sec. 5B.) For example, (16a) and (16b) reduce to (18a) and (18b) this way. However, as we see next, quasi-Hamiltonian equations are just what we need for investigating the effect of time-dependent transformations (2) on equations of Hamiltonian form.

B. How they transform

We now return to the case $n=1$ and consider the effect of transformations (2), with α an arbitrary (i.e., possibly time-dependent) automorphsim, on quasi-Hamiltonian equations. By Theorem 1 every $\boldsymbol{\alpha} \in \operatorname{aut} F$ is of form $\alpha_{\lambda,\gamma}$, with $\lambda = \boldsymbol{\alpha}(1)$ and $\gamma:(q,p)\to(\bar{q},\bar{p})$ canonical. Thus [also see (27)] we can write

$$(Q,P) = (\alpha_{\lambda,\gamma}(q), \alpha_{\lambda,\gamma}(p)) = (\lambda\bar{q},\bar{p}).$$

Note that λ, $\bar{q}$, $\bar{p}$ may all depend explicitly on time, but that, neverthless, the corresponding transformation (2) is not much more general than a canonical transformation. Now the properties of arbitrary canonical transformations are investigated by means of generating functions (see Goldstein[5], Chap. 8). A similar approach is adopted here.

Suppose γ has a generating function Γ (Its particular form is irrelevant here). Thus for any $H(q,p,t)$ there is a function $\bar{H}(\bar{q},\bar{p},t)$ such that

$$p\,dq - H(q,p,t)\,dt = \bar{p}\,d\bar{q} - \bar{H}(\bar{q},\bar{p},t) + d\Gamma.$$

(For notational convenience we use differentials instead of total derivatives.) As is well known $\bar{H}(\bar{q},\bar{p},t)$ is given in each case by

$$\bar{H}(\bar{q},\bar{p},t) = H(q,p,t) + \frac{\partial\Gamma}{\partial t}.$$

Thus for any $\mu(t)$ ($\neq 0$) and $H(q,p,t)$

$$\begin{aligned}
\mu[p\,dq - H(q,p,t)\,dt] &= \mu\bar{p}\,d\bar{q} - \mu\bar{H}(\bar{q},\bar{p},t)\,dt + \mu\,d\Gamma \\
&= \mu\bar{p}\,d\bar{q} - \mu\bar{H}(\bar{q},\bar{p},t)\,dt - \dot{\mu}\Gamma\,dt + d(\mu\Gamma) \\
&= \mu\bar{p}\,d\bar{q} - \left[\mu H(q,p,t) + \mu\frac{\partial\Gamma}{\partial t} + \dot{\mu}\Gamma\right]dt + d(\mu\Gamma) \\
&= \mu P\cdot d\left(\frac{Q}{\lambda}\right) - \left[\mu H(q,p,t) + \frac{\partial}{\partial t}(\mu\Gamma)\right]dt + d(\mu\Gamma) \\
&= \frac{\mu}{\lambda}P\,dQ - \frac{\mu}{\lambda}\left[\frac{\dot{\lambda}}{\lambda}PQ + \lambda H(q,p,t) + \frac{\lambda}{\mu}\frac{\partial}{\partial t}(\mu\Gamma)\right]dt \\
&\qquad + d(\mu\Gamma) \\
&= \mu'[P\,dQ - H'(Q,P,t)\,dt] + d(\mu\Gamma),
\end{aligned}$$

where

$$\mu' = \mu/\lambda \tag{35a}$$

and

$$H'(Q,P,t) = \frac{\dot{\lambda}}{\lambda}PQ + \lambda\left[H(q,p,t) + \frac{1}{\mu}\frac{\partial}{\partial t}(\mu\Gamma)\right]$$

expressed in (Q,P) terms. (35b)

(Notice that $\lambda, \Gamma, \gamma, \mu, H$ determine μ', H'.) Thus, by the usual arguments of the calculus of variations, (33) implies

$$\delta\int_{t_1}^{t_2}\mu'(t)[P\dot{Q} - H'(Q,P,t)]\,dt = 0 \quad \text{all } t_2 > t_1,$$

i.e., Eqs. (33a) and (33b), when expressed in (Q,P) terms, are again quasi-Hamiltonian.

Remark: We have tried, so far without success, to characterize the transformations $(q,p)\to(Q,P)$ which preserve the quasi-Hamiltonian form of equations. The difficulty seems to be that this condition does not, by itself, determine the transformations.

Reduction to Hamiltonian form: To reduce (33a) and (33b) to Hamiltonian form, all we do is set $\lambda[=\alpha(1)] = \mu$; the canonical transformation γ can be arbitrary (cf., remark after Lemma 5). Thus (34a) is just one of infinitely many possibilities.

Effect on equations of Hamiltonian form: Suppose Eqs. (33a) and (33b) are in fact of Hamiltonian form, that is, suppose $\mu(t)$ is constant. Then, by (35a), the transformed equations are of Hamiltonian form if and only if $\lambda[=\alpha(1)]$ is constant (as remarked previously, this does not imply that α is time-independent), in which case the new Hamiltonian is, by (35b),

$$H'(Q,P,t) = \alpha(1)\left[H(q,p,t) + \frac{\partial\Gamma}{\partial t}\right]$$

expressed in (Q,P) terms.

In particular, if α is time-in-independent, then

$$\frac{\partial\Gamma}{\partial t} = 0$$

and

$H'(Q,P,t) = \alpha(1)\cdot H(q,p,t)$ expressed in (Q,P) terms.

This result was derived, by a different method, in Sec. 1. When $\alpha(1) = 1$ both reduce to familiar results for canonical transformations.

C. Linear quasi-Hamiltonian equations with constant coefficients

(This section is formulated for the case of general n). We now consider quasi-Hamiltonian equations (33a) and (33b) with $\mu = \exp(-at)$ for some constant a and H time-independent and homogeneous quadratic in (q,p). Equations (33a) and (33b) will then be linear with constant coefficients, and it is convenient to express them in matrix terms. Before doing this we show that there exists for such equations a conserved quantity which is homogeneous quadratic in (q,p).

From (34a) and (34b), we know that the change to new variables

$$(\bar{\bar{q}},\bar{\bar{p}}) = (q,\mu p) = [q, p\exp(-at)]$$

will reduce (33a) and (33b) to Hamiltonian form with Hamiltonian

$$\bar{\bar{H}}(\bar{\bar{q}},\bar{\bar{p}},t) = \mu H(\bar{\bar{q}},\bar{\bar{p}}/\mu) = \exp(-at)\,H[\bar{\bar{q}},\bar{\bar{p}}\exp(at)] \tag{36}$$

[If $a=0$, then (33a) and (33b) are already in Hamiltonian form]. Now the explicit time dependence can be removed from (36), by means of the canonical transformation $(\bar{\bar{q}},\bar{\bar{p}})\to(\bar{\bar{q}}',\bar{\bar{p}}')$ generated by

$$\Gamma(\bar{\bar{q}},\bar{\bar{p}}',t) = \left(\sum_{i=1}^{n}\bar{\bar{q}}_i\bar{\bar{p}}_i'\right)\cdot\exp(-\tfrac{1}{2}at).$$

For this generates the transformation obtained (see Goldstein,[5] Chap. 8) by solving

$$\bar{\bar{p}}_i = \frac{\partial\Gamma}{\partial\bar{\bar{q}}_i} = \bar{\bar{p}}_i'\cdot\exp(-\tfrac{1}{2}at),$$

$$\bar{\bar{q}}_i' = \frac{\partial\Gamma}{\partial\bar{\bar{p}}_i'} = \bar{\bar{q}}_i\cdot\exp(-\tfrac{1}{2}at),$$

i.e.,

$$(\bar{\bar{q}}',\bar{\bar{p}}') = [\bar{\bar{q}}\exp(-\tfrac{1}{2}at),\ \bar{\bar{p}}\exp(\tfrac{1}{2}at)] = [q\exp(-\tfrac{1}{2}at), p\exp(-\tfrac{1}{2}at)]$$

and the corresponding new Hamiltonian is given by

$$\begin{aligned}
&\bar{\bar{H}}(\bar{\bar{q}},\bar{\bar{p}},t) + \frac{\partial\Gamma}{\partial t} \quad \text{expressed in } (\bar{\bar{q}}',\bar{\bar{p}}') \text{ terms} \\
&= \exp(-at)\,H[\bar{\bar{q}}'\exp(\tfrac{1}{2}at), \bar{\bar{p}}'\exp(\tfrac{1}{2}at)] - \tfrac{1}{2}a\sum_{i=1}^{n}\bar{\bar{q}}_i'\bar{\bar{p}}_i' \\
&= H(\bar{\bar{q}}',\bar{\bar{p}}') - \tfrac{1}{2}a\sum_{i=1}^{n}\bar{\bar{q}}_i'\bar{\bar{p}}_i' \quad \text{by homogeneity.}
\end{aligned}$$

Because the new Hamiltonian does not depend explicitly on time, it is conserved. In (q,p) terms it becomes

$$W(q,p,t)=[\mathcal{H}(q,p)-\tfrac{1}{2}a\sum_{i=1}^{n}q_ip_i]\exp(-at)$$

(again by homogeneity), which is of the form mentioned. If $a=0$ it reduces, as expected, to $\mathcal{H}(q,p)$.

It is interesting, and does not seem at all obvious, that $W(q,p,t)$ can be described in terms of the coefficient matrix arising when the linear equations (33a) and (33b) are expressed in matrix terms. To see this let u denote the $(2n\times 1)$ column matrix $\binom{q}{p}$, let J denote the $(2n\times 2n)$ block matrix $\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$, where I_n denotes the $(n\times n)$ identity matrix, and let prime denote matrix transpose.

Now Leung and Meyer[6] have given the following elegant characterization of linear Hamiltonian systems: a linear system $\dot{u}=Au$ is Hamiltonian [and the $(2n\times 2n)$ matrix A, whether constant or not, is called *Hamiltonian*] if and only if JA is symmetric, in which case the Hamiltonian is $-\frac{1}{2}u'JAu$. Thus Eqs. (33a) and (33b) can be written

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & aI_n\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}+B\begin{pmatrix}q\\ p\end{pmatrix},$$

where B is a (constant) Hamiltonian matrix and $\mathcal{H}(q,p)=-\frac{1}{2}u'JBu$, i.e.,

$$\dot{u}=Cu \tag{37}$$

with constant coefficient matrix

$$C=\begin{pmatrix}0 & 0\\ 0 & aI_n\end{pmatrix}+B.$$

In particular,

$$-\tfrac{1}{2}u'JCu=-\tfrac{1}{2}u'\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & aI_n\end{pmatrix}u-\tfrac{1}{2}u'JBu$$
$$=-\tfrac{1}{2}a\,(\sum_{i=1}^{n}q_ip_i)+\mathcal{H}(q,p).$$

Also, by the Hamiltonian matrix property, $(JB)'=B'J'=-B'J=JB$, i.e., $JB'J=-J^2B=B$ (this condition appears in Laub and Meyer[7] as a possible definition of the Hamiltonian property), and so standard properties of trace give us

$$\operatorname{tr}B=\operatorname{tr}(JB'J)=\operatorname{tr}(B'J^2)=\operatorname{tr}(-B')$$
$$=-\operatorname{tr}B'=-\operatorname{tr}B,$$

i.e.,

$$\operatorname{tr}B=0,$$

so that

$$\operatorname{tr}C=na+\operatorname{tr}B=na$$

and thus

$$W(q,p,t)=[-\tfrac{1}{2}u'JCu]\exp[-(1/n)(\operatorname{tr}C)t]$$

which is of the form mentioned. If $\operatorname{tr}C=0$, it reduces to $\mathcal{H}(q,p)$.

This discussion is useful if linear quasi-Hamiltonian systems with constant coefficients abound, and can be easily recognized. From previous considerations a linear system (37) (whether C is constant or not) is quasi-Hamiltonian if and only if the matrix

$$B=C-\frac{1}{n}(\operatorname{tr}C)\begin{pmatrix}0 & 0\\ 0 & I_n\end{pmatrix}$$

is Hamiltonian, that is (we omit the details) if and only if C satisfies

$$C'J+JC=(1/n)(\operatorname{tr}C)J. \tag{38}$$

It is not clear how plentiful the solutions to this are. However, if $n=1$, then *every* linear system (37) (whether C is constant or not) is quasi-Hamiltonian, for it can be verified that (38) is satisfied by *all* (2×2) matrices C. [This can be explained by the fact that, when $n=1$, Eqs. (32a) and (32b) take the form (33a) and (33b) if and only if

$$\frac{\partial M_1}{\partial q_1}+\frac{\partial N_1}{\partial p_1}=-\frac{\dot\mu}{\mu}\quad\text{for some }\mu(t)\neq 0,$$

and this last condition is satisfied in every linear case $\dot{u}=Cu$, whether C is constant or not.] We can now deduce:

Theorem 2: Let C be an *arbitrary* constant (2×2) matrix. Then the quantity

$$W(q,p,t)=[-\tfrac{1}{2}u'JCu]\exp[-(\operatorname{tr}C)t] \tag{39}$$

is conserved along trajectories of the system $\dot{u}=Cu$.

The advantage of (39) is that it can be written down immediately and does not require any knowledge of trajectories of the system, or of eigenvalues of C; but it is not yet clear if it will help us to decide the qualitative nature of trajectories of the system. If $\operatorname{tr}C=0$ the equations are of Hamiltonian form and $W(q,p,t)$ gives the Hamiltonian.

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APPENDIX

Proof of Lemma 2: [The proof is modelled on Goldstein,[5] p. 254–255. Equation (A1) is Goldstein's equation 8.49.] We begin with the relation

$$\{u,v\}_{(q,p)}=\sum_{k=1}^{n}\left(\{u,Q_k\}_{(q,p)}\cdot\frac{\partial v}{\partial Q_k}+\{u,P_k\}_{(q,p)}\cdot\frac{\partial v}{\partial P_k}\right)$$
$$\text{all } u,v\in F \tag{A1}$$

which is essentially a formula relating partial derivatives [i.e., it does not depend on the canonicity or otherwise of transformation (2)].

Putting $u=Q_i$ into (A1), we find that for all $v\in F$

$$\{Q_i,v\}_{(q,p)}=\sum_{k=1}^{n}\left(\{Q_i,Q_k\}_{(q,p)}\frac{\partial v}{\partial Q_k}+\{Q_i,P_k\}_{(q,p)}\frac{\partial v}{\partial P_k}\right)$$
$$=\sum_{k=1}^{n}\alpha(1)\delta_{ik}\frac{\partial v}{\partial P_k}\quad\text{using (3a) and (3c)}$$
$$=\alpha(1)\cdot\frac{\partial v}{\partial P_i}. \tag{A2}$$

Similarly, putting $u=P_i$ into (A1) and using (3b) and (3c) we find that for all $v\in F$

$$\{P_i, v\}_{(q,p)} = -\alpha(1)\cdot\frac{\partial v}{\partial Q_i}. \tag{A3}$$

It is convenient to write these, for all $f\in F$, as

$$\{f, Q_i\}_{(q,p)} = -\alpha(1)\cdot\frac{\partial f}{\partial P_i}, \tag{A2$'$}$$

$$\{f, P_i\}_{(q,p)} = \alpha(1)\cdot\frac{\partial f}{\partial Q_i}. \tag{A3$'$}$$

Now put $u=f$ and $v=g$ into (A1), and we obtain

$$\{f,g\}_{(q,p)} = \sum_{k=1}^{n}\left(\{f, Q_k\}_{(q,p)}\cdot\frac{\partial g}{\partial Q_k} + \{f, P_k\}_{(q,p)}\frac{\partial g}{\partial P_k}\right)$$

$$= \alpha(1)\cdot\sum_{k=1}^{n}\left(-\frac{\partial f}{\partial P_k}\cdot\frac{\partial g}{\partial Q_k} + \frac{\partial f}{\partial Q_k}\cdot\frac{\partial g}{\partial P_k}\right)$$

using (A2$'$) and (A3$'$)

$$= \alpha(1)\cdot\{f,g\}_{(Q,P)} \quad \text{all } f,g\in F.$$

This proves Lemma 2.

Remark: In proving Lemma 2 we have used only the properties (3a)–(3c), and not the automorphism property (1).

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On the linear connection and curvature in Newtonian mechanics

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The trajectories of a scleronomic, holonomic particle motion in an otherwise general force field are autoparallel curves in a linear connected, symmetric, "almost" semimetric space. The Riemann–Christoffel curvature tensor and its concomitants belonging to the dynamical affinity are defined, and the physical meaning is discussed.

1. INTRODUCTION

Since the birth and success of Einstein's general theory of relativity there have been many attempts to geometricize classical mechanics. When a field of forces is given, the idea is to find such a geometric structure for the configuration space that the path defined by the time development of the system turns out to be a geodesic line. In the case of conservative systems the problem was solved by Douglas[1] and Eisenhart,[2] who showed that the structure of a Riemannian space is sufficient. The first one to consider more general fields of forces was Lichnerowicz,[3] and he used a semisymmetric linear connection to define the geodesics. Vujanovic[4–6] presented the use of a semimetric, semisymmetric connection. The geometrization of classical mechanics with the help of a linearly connected, pseudometric space of paths is given in two recent papers. Some curvature properties related with the connection and their physical implications are also considered in these works.[7,8]

In this paper we consider curvature properties in linearly connected semimetric and almost semimetric spaces of paths, whose geodesics are the trajectories of classical holonomic, scleronomic mechanical systems. No restrictions are imposed on the nature of the forces, except those of smoothness. We restrict our consideration to one-particle systems, but the generalization to many-particle systems is straightforward.

2. A GEODESIC FORM OF THE NEWTONIAN EQUATION OF MOTION

The equation of motion in the coordinate system $\{q^\lambda\}$ reads

$$m(\ddot{q}^\lambda + \{{}^{\lambda}_{\mu\nu}\}\dot{q}^\mu\dot{q}^\nu) = f^\lambda, \tag{1}$$

where f^λ is the force field and the dot (·) denotes the time derivative. The kinetic energy of the particle is

$$T = \tfrac{1}{2} m g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu; \tag{2}$$

it follows that the force field can be written as

$$f^\lambda = f^\lambda m g_{\mu\nu}\dot{q}^\mu_\mu\dot{q}^\nu/2T, \tag{3}$$

and the Newtonian equation of motion (1) is

$$\ddot{q}^\lambda + [\{{}^{\lambda}_{\mu\nu}\} - f^\lambda g_{\mu\nu}/2T]\dot{q}^\mu\dot{q}^\nu = 0. \tag{4}$$

We see that this equation is identical with the geodesic equation

$$\ddot{q}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = 0, \tag{5}$$

if the coefficients of linear connection are defined by

$$\Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\} - \phi^\lambda g_{\mu\nu}, \tag{6}$$

where

$$\phi^\lambda = f^\lambda/2T. \tag{7}$$

Because of the kinetic energy T in the denominator of ϕ^λ (7), and the very general nature of the forces f^λ, this symmetric connection is only defined along real trajectories, and so our space is a space of paths.

To a connection we can add an antisymmetric tensor without changing the geodesics. So, the transformation

$$\Gamma^\lambda_{\mu\nu} \to \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + 2\phi_{[\mu}\delta^\lambda{}_{\nu]} \tag{8}$$

does not alter the geodesic lines, and gives the connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ used by Vujanovic[4–6] as a starting point. On the other hand, the projective transformation

$$\Gamma^\lambda_{\mu\nu} \to \bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + 2\phi_{(\mu}\delta^\lambda{}_{\nu)} \tag{9}$$

gives a connection $\bar{\Gamma}^\lambda_{\mu\nu}$, which makes our space a Weyl's semimetric, symmetric space. Moreover, this transformation conserves the geodesics, but time no longer remains a natural affine parameter.[9]

3. COVARIANT DERIVATIVE

On the trajectory of the particle we can now define the covariant derivative belonging to the connection $\Gamma^\lambda_{\mu\nu}$ of a tensor of any valence. The definition is the usual one, and we give only an example:

$$\nabla_\nu P^{\kappa\lambda}{}_\mu = \partial_\nu P^{\kappa\lambda}{}_\mu + \Gamma^\kappa_{\nu\rho}P^{\rho\lambda}{}_\mu + \Gamma^\lambda_{\nu\rho}P^{\kappa\rho}{}_\mu - \Gamma^\rho_{\nu\mu}P^{\kappa\lambda}{}_\rho. \tag{10}$$

The covariant derivative of the metric tensor $g_{\lambda\kappa}$ belonging to the symmetric connection (6) is

$$\nabla_\mu g_{\lambda\kappa} = 2g_{\mu(\lambda}\phi_{\kappa)}. \tag{11}$$

Thus, we could call the connection (6) almost semimetric. The connection (8) is semimetric, for

$$\tilde{\nabla}_\mu g_{\lambda\kappa} = 2\phi_\mu g_{\lambda\kappa}. \tag{12}$$

It is easy to show that the Newtonian equation of motion (5) can be written in the forms

$$\dot{q}^\mu\nabla_\mu\dot{q}^\lambda = 0 \tag{13}$$

or

$$\dot{q}^\mu\tilde{\nabla}_\mu\dot{q}^\lambda = 0. \tag{14}$$

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The covariant differential of a contravariant (covariant) vector v^λ (v_κ) in a displacement along a trajectory from the point q^ν to the point $q^\nu + dq^\nu$ is given by

$$\delta v^\lambda \equiv dv^\lambda + \Gamma^\lambda_{\mu\nu} v^\nu dq^\mu \tag{15}$$

and

$$\delta v_\kappa \equiv dv_\kappa - \Gamma^\nu_{\mu\kappa} v_\nu dq^\mu, \tag{16}$$

where d means the ordinary differential. If a vector is displaced in such a way that $\delta v^i = 0$, the displacement is said to be parallel.

When using the symmetric connection (6), the square of a vector $v^2 = v^\mu v_\mu$ changes in the parallel displacement dq^μ along a trajectory as

$$d(v^2) = 2v^\nu \phi_\nu v_\mu dq^\mu. \tag{17}$$

In the usual vector notation this expression reads as

$$d(v^2) = 2(\bar{v}\cdot\bar{f})(\bar{v}\cdot d\bar{r})/2T. \tag{18}$$

From the geodesic equation (5) we see that if the velocity vector $\mathring{q}^\mu$ at a point is displaced parallelly along a trajectory to another point, we obtain the velocity vector belonging to that point. So, in the case where $\bar{v}$ is the velocity vector, Eq. (18) can be interpreted as the power equation $\dot{T} = P$ ($P = \bar{v}\cdot\bar{f} =$ power). The connection (8) gives respectively

$$d(v^2) = 2v^2 \phi_\mu dq^\mu. \tag{19}$$

For the velocity this gives the same result as before, but in the different form $dT = \bar{f}\cdot d\bar{r}$.

It is worth noticing that some quantities, as for example the velocity $\mathring{q}^\mu$, the kinetic energy T, the forces f^μ, and the coefficients of the linear connection $\Gamma^\lambda_{\mu\nu}$, need not be globally defined as a function of position, but only along a trajectory $q^\mu = q^\mu(t)$. Nevertheless, by considering a whole (continuous) family of trajectories simply covering a neighborhood of the original path $q^\mu(t)$, we can extend the domain of definition of such quantities to make their partial derivatives meaningful. This is the case naturally occurring later, in the discussion of the stability of motion (Sec. 5). We can obtain the family of trajectories in various ways; we could, for example, take the trajectories starting from a fixed point to different directions with the same kinetic energy, or we could take the trajectories starting with equal kinetic energies from the points of a surface to the directions normal to that surface. We could let the initial kinetic energies vary for different trajectories, or, in the case of conservative systems, we could require the total energies to be equal for all trajectories, too. Since, however, the derivatives of velocity previously used were essentially directional derivatives in the direction of the trajectory, the formulas (13)–(19) are independent of the way, however all this is done. In the further discussion this will no longer be true.

4. CURVATURE

Possessing a family of trajectories, we can now define the curvature tensor on them as a function of position by

$$R_{\nu\mu\lambda}{}^\kappa = 2\partial_{[\nu}\Gamma^\kappa_{\mu]\lambda} + 2\Gamma^\kappa_{[\nu|\sigma|}\Gamma^\sigma_{\mu]\lambda}. \tag{20}$$

It is well known that if $T_{\mu\lambda}{}^\kappa$ is an arbitrary tensor, then $\Gamma'^\kappa_{\mu\lambda} = \Gamma^\kappa_{\mu\lambda} + T_{\mu\lambda}{}^\kappa$ represents another connection, and the relation between the curvature tensors of these two connections is

$$R'_{\nu\mu\lambda}{}^\kappa = R_{\nu\mu\lambda}{}^\kappa + 2\nabla_{[\nu}T_{\mu]\lambda}{}^\kappa + 2S_{\nu\mu}{}^\rho T_{\rho\lambda}{}^\kappa - 2T_{[\nu|\lambda|}{}^\rho T_{\mu]\rho}{}^\kappa, \tag{21}$$

where ∇_ν means the covariant derivative belonging to the connection $\Gamma^\kappa_{\mu\lambda}$, and the tensor $S_{\nu\mu}{}^\rho$ is defined by

$$S_{\nu\mu}{}^\rho = \Gamma^\rho_{[\nu\mu]}. \tag{22}$$

After some calculation the symmetric connection (6) gives

$$R_{\nu\mu\lambda}{}^\kappa = K_{\nu\mu\lambda}{}^\kappa + g_{\lambda[\mu}[\partial_{\nu]}\phi^\kappa + \{{}^{\kappa}_{\nu]\sigma}\}\phi^\sigma - \phi_{\nu]}\phi^\kappa], \tag{23}$$

where $K_{\nu\mu\lambda}{}^\kappa$ is the curvature tensor belonging to the connection, whose components are the Christoffel symbols $\{{}^{\kappa}_{\nu\sigma}\}$ obtained from the metric tensor. By introducing the tensor $F_\nu{}^\kappa$ as

$$F_\nu{}^\kappa = \partial_\nu\phi^\kappa + \{{}^{\kappa}_{\nu\sigma}\}\phi^\sigma - \phi_\nu\phi^\kappa = \nabla_\nu\phi^\kappa, \tag{24}$$

we can write the curvature tensor of the connection (6) as

$$R_{\nu\mu\lambda}{}^\kappa = K_{\nu\mu\lambda}{}^\kappa + 2g_{\lambda[\mu}F_{\nu]}{}^\kappa. \tag{25}$$

By contracting the tensor $R_{\nu\mu\lambda}{}^\kappa$ with respect to the indices $\nu\kappa$, we get the so-called Ricci tensor

$$R_{\mu\lambda} = R_{\kappa\mu\lambda}{}^\kappa = K_{\mu\lambda} + 2g_{\lambda[\mu}F_{\kappa]}{}^\kappa. \tag{26}$$

Another concomitant of the curvature tensor is formed by contracting the indices $\lambda\kappa$. This antisymmetric tensor reads

$$\begin{aligned} V_{\nu\mu} &= R_{\nu\mu\kappa}{}^\kappa \\ &= K_{\nu\mu\kappa}{}^\kappa + 2g_{\kappa[\mu}F_{\nu]}{}^\kappa \\ &= 2g_{\kappa[\mu}F_{\nu]}{}^\kappa \\ &= 2F_{[\nu\mu]}. \end{aligned} \tag{27}$$

An easy calculation gives the identity

$$2R_{[\mu\lambda]} = -V_{\mu\lambda}, \tag{28}$$

and in the terms of the vector ϕ_μ the tensor $V_{\mu\lambda}$ reads

$$V_{\mu\lambda} = \partial_\mu\phi_\lambda - \partial_\lambda\phi_\mu. \tag{29}$$

If the force field is a gradient field, and if the total energies of the trajectories belonging to the family (cf. the discussion in the end of Sec. 3) are equal, we can prove that the tensor $V_{\mu\lambda}$ vanishes identically. Indeed, we can write

$$E = T + U(q^\lambda), \tag{30}$$

$$f_\mu = -\partial_\mu U, \tag{31}$$

$$\phi_\mu = -\tfrac{1}{2}(E-U)^{-1}\partial_\mu U, \tag{32}$$

$$\partial_\mu T = \partial_\mu(E-U) = -\partial_\mu U = f_\mu, \tag{33}$$

and Eq. (29) gives

$$\begin{aligned}V_{\mu\lambda}&=\partial_\mu(f_\lambda/2T)-\partial_\lambda(f_\mu/2T)\\&=(1/2T^2)(f_\lambda\partial_\mu T+T\partial_\mu f_\lambda-f_\mu\partial_\lambda T-T\partial_\lambda f_\mu)\\&=(1/2T)(\partial_\lambda U\partial_\mu U-\partial_\mu U_\lambda U)\\&=0.\end{aligned}\tag{34}$$

It follows that in this case there is a scalar field $\sigma(q^\mu)$ such that

$$\phi_\mu=-\partial_\mu\sigma.\tag{35}$$

Conversely, we can write

$$\begin{aligned}2\partial_{[\mu}f_{\lambda]}&=2T[\partial_\mu\phi_\lambda-\partial_\lambda\phi_\mu]+2\phi_\lambda\partial_\mu T-2\phi_\mu\partial_\lambda T\\&=2TV_{\mu\lambda}+(1/T)(f_\lambda\partial_\mu T-f_\mu\partial_\lambda T).\end{aligned}\tag{36}$$

Thus provided that $V_{\mu\lambda}=0$, we see that the force field is a gradient field if and only if the vector fields f_μ and $\partial_\mu T$ are collinear.

Transvecting the metric tensor $g^{\mu\lambda}$ onto the Ricci tensor $R_{\mu\lambda}$, we obtain the scalar

$$R=R_{\mu\lambda}g^{\mu\lambda}.\tag{37}$$

A straightforward calculation gives

$$R=K+2F_\mu{}^\mu\tag{38}$$

or

$$\nabla_\mu\phi^\mu=(R-K)/2.\tag{39}$$

So, $(R-K)/2$ serves as a covariant source of the ϕ^μ-field.

For the curvature tensor of the symmetric connection (6) it is easy to verify the usual identities[9]

$$R_{(\nu\mu)\lambda}{}^\kappa=0,\tag{40}$$

$$R_{[\nu\mu\lambda]}{}^\kappa=0,\tag{41}$$

$$R_{\nu\mu(\lambda\kappa)}=-2\nabla_{[\nu}g_{\mu](\lambda}\phi_{\kappa)},\tag{42}$$

and the identity of Ricci-Bianchi

$$\nabla_\kappa R_{\nu\mu\lambda}{}^\kappa-2\nabla_{[\nu}R_{\mu]\lambda}=0.\tag{43}$$

The previous procedure can be applied to the curvature tensor of the semisymmetric connection (8), too, but the presence of the torsion part makes the calculations more complicated.

5. THE STABILITY OF MOTION

The absolute derivative of a vector field $v^\mu(q^\nu(t))$ along the trajectory is defined by

$$\begin{aligned}\frac{Dv^\mu}{dt}&=\frac{dq^\nu}{dt}\nabla_\nu v^\mu\\&=\dot v^\mu+\Gamma^\mu_{\nu\rho}v^\rho\dot q^\nu.\end{aligned}\tag{44}$$

Accordingly, the second absolute derivative along a geodesic is given by

$$\begin{aligned}\frac{D^2v^\mu}{dt^2}&=\ddot v^\mu+2\Gamma^\mu_{\nu\kappa}\dot v^\kappa\dot q^\nu\\&\quad+[\partial_\nu\Gamma^\mu_{\kappa\lambda}-\Gamma^\mu_{\tau\lambda}\Gamma^\tau_{\nu\kappa}+\Gamma^\mu_{\nu\tau}\Gamma^\tau_{\kappa\lambda}]v^\lambda\dot q^\kappa\dot q^\nu.\end{aligned}\tag{45}$$

If we are given two adjacent trajectories $q^\lambda(t)$ and $q^\lambda(t)+\epsilon^\lambda(t)$, where ϵ is the infinitesimal isochronous displacement from the trajectory $q_\lambda(t)$ to the trajectory $q^\lambda(t)+\epsilon^\lambda(t)$, the paths fulfill the equations

$$\ddot q^\lambda+\Gamma^\lambda_{\mu\nu}(q^\kappa)\dot q^\nu\dot q^\mu=0\tag{46}$$

and

$$\ddot q^\lambda+\ddot\epsilon^\lambda+\Gamma^\lambda_{\mu\nu}(q^\kappa+\epsilon^\kappa)[\dot q^\nu+\dot\epsilon^\nu][\dot q^\mu+\dot\epsilon^\mu]=0.\tag{47}$$

By subtracting Eq. (46) from Eq. (47) and neglecting terms of second or higher order in ϵ^λ and $\dot\epsilon^\lambda$, we get

$$\ddot\epsilon^\lambda+(\partial_\rho\Gamma^\lambda_{\mu\nu})\epsilon^\rho\dot q^\mu\dot q^\nu+2\Gamma^\lambda_{\mu\nu}\dot\epsilon^\nu\dot q^\mu=0.\tag{48}$$

Writing the second absolute derivative of ϵ^λ along the trajectory $q^\lambda(t)$ and using Eq. (48), we obtain

$$\frac{D^2\epsilon^\lambda}{dt^2}-R_{\nu\mu\kappa}{}^\lambda\dot q^\kappa\epsilon^\mu\dot q^\nu=0.\tag{49}$$

This equation is analogous to the Levi-Cività[10] equation for the geodesic deviation in a Riemannian space: The curvature tensor $K_{\nu\mu\kappa}{}^\lambda$ of the Christoffel symbols is merely replaced by the curvature tensor $R_{\nu\mu\kappa}{}^\lambda$ based on the coefficients of the dynamic connection (6).

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Tensor spherical harmonics and tensor multipoles. II. Minkowski space

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The bases of tensor spherical harmonics and of tensor multipoles discussed in the preceding paper are generalized in the Hilbert space of Minkowski tensor fields. The transformation properties of the tensor multipoles under Lorentz transformation lead to the notion of irreducible tensor multipoles. We show that the usual 4-vector multipoles are themselves irreducible, and we build the irreducible tensor multipoles of the second order. We also give their relations with the symmetric tensor multipoles defined by Zerilli for application to the gravitational radiation.

1. INTRODUCTION

In the preceding paper,[1] hereafter called I, we have studied bases in the Hilbert spaces of complex tensor fields on the unit sphere $\mathcal{S}^2$, embedded in the three-dimensional Euclidean space $\mathcal{E}^3$. In this work, our purpose is to give a relativistic generalization of these bases for the Minkowski space denoted by $\mathcal{M}$. We consider Hilbert spaces $\mathcal{L}_r^2[\mathcal{S}^2(e)]$ of rth-order Minkowski tensor fields on the unit sphere $\mathcal{S}^2(e)$ embedded in the subspace $\mathcal{E}^3(e)$ orthogonal to an arbitrary time like 4-vector e.

In these Hilbert spaces we build the tensor spherical harmonics and tensor multipole bases of the little group of the 4-vector e, by the method expounded in I. These two sets are orthonormal in $\mathcal{L}_r^2[\mathcal{S}^2(e)]$ but only the tensor multipoles are pairwise orthogonal in $\mathcal{M}^{\otimes r}$. By studying the properties of the tensor multipoles under pure Lorentz transformations, we introduce the concept of irreducible tensor multipoles which transform according to an irreducible representation of such transformations.

In Sec. 2, we define the spherical tetrads in the space $\mathcal{M}$ and from them we build the spherical basis tensors which transform according to an irreducible representation under a rotation of the little group of e.

Section 3 is devoted to rth-order tensor spherical harmonics. First the spherical harmonics on the unit sphere $\mathcal{S}^2(e)$ are derived. Then they are coupled with the basis tensors through Clebsch–Gordan coefficients to obtain the tensor spherical harmonics on the same sphere. We also give some properties of the first- and second-order tensor spherical harmonics.

In Sec. 4, the rth-order tensor multipoles are deduced from the tensor spherical harmonics by the orthogonal transformation defined in I. Then the expansion of the tensor product $\mathcal{M}^{\otimes r}$ in a direct sum of two-dimensional subspaces invariant under a pure Lorentz transformation allows us to introduce the notion of irreducible tensor multipoles which form bases of these subspaces. In particular, we show that the usual 4-vector multipoles are themselves irreducible and we build explicitly the irreducible tensor multipoles of the second order.

Finally, in two Appendices we give some results on the canonical decomposition of a Lorentz transformation and on representations of pure Lorentz transformations.

Throughout the paper we use the summation convention for repeated Minkowski and magnetic indices but we always write explicit summations over angular momentum indices. The scalar products in $\mathcal{M}$, $\mathcal{M}\otimes\mathcal{M}$, and $\mathcal{M}^{\otimes r}$ are denoted by a single dot (.), a double dot (:) and $(\overset{r}{.})$, respectively, e.g., $a.b$, $t:T$, $t(\overset{r}{.})T$. For rotation matrices, Clebsch–Gordan coefficients (CG coefficients) an extensive bibliography is given in I.

2. TENSOR SPHERICAL BASES

A. Spherical tetrads

In the Minkowski space $\mathcal{M}$, we use the metric tensor g, the components of which are $g^{00}=-g^{ii}=1$ $(i=1,2,3)$, $g^{\mu\nu}=0$ for $\mu\neq\nu$ $(\mu,\nu=0,1,2,3)$ and $g^{\mu\nu}=g^{\nu\mu}=g_{\mu\nu}$. We also use the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ which is the completely antisymmetric tensor such that $\epsilon_{0123}=1$. With this tensor, the determinant of four 4-vectors is defined by $\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma$ and it is denoted by $\{a,b,c,d\}$.

A tetrad $\{e\}$ is a set of four 4-vectors e_α $(\alpha=0,1,2,3)$, forming a basis of $\mathcal{M}$, satisfying the orthonormality, orientation, and closure relations

$$e_\alpha\cdot e_\beta=g_{\alpha\beta},\tag{1}$$

$$\{e_\alpha,e_\beta,e_\gamma,e_\delta\}=\epsilon_{\alpha\beta\gamma\delta},\tag{2}$$

$$g^{\alpha\beta}e_\alpha\otimes e_\beta=g.\tag{3}$$

The spherical tetrad associated with $\{e\}$ is the set of four 4-vectors e_n^j $(j=0, n=0; j=1, n=-,0,+)$ defined by

$$e_0^0=e_0,\quad e_0^1=e_3,\tag{4a}$$

$$e_\pm^1=\mp(1/\sqrt{2})(e_1\pm ie_2).\tag{4b}$$

These 4-vectors are complex, they belong to the complexification $\mathcal{M}_c$ of $\mathcal{M}$. They satisfy the identity

$$e_n^{j*}=(-1)^n e_{-n}^j,\tag{5}$$

where the symbol (*) means complex conjugation, and the following relations corresponding to Eqs. (1), (2), (3) above

$$e_n^{j*}\cdot e_{n'}^{j'}=(-1)^j\delta_{jj'}\delta_{nn'},\tag{6}$$

$$\{e_0^0,e_m^1,e_n^1,e_r^1\}=-i\epsilon_{mnr},\tag{7}$$

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$$\sum_{j=0}^{1} (-1)^j e_n^{j*} \otimes e_n^j = g, \tag{8}$$

where ϵ_{mnr} is the spherical Levi–Civita tensor defined in Eq. (4) of I.

The spherical components a_n^j of a 4-vector a on the tetrad $\{e\}$ are defined by

$$a_n^j = (-1)^j a \cdot e_n^j, \quad a = \sum_{j=0}^{1} a_n^j e_n^{j*}. \tag{9}$$

Then the scalar product of two 4-vectors can be written with the spherical components

$$a \cdot b = a_0^0 b_0^0 + a_+^1 b_-^1 + a_-^1 b_+^1 - a_0^1 b_0^1. \tag{10}$$

In the following, when no confusion is possible, we drop out the index of the unit timelike vector of tetrads, e.g., we write e for e_0 or e_0^0.

In the rest frame of e, the time component of e_n^1 vanishes and e_n^1 reduces to the vector spherical basis of the Euclidean space, i.e., $e = (1, 0)$, $e_n^1 = (0, \mathbf{e}_n)$, see Sec. 2 of I.

Let Λ be a Lorentz transformation of the restricted group. By using the canonical decomposition of Λ with respect to the 4-vector e, see Appendix A1, the transformation law of the spherical basis 4-vectors reads

$$\Lambda e_n^j = L_\Lambda e_{n'}^j D^j (R_\Lambda)_n^{n'}, \tag{11}$$

where the pure Lorentz transformation L_Λ and the rotation R_Λ are defined in Eqs. (A2) and (A3).

In the tetrad $\{e\}$, the parity operator P is defined by

$$P = 2e \otimes e \cdot - g \cdot \tag{12}$$

and it acts on the spherical basis 4-vectors according to

$$P e_n^j = (-1)^j e_n^j. \tag{13}$$

B. Tensor spherical basis of rth order

The tensor products $e_{n_1}^{j_1} \otimes e_{n_2}^{j_2} \otimes \cdots \otimes e_{n_r}^{j_r}$ form a basis of the space $\mathcal{M}_c^{\otimes r}$ but under a rotation of the little group of e, they transform according to the product of representations $D^{j_1}(R) \otimes D^{j_2}(R) \otimes \cdots \otimes D^{j_r}(R)$. The tensors of the spherical basis are built by coupling the tensor products of 4-vectors through CG coefficients such that they transform according to an irreducible representation of a rotation.

The rth-order tensors are built by a recurrence relation from the $(r-1)$th-order tensors and the 4-vectors of the spherical tetrad

$$t_m^{j_r \cdots j_2 k_1 \cdots k_r} = \langle j_{r-1} m' k_r n | j_r m \rangle t_{m'}^{j_{r-1} \cdots j_2 k_1 \cdots k_{r-1}} \otimes e_n^{k_r}. \tag{14}$$

For instance the second-order tensors are defined by

$$t_m^{j_2 k_1 k_2} = \langle k_1 n_1 k_2 n_2 | j_2 m \rangle e_{n_1}^{k_1} \otimes e_{n_2}^{k_2}. \tag{15}$$

Often in the following, we shall denote for short these tensors by $t_m^{j_r \cdots}$ when no confusion is possible.

Let us consider some of their properties. They satisfy the identity

$$(t_m^{j_r \cdots})^* = (-1)^{j_r + m + K} t_{-m}^{j_r \cdots}, \quad \text{with } K = \sum_{i=1}^{r} k_i, \tag{16}$$

and the orthonormality relation

$$(t_m^{j_r \cdots})^* (\vdots)(t_{m'}^{j_r' \cdots}) = (-1)^K \left(\prod_{i=2}^{r} \delta_{j_i j_i'}\right)\left(\prod_{i=1}^{r} \delta_{k_i k_i'}\right) \delta_{mm'}. \tag{17}$$

Under a Lorentz transformation Λ of the restricted group, these tensors become

$$\Lambda^{\otimes r} t_m^{j_r \cdots} = (L_\Lambda^{\otimes r} t_{m'}^{j_r \cdots}) D^{j_r}(R_\Lambda)_m^{m'}, \tag{18}$$

and the parity operator acts on them according to

$$P^{\otimes r} t_m^{j_r \cdots} = (-1)^K t_m^{j_r \cdots}. \tag{19}$$

The spherical components $T_m^{j_r \cdots}$ of an arbitrary tensor T are defined by

$$T_m^{j_r \cdots k_r} = (-1)^K T(\vdots) t^{j_r \cdots k_r}, \tag{20a}$$

$$T = \sum_{\substack{k_1 \cdots k_r \\ j_2 \cdots j_r}} T_m^{j_r \cdots k_r} t_m^{j_r \cdots k_r *}. \tag{20b}$$

The tensors obtained for the maximal couplings (i.e., $k_i = 1$ for $i = 1$ to r and $j_i = i$ for $i = 2$ to r) are simply denoted by t_m^r. They are completely symmetric, orthogonal to e, and they have a vanishing trace[2]

$$(t_m^r)^{\mu_1 \cdots \mu_i \cdots \mu_l \cdots \mu_r} = (t_m^r)^{\mu_1 \cdots \mu_l \cdots \mu_i \cdots \mu_r}, \tag{21}$$

$$t_m^r \cdot e = 0, \tag{22}$$

$$t_m^r : g = 0. \tag{23}$$

Furthermore the integer order representations of the rotation of the little group of the 4-vector e, can be obtained as

$$D^j(R)_n^m = (-1)^j t_m^{j*} (\vdots) R^{\otimes j} t_n^j. \tag{24}$$

3. TENSOR SPHERICAL HARMONICS

A. Spherical harmonics on $\mathcal{S}^2(e)$

Consider the unit sphere $\mathcal{S}^2(e)$ orthogonal to the unit timelike 4-vector e. A point on $\mathcal{S}^2(e)$ is characterized by a unit spacelike 4-vector u orthogonal to e

$$u^2 = -1, \quad u \cdot e = 0. \tag{25}$$

Hence, the tensor fields on $\mathcal{S}^2(e)$ can be parametrized by such a 4-vector u in a tetrad $\{e\}$.

The scalar fields on $\mathcal{S}^2(e)$ form the Hilbert space $\mathcal{L}_0^2[\mathcal{S}^2(e)]$ with the scalar product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{S}^2(e)} f_1^*(u) f_2(u)\, d\Omega(u), \tag{26}$$

where the invariant measure $d\Omega(u)$ on $\mathcal{S}^2(e)$ is

$$d\Omega(u) = 2\delta(u \cdot e)\, \delta(u^2 + 1)\, du. \tag{27}$$

The spherical harmonics of the 4-vector u in the tetrad $\{e\}$, $Y_m^l(u, \{e\}) \equiv Y_m^l(u)$, are obtained from the maximal coupling tensors t_m^l by

$$Y^l_m(u) = (-1)^l \left[\frac{(2l+1)!!}{4\pi l!}\right]^{1/2} \overset{l}{\otimes} u(\,^l_\cdot\,) t^l_m. \tag{28}$$

They form an orthonormal basis in the space $\mathcal{L}^2_0[\mathcal{S}^2(e)]$

$$\langle Y^l_m, Y^{l'}_{m'}\rangle = \delta_{ll'}\delta_{mm'}. \tag{29}$$

Let Λ be a Lorentz transformation of the restricted group, L_Λ and R_Λ the pure transformation and the rotation of the canonical decomposition of Λ with respect to e. Then, we have the identity

$$(L_\Lambda u)\cdot(\Lambda e^j_n) = u\cdot(R_\Lambda e^j_n) = u\cdot e^j_{n'} D^j(R_\Lambda)^{n'}{}_n, \tag{30}$$

i.e., the spherical components of the 4-vector $L_\Lambda u$ on the tetrad $\Lambda\{e\}$ are also those of the 4-vector u on the tetrad $R_\Lambda\{e\}$. These relations are generalized to the spherical harmonics according to

$$Y^l_m(L_\Lambda u, \Lambda\{e\}) = Y^l_m(u, R_\Lambda\{e\}) = Y^l_{m'}(u,\{e\}) D^l(R_\Lambda)^{m'}{}_m. \tag{31}$$

Under the parity operation, the transformation law of the spherical harmonics is

$$Y^l_m(Pu,\{e\}) = Y^l_m(u, P\{e\}) = (-1)^l Y^l_m(u,\{e\}). \tag{32}$$

In the frame where $e=(1,0)$ the time components of the 4-vectors u and e^1_n vanish and we have

$$Y^l_m(u,\{e\}) = Y^l_m(\mathbf{u},\{\mathbf{e}\}), \tag{33}$$

i.e., the spherical harmonics on $\mathcal{S}^2(e)$ are identical to the usual spherical harmonics. For a brief review of their properties the reader is referred to Sec. 2 of I.

B. Tensor spherical harmonics of rth order

The Hilbert space $\mathcal{L}^2_r[\mathcal{S}^2(e)]$ of complex tensor fields on the sphere $\mathcal{S}^2(e)$ has the scalar product

$$\langle f, g\rangle = \int f^*(u)(\,^r_\cdot\,) g(u)\, d\Omega(u). \tag{34}$$

As for the tensor fields on $\mathcal{S}^2$, the rth-order tensor spherical harmonics (TSH) on $\mathcal{S}^2(e)$ are built by coupling basis tensors and spherical harmonics through CG coefficients

$$Y^{l\,j_r\cdots k_r J}{}_M(u) = \langle lmj_rn|JM\rangle Y^l_m(u) t^{j_r\cdots k_r}_n. \tag{35}$$

By construction these TSH form an orthonormal basis of the space $\mathcal{L}^2_r[\mathcal{S}^2(e)]$

$$\langle Y^{l\,j_r\cdots k_r J}{}_M, Y^{l'\,j'_r\cdots k'_r J'}{}_{M'}\rangle$$
$$= (-1)^K \left(\prod_{i=1}^r \delta_{k_ik'_i}\right)\left(\prod_{i=2}^r \delta_{j_ij'_i}\right)\delta_{ll'}\delta_{JJ'}\delta_{MM'} \tag{36}$$

and they satisfy the identity

$$Y^{l\,j_r\cdots J}{}_M(u)^* = (-1)^{l+K+J+M} Y^{l\,j_r\cdots J}{}_{-M}(u). \tag{37}$$

Let Λ be a Lorentz transformation of the restricted group. The TSH of the 4-vector $L_\Lambda u$ in the tetrad $\Lambda\{e\}$ are related to those of the 4-vector u in the tetrad $\{e\}$ by

$$Y^{l\,j_r\cdots J}{}_M(L_\Lambda u, \Lambda\{e\}) = L_\Lambda^{\otimes r} Y^{l\,j_r\cdots J}{}_{M'}(u,\{e\}) D^J(R_\Lambda)^{M'}{}_M. \tag{38}$$

In the cases where Λ is a rotation R of the little group of e, or a pure Lorentz transformation L, the preceding relation becomes

$$Y^{l\,j_r\cdots J}{}_M(u, R\{e\}) = Y^{l\,j_r\cdots J}{}_{M'}(u,\{e\}) D^J(R)^{M'}{}_M, \tag{39a}$$

$$Y^{l\,j_r\cdots J}{}_M(Lu, L\{e\}) = Y^{l\,j_r\cdots J}{}_M(u,\{e\}). \tag{39b}$$

Under the parity operation, the transformation law of the TSH is

$$Y^{l\,j_r\cdots J}{}_M(u, P\{e\}) = (-1)^{l+K} Y^{l\,j_r\cdots J}{}_M(u,\{e\}), \tag{40}$$

where the sign $(-1)^{l+K}$ is the parity of the TSH.[3]

By using $6j$ symbols and the notation $\hat{x} = (2x+1)^{1/2}$, the scalar product in $\mathcal{M}_c^{\otimes r}$ of two TSH reads

$$Y^{l\,j_r\cdots J}{}_M(u)(\,^r_\cdot\,)Y^{l'\,j'_r\cdots J'}{}_{M'}(u) = \left(\prod_{i=2}^r \delta_{j_ij'_i}\right)\left(\prod_{i=1}^r \delta_{k_ik'_i}\right)(-1)^{K+l'+j'_r+J'}$$
$$\times \hat{J}\hat{J}'\hat{l}\hat{l}'\sum_k (4\pi)^{-1/2}/\hat{k}\begin{Bmatrix} k & l & l' \\ j_r & J' & J\end{Bmatrix}\langle l0l'0|k0\rangle$$
$$\times\langle JMJ'M'|kn\rangle Y^k_n(u). \tag{41}$$

This equation for $J=J'$ and $M=M'$ exhibits the geometric properties of the TSH in $\mathcal{M}_c^{\otimes r}$ for fixed values of J and M. The symmetry of the CG coefficients implies that the TSH with different j_i or k_i, or with opposite parity are orthogonal in $\mathcal{M}_c^{\otimes r}$, but the TSH with the same j_i, k_i and parity are not. In Sec. 4, we shall define the tensor multipoles which are pairwise orthogonal in $\mathcal{M}_c^{\otimes r}$.

C. Tensor spherical harmonics of first and second order

Let us consider the lower-order TSH. The first-order ones are the 4-vector spherical harmonics, defined by

$$Y^{ljJ}{}_M(u) = \langle lmjn|JM\rangle Y^l_m(u) e^j_n. \tag{42}$$

For J and M fixed, the possible values of j and l are $j=0$, $l=J$; $j=1$, $l=J-1$, J, $J+1$. Hence one has four 4-vector spherical harmonics. The first one is proportional to the timelike 4-vector e, while the three others are combinations of spacelike 4-vectors. In the rest frame of e, they are

$$Y^{J0J}{}_M(u) \equiv Y^J_M(u) e = [Y^J_M(\mathbf{u}), 0], \tag{43a}$$

$$Y^{l1J}{}_M(u) = [0, \mathbf{Y}^{lJ}{}_M(\mathbf{u})]. \tag{43b}$$

The 16 tensors of the second-order spherical basis, defined in Eq. (15), satisfy the symmetry relation

$$(t^{jk_1k_2}_n)^{\mu\nu} = (-1)^{j+k_1+k_2}(t^{jk_2k_1}_n)^{\nu\mu}. \tag{44}$$

All tensors with $k_1=k_2$ are either symmetric or antisymmetric according to the value of j but the tensors with $k_1\neq k_2$ have no well-defined symmetry property. The trace of these tensors is

$$\mathrm{tr}(t^{jk_1k_2}_n) \equiv g : t^{jk_1k_2}_n = \hat{k}_1\delta_{k_1k_2}\delta_{j0}\delta_{n0}. \tag{45}$$

All tensors but t^{000}_0 and t^{011}_0 have a vanishing trace.

The second-order TSH are built from these tensors by

$$Y^{l\,jk_1k_2J}{}_M(u) = \langle lmjn|JM\rangle Y^l_m(u) t^{jk_1k_2}_n. \tag{46}$$

By construction they have the same symmetry and trace properties as the basis tensors, namely,

$$[Y^{ljk_1k_2J}_M(u)]^{\mu\nu} = (-1)^{j+k_1+k_2}[Y^{ljk_2k_1J}_M(u)]^{\nu\mu}, \tag{47}$$

$$\mathrm{tr}[Y^{ljk_1k_2J}_M(u)] = \delta_{lJ}\,\delta_{j0}\delta_{k_1k_2}\hat{k}_1 Y^J_M(u). \tag{48}$$

For J and M fixed one has 16 TSH according to the values of k_1, k_2, j and l. For $k_1=k_2=j=0$, the first one is proportional to the tensor product $e\otimes e$

$$Y^{J\,000J}_M(u) = Y^J_M(u)e\otimes e. \tag{49}$$

For $k_1\neq k_2$, $j=1$, $l=J-1$, J, $J+1$, there are 6 TSH which are the tensor product of a 4-vector harmonics by e

$$Y^{l110J}_M(u) = Y^{l1J}_M(u)\otimes e, \tag{50a}$$

$$Y^{l101J}_M(u) = e\otimes Y^{l1J}_M(u). \tag{50b}$$

For $k_1=k_2=1$ we have 9 TSH which correspond to the 9 TSH defined in the space $\mathcal{L}^2_2(\mathcal{S}^2)$, see Sec. 3 of I.

4. TENSOR MULTIPOLES

A. Tensor multipoles of rth order

As for the Euclidean space, the tensor multipoles (TM) of rth order are orthonormal linear combinations of TSH with the same j_i, k_i and parity. They are defined by the orthogonal transformation

$$X^{j_r\cdots k_rJ}_{\mu}{}_M(u) = \sum_l M(j_r, J)_{\mu l}\, Y^{lj_r\cdots k_rJ}_M(u), \tag{51}$$

where the matrix elements are CG coefficients and are defined in Eq. (66) of I.

By construction the TM form an orthonormal basis of $\mathcal{L}^2_r[\mathcal{S}^2(e)]$

$$\langle X^{j_r\cdots k_rJ}_{\mu}{}_M, X^{j'_r\cdots k'_rJ'}_{\mu'}{}_{M'}\rangle = (-1)^K\left(\prod_{i=1}^r \delta_{k_ik'_i}\right)\left(\prod_{i=2}^r \delta_{j_ij'_i}\right) \times \delta_{\mu\mu'}\delta_{JJ'}\delta_{MM'} \tag{52}$$

and for fixed values of J and M, they are pairwise orthogonal in $\mathcal{M}^{\otimes r}_c$

$$X^{j_r\cdots k_rJ}_{\mu}{}_M(\overset{r}{.})X^{j'_r\cdots k'_rJ}_{\mu'}{}_M = (-1)^K\left(\prod_{i=1}^r \delta_{k_ik'_i}\right)\left(\prod_{i=2}^r \delta_{j_ij'_i}\right)$$
$$\times\,\delta_{\mu\mu'}\epsilon^{j_r}_\mu(4\pi)^{-1/2}\sum_k(\hat{J}^2/\hat{k})\langle J\mu J-\mu\,|\,k0\rangle$$
$$\times\langle JMJM\,|\,kn\rangle Y^k_n(u), \tag{53}$$

where the sign $\epsilon^{j_r}_\mu$ is defined in Eq. (70) of I. Another geometrical property of the TM is that their products by the 4-vectors e and u are either vanishing or proportional to a TM of $(r-1)$th order

$$X^{j_r\cdots k_rJ}_{\mu}{}_M(u)\cdot e = \delta_{k_r0}\delta_{j_rj_{r-1}}X^{j_{r-1}\cdots k_{r-1}J}_{\mu}{}_M(u), \tag{54}$$

$$X^{j_r\cdots k_rJ}_{0}{}_M(u)\cdot u = -\,\delta_{k_r1}\langle 10j_{r-1}0\,|\,j_r0\rangle X^{j_{r-1}\cdots k_{r-1}J}_{0}{}_M(u), \tag{55a}$$

$$X^{j_r\cdots k_rJ}_{\pm\mu}{}_M(u)\cdot u = \mp\,\delta_{k_r1}\langle 10j_{r-1}\mu\,|\,j_r\mu\rangle X^{j_{r-1}\cdots k_{r-1}J}_{\mp\mu}{}_M(u),$$
$$\mu>0. \tag{55b}$$

Under a Lorentz transformation Λ, a rotation R of the little group of e or a pure Lorentz transformation L, the TM transform like the TSH

$$X^{j_r\cdots J}_{\mu}{}_M(L_\Lambda u, \Lambda\{e\}) = L_\Lambda^{\otimes r}X^{j_r\cdots J}_{\mu}{}_{M'}(u,\{e\})D^J(R_\Lambda)^{M'}{}_M, \tag{56a}$$

$$X^{j_r\cdots J}_{\mu}{}_M(u, R\{e\}) = X^{j_r\cdots J}_{\mu}{}_{M'}(u,\{e\})D^J(R_\Lambda)^{M'}{}_M, \tag{56b}$$

$$X^{j_r\cdots J}_{\mu}{}_M(Lu, L\{e\}) = L^{\otimes r}X^{j_r\cdots J}_{\mu}{}_M(u,\{e\}). \tag{56c}$$

The TM have a well-defined parity[3] η_μ

$$X^{j_r\cdots J}_{\mu}{}_M(u, P\{e\}) = \eta_\mu X^{j_r\cdots J}_{\mu}{}_M(u,\{e\}) \tag{57}$$

with $\eta_\mu=(-1)^{J+K}$ for $\mu>0$, $(-1)^{J+K+j_r}$ for $\mu=0$ and $-(-1)^{J+K}$ for $\mu<0$.

By analogy with electromagnetism, we can call

(i) magnetic the TM having parity $(-1)^{J+1}$, namely, $X^{j_r\cdots}_0$ with $(-1)^K=(-1)^{j_r+1}$ and $X^{j_r\cdots}_{\pm\mu}$ $(\mu>0)$ with $(-1)^K=\mp 1$;

(ii) electric those having parity $(-1)^J$, namely, $X^{j_r\cdots}_0$ with $(-1)^K=(-1)^{j_r}$ and $X^{j_r\cdots}_{\pm\mu}$ $(\mu>0)$ with $(-1)^K=\pm 1$.

B. Notion of irreducible tensor multipoles

Let $L(\kappa)$ be a pure Lorentz transformation in the 2-plane (e, u), see Appendix A2. In this subsection of the Appendix, we show that the space $\mathcal{M}^{\otimes r}_c$ can be reduced in a direct sum of two-dimensional subspaces which are invariant under such a transformation. We call "irreducible tensor multipoles" (ITM) tensor fields which have all the properties of the TM as well in the space $\mathcal{L}^2_r[\mathcal{S}^2(e)]$ as in $\mathcal{M}^{\otimes r}_c$ but which belong to these invariant subspaces and which transform according to a single representation $B(n\kappa)$ of $L(\kappa)$. Since these subspaces are two-dimensional, the ITM will be defined by pairs, each pair being a basis of the considered subspace.

Consider the transformation law (56c) of the TM, it involves the tensor product of transformations $L(\kappa)^{\otimes r}$. $L(\kappa)$ is represented by the direct sum of 2×2 matrices, $B(\kappa)\oplus I$, then $L(\kappa)^{\otimes r}$ is represented by the tensor product of matrices $[B(\kappa)\oplus I]^{\otimes r}$ which can be reduced in a direct sum of matrices $B(n\kappa)$ with $0\leq n\leq r$, see Eq. (A14). By definition each pair of ITM, denoted by $X^{i}_{n}{}^J_M(u,\{e\})$ with $i=0, 1$ and $0\leq n\leq r$, transforms according to the representation $B(n\kappa)$,

$$X^{i'}_{n}{}^J_M[L(\kappa)u, L(\kappa)\{e\}] = \sum_i B(n\kappa)^{i'}{}_i X^{i}_{n}{}^J_M(u,\{e\}). \tag{58}$$

Note that the invariant ITM, i.e., those transforming according to $B(0)\equiv I$ can be defined individually.

To keep all the properties of the TM {i.e., orthonormality in $\mathcal{L}^2_r[\mathcal{S}^2(e)]$, orthogonality in $\mathcal{M}^{\otimes r}_c$, irreducibility under rotations and well-defined parity} the ITM must be linear orthonormal combinations of TM with the same J, M and parity. In the following, we show that the 4-vector multipoles $X^j_\mu{}^J_M(u)$ are themselves irreducible and we build the ITM of the second order by using their geometrical properties and more precisely their orientation with respect to the two 4-vectors e and u.

C. Irreducible 4-vector multipoles

The 4-vector multipoles are deduced from the 4-vector harmonics by the orthogonal transformation

$$X^{0\,J}_{0\,M}(u)=Y^{J\,0\,J}_{\;\;\;M}(u),\quad X^{1\,J}_{+\,M}(u)=Y^{J\,1\,J}_{\;\;\;M}(u),$$

$$\begin{pmatrix}X^{1\;\;J}_{0\;\;M}(u)\\ X^{1\;\;J}_{-\;\;M}(u)\end{pmatrix}=\begin{pmatrix}-\sqrt{(J+1)}/\hat{J} & \sqrt{J}/\hat{J}\\ \sqrt{J}/\hat{J} & \sqrt{(J+1)}/\hat{J}\end{pmatrix}\begin{pmatrix}Y^{J+1\;1\;J}_{\;\;\;\;\;\;\;M}(u)\\ Y^{J-1\;1\;J}_{\;\;\;\;\;\;\;M}(u)\end{pmatrix}. \tag{59}$$

Two of these multipoles are proportional to the 4-vectors e or u while the two others are orthogonal to them

$$X^{0\,J}_{0\,M}(u)=Y^J_M(u)e, \tag{60}$$

$$X^{1\,J}_{0\,M}(u)=Y^J_M(u)u, \tag{61}$$

$$X^{1\,J}_{\pm\,M}(u)\cdot e=X^{1\,J}_{\pm\,M}(u)\cdot u=0. \tag{62}$$

This last equation implies that the multipoles $X^1_\pm$ are invariant under the transformation $L(\kappa)$ in the 2-plane (e,u)

$$X^1_\pm[L(\kappa)u,L(\kappa)\{e\}]=X^1_\pm(u,\{e\}). \tag{63}$$

But Eqs. (60) and (61) show that the multipoles X^0_0 and X^1_0 transform according to $B(\kappa)$ and we can write their transformation law

$$X^j_0[L(\kappa)u,L(\kappa)\{e\}]^L_M=\sum_i B(\kappa)^j_i X^i_0(u,\{e\})^L_M \tag{64}$$

with $j=0$ or 1 and summation over $i=0$ and 1. The relations (63) and (64) mean that the $X^{j\;\;J}_{\mu\;\;M}(u)$ are themselves irreducible for $L(\kappa)$.

By using the parity relation (57) we obtain that the multipole X^1_+ is magnetic and the three others X^0_0, X^1_0, and X^1_- are electric. By setting the indices t for timelike (proportional to e), $\mathcal{L}$ for longitudinal (proportional to u), and $\mathcal{T}$ for transverse (perpendicular to e and u), we denote the 4-vector multipoles by

$$\mathcal{E}_{tM}^{\;J}(u)=X^{0\,J}_{0\,M}(u),\quad \mathcal{E}_{\mathcal{L}\;M}^{\;\;J}(u)=X^{1\,J}_{0\,M}(u),$$

$$\mathcal{E}_{\mathcal{T}\;M}^{\;\;J}(u)=X^{1\,J}_{-\,M}(u),\quad \mathcal{M}_{\mathcal{T}\;M}^{\;\;J}(u)=X^{1\,J}_{+\,M}(u). \tag{65}$$

In the rest frame of e, these 4-vector multipoles can be written in the form

$$\mathcal{E}_{tM}^{\;J}=[Y^J_M(\mathbf{u}),0],\quad \mathcal{E}_{\mathcal{L}\;M}^{\;\;J}=[0,Y^J_M(\mathbf{u})\mathbf{u}],$$

$$\mathcal{E}_{\mathcal{T}\;M}^{\;\;J}=\left[0,\frac{r\nabla Y^J_M(\mathbf{u})}{\sqrt{J(J+1)}}\right],\quad \mathcal{M}_{\mathcal{L}\;M}^{\;\;J}=\left[0,\frac{\mathbf{L}Y^J_M(\mathbf{u})}{\sqrt{J(J+1)}}\right], \tag{66}$$

where the space components are the usual three-dimensional vector multipoles.[4]

D. Irreducible tensor multipoles of second order

Consider the second-order TM $X^{jk_1k_2J}_{\mu\;\;\;\;\;M}(u)$. For fixed values of $J\geqslant 2$ and M, we have 16 TM: for $k_1=k_2=j=\mu=0$, the first one is proportional to $e\otimes e$

$$X^{000J}_{0\;\;\;\;\;M}(u)=Y^J_M(u)e\otimes e. \tag{67}$$

For $k_1\neq k_2$, $j=1$, $\mu=-1$, 0, $+1$, there are 6 TM which are tensor products of a 4-vector multipole by e

$$X^{101J}_{\mu\;\;\;\;\;M}(u)=e\otimes X^{1\;\;J}_{\mu\;\;M}(u),\quad X^{110J}_{\mu\;\;\;\;\;M}(u)=X^{1\;\;J}_{\mu\;\;M}(u)\otimes e. \tag{68}$$

For $k_1=k_2=1$; $j=0,1,2$; $\mu=-j,\ldots,+j$; we have 9 TM which correspond to the 9 TM defined in the space $\mathcal{L}^2_2(\mathcal{S}^2)$, see Sec. 3 of I.

Equations (A14) and (A15) show that we have to define 1 pair of ITM transforming according to $B(2\kappa)$, 4 pairs according to $B(\kappa)$ and 6 invariant ITM. We denote them by $Z^{xiJ}_{nM}(u)$ or $Z^{x\;J}_{\eta\;M}(u)$ with the following conventions:

(i) the letter Z is $\mathcal{M}$ or $\mathcal{E}$ for magnetic [parity $(-1)^{J+1}$] or electric [parity $(-1)^J$],

(ii) the upper index x is s or a for symmetric or antisymmetric,

(iii) the lower index $n=1,2$ indicates the representation $B(n\kappa)$ by which the ITM transform,

(iv) the upper indices $i=0,1$ label each ITM in a pair,

(v) for the invariant ITM, the two indices n and i are omitted, while the index η gives their geometrical properties in $\mathcal{M}_c^{\otimes 2}$: $\mathcal{S}$ for scalar (proportional to g), $\mathcal{T}$ for transverse (orthogonal to e and u), $\mathcal{L}$ for longitudinal [with components in the 2-plane (e,u)].

Then the transformation law of the ITM under a transformation $L(\kappa)$ reads

$$L(\kappa)^{\otimes r}Z^{xi\;J}_{\;\;n\;M}=\sum_j B(n\kappa)^i{}_j Z^{xj\;J}_{\;\;n\;M}, \tag{69a}$$

$$L(\kappa)^{\otimes r}Z^{x\;J}_{\eta\;M}=Z^{x\;J}_{\eta\;M}. \tag{69b}$$

To build these ITM, we use the transformation laws of tensors given in Eqs. (A20), the geometrical properties of the VM and their transformation laws. The 2 ITM transforming according to $B(2\kappa)$ are electric, symmetric, and traceless

$$\begin{cases}\mathcal{E}^{s0\,J}_{\;\;2\,M}(u)=\sqrt{2^{-1}}\,Y^J_M(u)[e\otimes e+u\otimes u],\\ \mathcal{E}^{s1\,J}_{\;\;2\,M}(u)=\sqrt{2^{-1}}\,Y^J_M(u)[e\otimes u+u\otimes e].\end{cases} \tag{70}$$

We have already seen that 4 pairs of ITM transform according to $B(\kappa)$. These 8 ITM are also traceless and in each pair both ITM are either symmetric or antisymmetric and they are either magnetic or electric

$$\begin{cases}\mathcal{E}^{x0\,J}_{\;\;1\,M}(u)=\sqrt{2}\,[e\otimes X^{1\,J}_{-\,M}(u)]^x,\\ \mathcal{E}^{x1\,J}_{\;\;1\,M}(u)=\sqrt{2}[u\otimes X^{1\,J}_{-\,M}(u)]^x,\end{cases} \tag{71}$$

$$\begin{cases}\mathcal{M}^{x0\,J}_{\;\;1\,M}(u)=\sqrt{2}[e\otimes X^{1\,J}_{+\,M}(u)]^x,\\ \mathcal{M}^{x1\,J}_{\;\;1\,M}(u)=\sqrt{2}[u\otimes X^{1\,J}_{+\,M}(u)]^x,\end{cases} \tag{72}$$

where the symbol $[\;]^x$ means symmetric or antisymmetric part according to whether x is s or a.

It remains to define the six invariant ITM. The only one with trace, called scalar, is proportional to the tensor g and symmetric

$$\mathcal{E}^{s\;J}_{\mathcal{S}\;M}(u)=\tfrac{1}{2}Y^J_M(u)g. \tag{73}$$

The two longitudinal invariant ITM are built by means of the 4-vectors e and u and of the tensor g. They are traceless and electric, one being symmetric and the other antisymmetric

TABLE I. The orthogonal change of tensor functions from the tensor multipoles to the irreducible tensor multipoles. (The indices J and M and the u dependence are omitted.)

$$\begin{bmatrix} \mathcal{E}^{s0}_{2} \\ \mathcal{E}^{s}_{L} \\ \mathcal{E}^{s}_{S} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \begin{bmatrix} X^{000}_{0} \\ X^{011}_{0} \\ X^{211}_{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{E}^{s1}_{2} \\ \mathcal{E}^{a}_{L} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X^{101}_{0} \\ X^{110}_{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{E}^{s0}_{1} \\ \mathcal{E}^{a0}_{1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X^{101}_{-} \\ X^{110}_{-} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{M}^{s0}_{1} \\ \mathcal{M}^{a0}_{1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X^{101}_{+} \\ X^{110}_{+} \end{bmatrix}$$

$$\mathcal{E}^{s1}_{1} = X^{211}_{+1}, \quad \mathcal{M}^{s1}_{1} = X^{211}_{-1}$$

$$\mathcal{E}^{a1}_{1} = X^{111}_{+1}, \quad \mathcal{M}^{a1}_{1} = X^{111}_{-1}$$

$$\mathcal{E}^{s}_{T} = X^{211}_{+2}, \quad \mathcal{M}^{s}_{T} = X^{211}_{-2}, \quad \mathcal{M}^{a}_{T} = X^{111}_{0}$$

$$\mathcal{E}^{s}_{L}{}^{J}_{M}(u) = Y^{J}_{M}(u)[e\otimes e - u\otimes u - \tfrac{1}{2}g], \tag{74}$$

$$\mathcal{E}^{a}_{L}{}^{J}_{M}(u) = \sqrt{2^{-1}}\, Y^{J}_{M}(u)[e\otimes u - u\otimes e]. \tag{75}$$

The three last ITM are transverse, i.e., orthogonal to the 4-vectors e and u. They are the traceless TM $X^{211}_{\pm 2}$ and X^{111}_{0} themselves; X^{211}_{+2} is electric and symmetric, X^{211}_{-2} and X^{111}_{0} are magnetic, the first one being symmetric and the second one antisymmetric.

Then, using the identities (38)—(47) of Ref. 5 one deduces the change of tensor functions from the TM to the ITM. This change is given in Table I.

For application to the gravitational radiation, Zerilli[6] defined a set of symmetric multipoles on $(\mathcal{M}_c)^{\otimes 2}$ which generalizes his set on $(\mathcal{E}^3_c)^{\otimes 2}$, see Eqs. (48) of I. The relations between the Zerilli's multipoles and our ITM are gathered in Table II. The Zerilli's multipoles have good transformation properties under rotation and form an orthonormal set in the Hilbert space. However, they are not pairwise orthogonal in $(\mathcal{M}_c)^{\otimes r}$ and they do not transform according to an irreducible representation under a pure Lorentz transformation. Furthermore, three multipoles $a^{(0)}_{JM}$, a_{JM}, h_{JM} have nonvanishing trace while only the ITM $\mathcal{E}_S$ has a trace.

5. CONCLUSION

In this paper and in the preceding one (I) we have achieved a study of bases of tensor spherical harmonics and tensor multipoles for Euclidean space and Minkowski space.

The tensor spherical harmonics are a direct generalization of the well known vector spherical harmonics. By noticing that the coefficients of these linear combinations are Clebsch-Gordan coefficients one easily generalizes the concept of multipole basis to arbitrary tensor order.

For Minkowski space the transformation properties of the tensor multipoles under Lorentz transformation lead to the notion of irreducible tensor multipoles which transform according to an irreducible representation of the group of pure Lorentz transformations in a 2-plane. These irreducible tensor multipoles can be used to perform the multipole expansion of vertex functions in arbitrary frames.[5,7] The vertex functions depend on two independent momenta which define the 2-plane of the vertex. The coefficients of the expansion, called form factors, depend on the square of the momentum transfer and on the choice of a timelike reference 4-vector in the 2-plane. The basis of irreducible tensor multipoles yields form factors having very simple transformation laws in changes of reference 4-vector in this 2-plane, i.e., pure Lorentz transformations.

APPENDIX

1. Canonical decomposition of a Lorentz transformation

For a given timelike 4-vector, any Lorentz transformation can be split into the product of a pure Lorentz transformation and a rotation of the little group of this 4-vector.[8,9]

Let e be a timelike unit 4-vector and Λ a Lorentz transformation of the restricted group. The canonical decomposition of Λ is

$$\Lambda = L_\Lambda R_\Lambda, \tag{A1}$$

where L_Λ is the pure Lorentz transformation which maps e on Λe,

$$L_\Lambda = g\cdot - \frac{(\Lambda e + e)\otimes(\Lambda e\cdot + e\cdot)}{e\cdot\Lambda e + 1} + 2\Lambda e\otimes e\cdot, \tag{A2}$$

and where R_Λ is the rotation of the little group of e defined by

$$R_\Lambda = L_\Lambda^{-1}\Lambda = \Lambda - \frac{(\Lambda e + e)\otimes(e\cdot + \Lambda^{-1}e\cdot)}{e\cdot\Lambda e + 1} + 2e\otimes e\cdot. \tag{A3}$$

From the preceding expression one deduces the identities

$$L_{\Lambda^{-1}} = \Lambda^{-1}(L_\Lambda)^{-1}\Lambda, \tag{A4}$$

TABLE II. Relation between the Zerilli's multipoles and the symmetric irreducible tensor multipoles. (The indices J, M, s and the u dependence are omitted.)

$$a^{(0)} = \sqrt{2}^{-1}\,\mathcal{E}^0_2 + \tfrac{1}{2}\mathcal{E}_L + \tfrac{1}{2}\mathcal{E}_S$$

$$a^{(1)} = \mathcal{E}^1_2$$

$$a = \sqrt{2}^{-1}\,\mathcal{E}^0_2 - \tfrac{1}{2}\mathcal{E}_L - \tfrac{1}{2}\mathcal{E}_S$$

$$b^{(0)} = \mathcal{E}^0_1, \quad b = \mathcal{E}^1_1$$

$$c^{(0)} = \mathcal{M}^0_1, \quad c = \mathcal{M}^1_1$$

$$d = \mathcal{M}_T, \quad f = \mathcal{E}_T$$

$$h = -\sqrt{2}^{-1}\,\mathcal{E}_L + \sqrt{2}^{-1}\,\mathcal{E}_S$$

$$R_{\Lambda^{-1}} = (R_\Lambda)^{-1}. \tag{A5}$$

2. Representations of pure Lorentz transformations

Consider the pure Lorentz transformation L_Λ defined in (A2) and u, v, w three spacelike unit 4-vectors such that they form a tetrad with e and that u belongs to the 2-plane $(e, \Lambda e)$. Then the transformation L_Λ can be parametrized by the argument κ defined by

$$\cosh\kappa = e\cdot\Lambda e, \quad \sinh\kappa = u\cdot\Lambda e, \tag{A6}$$

and $L(\kappa)\equiv L_\Lambda$ reads

$$L(\kappa) = [\cosh\kappa(e\otimes e\cdot - u\otimes u\cdot) - \sinh\kappa(e\otimes u\cdot - u\otimes e\cdot)] + [-v\otimes v\cdot - w\otimes w\cdot]. \tag{A7}$$

This is a sum of two operators: the first one acts in the 2-plane (e, u) and the second one is the identity operator in the 2-plane (v, w). In the basis (e, u), the first operator is represented by the 2×2 symmetric matrix

$$B(\kappa) = \begin{pmatrix} \cosh\kappa & \sinh\kappa \\ \sinh\kappa & \cosh\kappa \end{pmatrix}, \tag{A8}$$

while the identity operator is represented, in any basis, by the 2×2 identity matrix $I\equiv B(0)$. The matrix elements of $B(\kappa)$ are denoted by $B(\kappa)^i_j$ where the indices i and j take the values 0 and 1. If we write $e^0\equiv e$ and $e^1\equiv u$, the transformation law of these 4-vectors reads

$$L(\kappa)e^j = \sum_{i=0}^{1} B(\kappa)^j_i e^i. \tag{A9}$$

The matrix elements can be defined analytically by

$$B(\kappa)^i{}_j = \tfrac{1}{2}[e^{+\kappa} + (-1)^{i+j}e^{-\kappa}] \tag{A10}$$

and they satisfy the identity

$$B(-\kappa)^i{}_j = (-1)^{i+j}B(\kappa)^i{}_j. \tag{A11}$$

The hyperbolic trigonometry allows to reduce the tensor product of two B matrices in a sum of two other B matrices. Let m and n be two arbitrary integer numbers, then we have

$$B(m\kappa)\otimes B(n\kappa) \approx B[(m+n)\kappa]\oplus B[(m-n)\kappa], \tag{A12}$$

and the explicit relations between the matrix elements are

$$B(m\kappa)^i{}_j B(n\kappa)^k{}_l = \tfrac{1}{2}\{B[(m+n)\kappa]^{i+k}{}_{j+l} + (-1)^{k+l}B[(m-n)\kappa]^{i+k}{}_{j-l}\}, \tag{A13a}$$

$$B[(m+n)\kappa]^{i+k}{}_{j+l} = B(m\kappa)^i{}_j B(n\kappa)^k{}_l + B(m\kappa)^{\bar{i}}{}_j B(n\kappa)^{\bar{k}}{}_l, \tag{A13b}$$

where the sum on the indices is defined modulus 2 and we use the notation $\bar{i} = 0, 1$ if $i = 1, 0$.

A rth-order tensor transforms according to $L(\kappa)^{\otimes r}$. That leads us to consider the matrix tensor product $[B(\kappa)\oplus I]^{\otimes r}$. This product can be reduced by means of Eq. (A12), one obtains

$$[B(\kappa)\oplus I]^{\otimes r} = \bigoplus_{n=0}^{r} \bigoplus^{\alpha_r(n)} B(n\kappa), \tag{A14}$$

where $\alpha_r(n)$ is the multiplicity of the representation $B(n\kappa)$,

$$\alpha_r(n) = (\tfrac{1}{2})^{\delta_{n0}} C^{2r}_{r-n} = (\tfrac{1}{2})^{\delta_{n0}} \frac{(2r)!}{(r-n)!\,(r+n)!}. \tag{A15}$$

Equation (A14) means that the tensor product $M^{\otimes r}$ can be reduced into the direct sum of 2^{2r-1} two-dimensional invariant subspaces and one verifies

$$\sum_{n=0}^{r} \alpha_r(n) = 2^{2r-1}. \tag{A16}$$

Among the globally invariant subspaces, those which transform according to the representation $B(0\kappa) = I$ are locally invariant. Hence the numbers N_r of invariant tensors under the transformation $L(\kappa)$ is

$$N_r = 2\alpha_r(0) = C^{2r}_r = (2r)!/(r!)^2 \tag{A17}$$

for instance $N_1 = 2$ and $N_2 = 6$.

As an example, consider the four second-order tensors made by the tensor products $t^{ij} = e^i\otimes e^j$ $(i, j = 0, 1)$. Under the transformation $L(\kappa)$ of the tetrad $\{e\}$, they transform according to

$$L(\kappa)^{\otimes 2}t^{ij} = B(\kappa)^i{}_k B(\kappa)^j{}_l t^{kl}. \tag{A18}$$

From the reduction formulas (A13) one easily builds the linear combinations of these tensors which are irreducible, namely,

$$L(\kappa)^{\otimes 2}(t^{ij} + t^{\bar{i}\bar{j}}) = \sum_k B(2\kappa)^j{}_k(t^{ik} + t^{\bar{i}\bar{k}}), \tag{A19a}$$

$$L(\kappa)^{\otimes 2}(t^{ij} - t^{\bar{i}\bar{j}}) = t^{ij} - t^{\bar{i}\bar{j}}. \tag{A19b}$$

If one writes explicitly the matrix elements of $B(\kappa)$ and the tensors t^{ij} as functions of the 4-vectors e and u, one gets

$$L(\kappa)^{\otimes 2}(e\otimes e + u\otimes u) = \cosh 2\kappa(e\otimes e + u\otimes u) + \sinh 2\kappa(e\otimes u + u\otimes e), \tag{A20a}$$

$$L(\kappa)^{\otimes 2}(e\otimes u + u\otimes e) = \sinh 2\kappa(e\otimes e + u\otimes u) + \cosh 2\kappa(e\otimes u + u\otimes e), \tag{A20b}$$

$$L(\kappa)^{\otimes 2}(e\otimes e - u\otimes u) = e\otimes e - u\otimes u, \tag{A20c}$$

$$L(\kappa)^{\otimes 2}(e\otimes u - u\otimes e) = e\otimes u - u\otimes e. \tag{A20d}$$

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Solution of the multigroup transport equation in L^p spaces

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The isotropic multigroup transport equation is solved in L^p, $p>1$, for both half range and full range problems, using resolvent integration techniques. The connection between these techniques and a spectral decomposition of the transport operator is indicated.

I. INTRODUCTION

Since Larsen and Habetler introduced a resolvent integral technique to solve the one-dimensional one-speed isotropic linear transport equation,[1] this method has been extended to study a variety of problems. In particular, Bowden, Sancaktar, and Zweifel have obtained a solution of the multigroup problem in Hilbert space,[2,3] and Larsen, Sancaktar, and Zweifel have extended the one-group results to L^p spaces.[4]

The purpose of this note is to indicate how these ideas can be combined to obtain a solution of the isotropic multigroup equation in L^p, $p>1$, for both half range and full range problems. The analysis demonstrates that the problem is reduced largely to estimating some relevant operator norms in the solution space $L^p(\mathbf{I})$ and in the spectral decomposition space $L^p(N,\sigma)$. These estimates are carried out in Lemmas 2–8, and lead to the representation theorem, Theorem 9.

We may point out that the elegant spectral analysis of Hangelbroek[5] to this problem does not appear to afford an alternate approach, except for the two-group, since, with the exception noted, it is not possible to symmetrize the production matrix C and simultaneously maintain the scattering matrix Σ diagonal. In Theorem 10 and the discussion preceding it, we indicate the connection between the von Neumann spectral theory utilized by Hangelbroek and the resolution of the identity obtained from the resolvent integrations.

Finally, Theorem 11 deals with the application of these results to half space theory.

II. THE MULTIGROUP PROBLEM

Let us define the Banach space $X_p(\mathbf{I})$ to be the space of (equivalence classes of) Lebesque measurable vector valued functions $\mathbf{f}$ from the real interval $\mathbf{I}=[-1,1]$ to $\mathbb{C}^n$ with norm

$$\|\mathbf{f}\|_p=\left\{\sum_{i=1}^n \int_{-1}^1 d\mu\,|\mu \mathbf{f}_i(\mu)|^p\right\}^{1/p}.$$

We distinguish the subspace of constant vectors X_p^c to be functions $\mathbf{f}\in X_p$ such that, for each i, $1\le i\le n$, $\mathbf{f}_i(u)$ is independent of μ. In particular let

$$(\mathbf{e}_{(j)})_i(\mu)=\delta_{ij}$$

for $1\le j\le n$. Then $\mathbf{f}\in X_p^c$ precisely if there are constants a_j, $1\le j\le n$, such that

$$\mathbf{f}=\sum_{j=1}^n a_j \mathbf{e}_{(j)}.$$

On X_p^c an inner product may be defined:

$$[\mathbf{f},\mathbf{g}]=\sum_{i=1}^n \mathbf{f}_i\bar{\mathbf{g}}_i,\quad \mathbf{f},\mathbf{g}\in X_p^c$$

By a solution of the (full range) multigroup transport equation is meant a differentiable function $\psi:\mathbb{R}\to X_p(\mathbf{I})$ satisfying

$$\mu\frac{\partial}{\partial x}\psi(x)=-\Sigma\psi(x)+C\int_{-1}^1 d\mu'\,\psi(x,\mu')+\mathbf{q}(x) \tag{1}$$

where Σ is an $n\times n$ diagonal matrix with positive entries, C is any $n\times n$ matrix with nonnegative entries, μ indicates multiplication by the independent variable in $X_p(\mathbf{I})$,

$$(\mu\mathbf{f})_i(\mu)=\mu\mathbf{f}_i(\mu),$$

and $\mathbf{q}$ is the inhomogeneous source term, which we assume to be a Hölder continuous function $\mathbf{q}:\mathbb{R}\to X_p(\mathbf{I})$. We have written $\psi(x,\mu')$ for $\psi(x)$ evaluated at μ', and in the remainder, we will omit the x dependence altogether, writing $\psi(\mu')$. The solution of Eq. (1) is also understood to satisfy specified boundary conditions, typically $\|\psi(x)\|_p\to 0$ as $x\to\pm\infty$.

The transport operator, or more correctly, the reduced transport operator, K, is the bounded linear transformation on $X_p(\mathbf{I})$,

$$Kf=\Sigma^{-1}\mu f+\Sigma^{-1}C(\Sigma-2C)^{-1}\int_{-1}^1 d\mu'\,\mu' f(\mu')$$

and its (unbounded) inverse is

$$(K^{-1}\mathbf{f})(\mu)=(1/\mu)\Sigma\mathbf{f}(\mu)-(1/\mu)C\int_{-1}^1 d\mu'\,\mathbf{f}(\mu').$$

We may assume, without loss of generality, that $\Sigma_{ii}\ge 1$, $1\le i\le n$, and $\|\Sigma^{-1}\|=1$, and we shall do so. It is also necessary to make the noncriticality assumption[10]

$$\det(\Sigma-2C)\ne 0.$$

The spectrum N of K as an operator on $X_p(\mathbf{I})$ consists of the interval $\mathbf{I}$, which is continuous spectrum, and of point spectrum $\sigma_p(K)$.

In Ref. 2, the Case transform $\mathbf{F}:\mathbf{f}\to\mathbf{A}$ is derived for $\mathbf{f}$ Hölder continuous, where

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$$F(\mathbf{f})(\nu)=\begin{cases}\dfrac{1}{2\pi i\nu}\left\{\Lambda^{-1}(\nu)\displaystyle\int_{-1}^{1}ds\, s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\right\}^{+} \\ -\left\{\Lambda^{-1}(\nu)\displaystyle\int_{-1}^{1}ds\, s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\right\}^{-}, \quad \nu\in\mathbf{I}, \qquad (2a) \\ \dfrac{1}{\Omega'(\nu)}\left[\displaystyle\int_{-1}^{1}ds\; s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s),\boldsymbol{\alpha}\right]\boldsymbol{\beta}_\nu,\ \nu\in\sigma_p(K). \end{cases} \tag{2b}$$

Here the dispersion matrix $\Lambda(z)=B+T(z)$ and its determinant $\Omega(z)=\det\Lambda(z)$ are given by

$$B=(\Sigma-2C)C^{-1}\Sigma,\quad T(z)=-\int_{-1}^{1}ds\; s(zI-s\Sigma^{-1})^{-1}, \tag{3}$$

and the superscripts $\pm$ indicate boundary values obtained as z converges to Rez from above (below) the real axis. The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_\nu$ are defined as follows. Since $\nu\in\sigma_p(K)$ if and only if $\Omega(\nu)=0$, let us take $\boldsymbol{\beta}_\nu$ to satisfy

$$\Lambda(\nu)\boldsymbol{\beta}_\nu=0 \tag{4}$$

for each $\nu\in\sigma_p(K)$. Then for any $\nu\in\sigma_p(K)$, we may define $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha}=\Lambda_c(\nu)\boldsymbol{\beta}_\nu$$

where $(\Lambda_c)_{ij}=\mathrm{cof}_{ij}\Lambda$ differs from the notation in Ref. 2 by a transpose.

In the above it has been assumed that $\Omega^{\pm}(\nu)$ does not vanish on the interval $\mathbf{I}=[-1,1]$ and that $\nu\in\sigma_p(K)$ has multiplicity one. We shall also assume that

$$\Gamma(\nu)=\begin{cases}\tfrac12(\Lambda(\nu)^{+}+\Lambda(\nu)^{-}), & \nu\in\mathbf{I},\\ 1, & \nu\in\sigma_p(K)\end{cases} \tag{5}$$

does not vanish on $\mathbf{I}$, although we do believe that all of these restrictions could be removed without difficulty (see, for example, the treatment of a similar problem in Ref. 6.)

The importance of the Case transform lies in the completeness theorem and in its "spectral" behavior under K. Namely, if

$$\Phi(\nu,\mu)=\quad ⨍\, \nu(\nu I-\mu\Sigma^{-1})^{-1}+\Delta(\Sigma\nu-\mu)\Sigma^{-1}\Gamma(\nu),\quad \nu\in\mathbf{I}, \tag{6a}$$

$$\Phi(\nu,\mu)=\quad (\nu I-\mu\Sigma^{-1})^{-1},\qquad \nu\in\sigma_p(K), \tag{6b}$$

where $⨍$ indicates a principal value integral is to be taken and

$$\Delta(\Sigma\nu-\mu)_{jk}=\delta_{jk}\delta(\sigma_j\nu-\mu),$$

then

$$\mathbf{f}=\int_N\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu) \tag{7}$$

for $\mathbf{A}=F(\mathbf{f})$ defined in Eqs. (2), and $\sigma(\nu)$ Lebesque measure on $\mathbf{I}$, $\sigma(\nu)=1$ for $\nu\in\sigma_p(K)$. Moreover,

$$K\mathbf{f}=\int_N\nu\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu).$$

Thus if we define F': $\mathbf{A}\to\mathbf{f}$ by

$$F'(\mathbf{A})(\mu)=\int_N\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu)$$

for $\mathbf{A}$ Hölder continuous, then Ref. 2 proves the following theorem.

Theorem 1: On Hölder continuous functions in $X_p(\mathbf{I})$,

$F'F=I$ and $F'\nu F=K$.

III. OPERATOR BOUNDS

Equation (2a) makes sense pointwise if f is Hölder continuous; in order to extend F to all of $X_p(\mathbf{I})$, we introduce the Banach space $X_p(N)$, where $\mathbf{A}\in X_p(N)$ if $\mathbf{A}$ is Lebesque measurable on $I\subset N$, $\mathbf{A}$ is proportional to $\boldsymbol{\beta}_\nu$ at each $\nu\in\sigma_p(K)$, and

$$\|\mathbf{A}\|_{p,\Gamma}\equiv\left\{\sum_{i=1}^{n}\int_N|\nu\Gamma(\nu)\mathbf{A}_i(\nu)|^p\,d\nu\right\}^{1/p}<\infty.$$

In other words,

$$\|\mathbf{A}\|_{p,\Gamma}=\left\{\sum_{\nu\in\sigma_p(K)}\sum_{i=1}^{n}|\mathbf{A}_i(\nu)|^p+\|\Gamma\mathbf{A}\|_p\right\}^{1/p}.$$

For a proper extension to all of $X_p(\mathbf{I})$ then, it is sufficient to prove:

(i) F:$H_p(\mathbf{I})\to X_p(N)$ is a bounded, densely defined operator, where

$$H_p(\mathbf{I})=\{f\in X_p(\mathbf{I})f \text{ Hölder continuous on } \mathbf{I}\}; \tag{8a}$$

(ii) F':$H_p(N)\to X_p(\mathbf{I})$ is bounded, where

$$H_p(N)=\{A\in X_p(N)\,|\,f \text{ Hölder continuous on } \mathbf{I}\} \tag{8b}$$

(iii) Ran$F$ is dense in $X_p(N)$.

As there has been, in our opinion, some continuing confusion in the literature over these rather simple observations, we reiterate the following. In the transformed space $X_p(N)$, the transport operator K acts simply as a multiplication operator. Hence transport problems can be related to problems involving the much simpler, and necessarily normal, multiplication operator. However, unless Ran$F$ is demonstrated to be dense in $X_p(N)$, there is no assurance that the solution of a transport problem solved in $X_p(N)$ will be the image under F of a vector in $X_p(\mathbf{I})$. This is, of course, equally true for the one group. If we consider, for example, the uniform slab problem, where the function A is given in Ref. 7 implicitly as the solution of a Fredholm integral equation, then unless A is known to be contained in Ran$F$, it cannot be assumed that $F'A=\phi$ satisfies $F\phi=A$, and hence that it is the desired solution of the slab problem. Note also that the boundedness of F and its invertibility on a dense set is not sufficient to deduce the invertibility of F on $X_p(\mathbf{I})$, unless it has been established that F' is bounded.

The analysis of Ref. 2 hinges on the following theorem concerning Hilbert transforms, which we quote in a form useful for our purposes.[8]

Lemma 2: Let $\mathbf{f}\in X_p(\mathbf{I})$. Then the formula

$$\mathbf{g}(\mu)=⨍_{-1}^{1}s(\mu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\,ds$$

defines almost everywhere a function $\mathbf{g}$ also belonging to $X_p(\mathbf{I})$, and for a constant M_p depending only upon p and $\|\Sigma^{-1}\|$,

$\|\mathbf{g}\|_p \leq M_p \|\mathbf{f}\|_p$.

Before proceeding to study F and F', we collect some important properties of the dispersion matrix.

Lemma 3:

(i) On $X_p(\mathbf{I})$,

$$\Lambda(\nu)^+ - \Lambda(\nu)^- = -2\pi i \nu \Sigma^2 \Delta_\Sigma(\nu), \tag{9a}$$

where $\Delta_\Sigma(\nu)_{ij} = 1$ if $i = j$ and $|\nu| \leq \sigma_i$, zero other-otherwise.

(ii) $\Gamma(\nu)$ is continuously differentiable on $\mathbf{I}/\mathbf{T}$, where $\mathbf{T} = \{\pm \sigma_i^{-1}\}_{i=1}^n$, and $\Gamma^{-1}(\nu)$ defines a bounded operator Γ^{-1} on $X_p(\mathbf{I})$.

(iii) On $X_p(\mathbf{I})$,

$$\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^- = 2\pi i \Lambda^{-1}(\nu)^+ \Sigma^2 \Delta_\Sigma(\nu) \Lambda^{-1}(\nu)^- \tag{9b}$$

is bounded, and on its range, Γ is bounded.

(iv) On $X_p(\mathbf{I})$,

$$\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^- = -2\Lambda^{-1}(\nu)^+ \Gamma(\nu) \Lambda^{-1}(\nu)^- \tag{9c}$$

is bounded, and on its range, Γ is bounded.

Proof: Let us consider (iii)—(iv) first. If $T: \mathbf{I} \to \mathbb{C}^n$ is a continuous map, then $\hat{T}: X_p(\mathbf{I}) \to X_n(\mathbf{I})$ given by

$$(\hat{T}f)(\mu) = T(\mu)\mathbf{f}(\mu)$$

is bounded. Therefore, the problem reduces to studying $\Lambda(\nu)$ on X_p^c for fixed ν in a neighborhood of the "endpoints" $\mathbf{T}$.

Let $b = +\sigma_i^{-1}$ or $-\sigma_i^{-1}$ be such an endpoint, and let $\lim_{\nu \to b\pm}$ indicate a limit taken along $\nu = b \pm i\epsilon$ with $\epsilon \to 0^+$. Suppose $M_b^\pm = N_b \cap \mathbb{C}_\pm$ with N_b a neighborhood of b such that $\overline{N_b}$ contains no other endpoints, and $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z \geq 0\}$. Since $\lim_{\nu \to b\pm} \|\Lambda(\nu) e_i\| = \infty$, we claim that

$$\lim_{\nu \to b\pm} \Lambda^{-1}(\nu) e_i = 0.$$

For,

$$\Lambda(\nu) e_i = \boldsymbol{\alpha}_i(\nu) e_i + B e_i$$

$$= (\boldsymbol{\alpha}_i(\lambda) + \tfrac{1}{2}[Be_i, e_i]) e_i + (I - P_i) B e_i$$

where $P_i: \psi \to \frac{1}{2}[\psi, e_i] e_i$ is the projection onto e_i and $\alpha_i(\nu) \to \pm\infty$, and therefore, since $\Lambda^{-1}(\nu)(I - P_i)$ is a continuous function of ν for $\nu \in M_b^\pm$, we see that

$$\Lambda^{-1}(\nu) e_i = (\boldsymbol{\alpha}_i(\lambda) + \tfrac{1}{2}[Be_i, e_i])^{-1}$$

$$\times (e_i - \Lambda^{-1}(\nu)(I - P_i) B e_i) \to 0.$$

In other words,

$$\operatorname{Sp}\{e_i\} \subset \operatorname{Ker} \Lambda^{-1}(b)^\pm. \tag{10}$$

$\Lambda(\nu)^\pm$ is invertible and, as we have noted, a continuous function of $\nu \in M_b^\pm$ on the subspace $(I - P_i) X_p^c$. Since X_p^c is finite dimensional, $X_p^c = \operatorname{Ran} \Lambda(b) + \operatorname{Sp}\{e_i\}$, and this along with Eq. (10) enables us to conclude that $\Lambda^{-1}(b) = \lim_{\nu \to b\pm} \Lambda^{-1}(\nu)$ on X_p^c. Then, Eq. (10) and the continuity and boundedness of $\Lambda(\nu)^+ + \Lambda(\nu)^-$ on $(I - P_i) X_p$ for $\nu \in M_b^\pm$ gives us directly that

$$(\Lambda(\nu)^+ + \Lambda(\nu)^-) \Lambda^{-1}(\nu)^+ (\Lambda(\nu)^+ \pm (\Lambda(\nu)^-) \Lambda^{-1}(\nu)^-$$

is also a bounded operator on X_p.

Equation (9a) results immediately from the Plemelj formulas applied componentwise to Eq. (3). Since $\Lambda(\nu)^\pm$ is analytic off the real axis and continuous on $[-1, 1] \setminus \mathbf{T}$, the integration along $[-1, 1]$ between neighboring endpoints $b_1, b_2 \in \mathbf{T}$ may be replaced by integration along the complex contours $\Gamma(\theta) = \frac{1}{2}(b_1 + b_2 + (b_1 - b_2) e^{i\theta})$, $\theta \in [0, \pm\pi]$. From this the continuity properties of Γ may be deduced. Finally, we note that the analysis of $\Lambda^{-1}(\nu)$ may also be applied to $\Gamma(\nu) = \Lambda(\nu)^+ + \Lambda(\nu)^-$ to obtain the existence of $\Gamma(\nu)$.

Corollary 4: $H_p(N)$ is dense in $X_p(N)$.

Lemma 5: $F \mid H_p(\mathbf{I})$ is bounded.

Proof: Using the Plemelj formulas for $\nu \in \mathbf{I}$,

$$\nu F(f)(\nu) = \frac{1}{2\pi i} (\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^-) \oint_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1} \mathbf{f}(s)$$
$$+ \tfrac{1}{2}(\Lambda^{-1}(\nu)^+ + \Lambda^{-1}(\nu)^-) \Sigma^2 \nu (V_\Sigma \mathbf{f})(\nu), \tag{11}$$

where V_Σ is defined by

$$(V_\Sigma \mathbf{f})_i(\mu) = \begin{cases} \mathbf{f}_i(\mu\sigma_i), & |\mu| \leq 1/\sigma_i, \\ 0, & |\mu| > 1/\sigma_i. \end{cases}$$

By applying Γ to both sides of the equation, and integrating pth powers of each term over $\mathbf{I}$, $\|F(\mathbf{f})\|_{p,\Gamma}$ may be estimated by a sum of norms. Thus, the norm obtained from the first term on the right-hand side of the equation is bounded by $\|\Gamma(\Lambda^{-1})^+ \Sigma^2 \Delta_\Sigma (\Lambda^{-1})^-\| M_p \|\mathbf{f}\|_p$, and the second by $\|\Sigma\|^{1-1/p} \|\Gamma(\Lambda^{-1})^+ \Gamma(\Lambda^{-1})^-\| \, \|\mathbf{f}\|_p$ since $\|\Delta_\Sigma\| = 1$ and $\|V_\Sigma\| = \|\Sigma^{-1-1/p}\|$. Then the contribution to $\|F(\mathbf{f})\|_{p,\Gamma}$ of the continuous spectrum is

$$(\|F(\mathbf{f})\|_{p,\Gamma})_{\sigma_c} \leq \{\|\Gamma(\Lambda^{-1})^+ \Sigma^2 \Delta_\Sigma (\Lambda^{-1})^-\| M_p$$
$$+ \|\Sigma\|^{1-1/p} \|\Gamma(\Lambda^{-1})^+ \Gamma(\Lambda^{-1})^-\|\} \|\mathbf{f}\|_p.$$

If $\nu \in \sigma_p(K)$, then for q satisfying $1/p + 1/q = 1$, the Hölder inequality gives

$$\left| \left[\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1} \mathbf{f}(s), \boldsymbol{\alpha} \right] \right| \leq \|\mathbf{f}\|_p$$

$$\times \left\{ \sum_i \int_{-1}^{1} ds \left| \frac{\alpha_i}{\nu - s\sigma_{ii} - 1} \right|^q \right\}^{1/q} \leq 2^{1/q} \sup_{\nu \in \sigma_p(K)} \frac{1}{d(\nu)} \|\boldsymbol{\alpha}\|_{[q]},$$

where

$$d(\nu) = \inf_{s \in \mathbf{I}} |\nu - s|$$

and $\|\ \|_{[q]}$ is the q-norm on $\mathbb{C}^n$,

$$\|\boldsymbol{\xi}\|_{[q]} = \left\{ \sum_{i=1}^{n} |\boldsymbol{\xi}_i|^q \right\}^{1/q}, \quad \boldsymbol{\xi} \in \mathbb{C}^n.$$

Thus, the contribution to $\|F(\mathbf{f})\|_{p,\Gamma}$ of the point spectrum is

$$(\|F(f)\|_{p,\Gamma})_{\sigma_p} \le \sup_{\nu' \in \sigma_p(K)} \frac{1}{d(\nu')} 2^{1-1/p} \|\boldsymbol{\alpha}\|_{[q]}$$

$$\times \Big(\sum_{\nu \in \sigma_p(K)} \|\boldsymbol{\beta}_\nu\|^p_{[q]} \Big)^{1/p} \|\mathbf{f}\|_p.$$

This completes the proof.

Lemma 6: $F' | H_p(N)$ is bounded.

Proof: We have from Eqs. (6) and (7):

$$\mu(F^{-1}\mathbf{A})(\mu) = \mu \oint_{-1}^{1} \nu(\nu I - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu + \mu\Sigma^{-2}(V_{\Sigma^{-1}}\Gamma\mathbf{A})(\mu)$$

$$+ \sum_{\nu \in \sigma_p(K)} \mu(\nu I - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu). \qquad (12)$$

The norm of the first term on the right-hand side is bounded by

$$M_p \|\mathbf{A}\|_p \le M_p \|\Gamma^{-1}\| \; \|\Gamma\mathbf{A}\|_p = M_p \|\Gamma^{-1}\| \; \|A\|_{p,\Gamma}$$

and the second term by

$$\|\Sigma^{1/p-1}\| \; \|\mathbf{A}\|_{p,\Gamma}.$$

The third term may be estimated by

$$\left\{ \sum_{\nu \in \sigma_p(K)} \sum_{i=1}^{n} \int_{-1}^{1} d\mu \left| \frac{\mu}{\nu^2 - \mu\nu\sigma_i^{-1}} \right|^p |\nu \mathbf{A}_i(\nu)|^p \right\}^{1/p}$$

$$\le \left(\frac{2}{p+1} \right)^{1/p} \sup_{\nu \in \sigma_p(K)} \frac{1}{\nu\, d(\nu)} \|\mathbf{A}\|_{p,\Gamma}.$$

Lemma 7: Let $J_p(N) = \{A \in H_p(N) | \Gamma A \in H_p(N)\}$. Then $F: H_p(\mathbf{I}) \to H_p(N)$ and $F': J_p(N) \to H_p(\mathbf{I})$.

Proof: Since the Cauchy integral of a Hölder continuous function is Hölder continuous on the interior of a Liapunov contour,[9] the only potential difficulty is at the boundary points $\pm 1 \backslash \sigma_i$. A typical term in the expression for $F(f)$ is

$$\sum_{k=1}^{n} \Lambda^{-1}(\nu)^{\pm} \int_{-1}^{1} ds (\nu I - s\Sigma_{kk}^{-1})^{-1} f_k(s)\mathbf{e}_k,$$

which is explicitly Hölder continous at $\nu = 1\backslash\sigma_i$ except possibly for $k = i$. From Eq. (10), however,

$$\Lambda^{-1}(1/\sigma_i)^{\pm}\mathbf{e}_i = 0.$$

The second part of the lemma may be proved immediately from Eq. (12).

Lemma 8: Ran$F$ is dense in $H_p(N)$.

Proof: We first wish to reduce the transformation F between Banach spaces $X_p(N)$ and $X_p(\mathbf{I})$ by subspaces corresponding to the eigenspace of K and appropriate topological supplements. Thus, let us define the bounded linear forms ρ_ν on $H_p(\mathbf{I})$ by

$$\rho_\nu(f) = \Big[\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1}\mathbf{f}(s), \boldsymbol{\alpha} \Big]$$

for $\nu \in \sigma_p(K)$, and let $H_p(\mathbf{I})$ be the submanifold

$$H_p(\mathbf{I}) = \{ f \in H_p(\mathbf{I}) | \rho_\nu(f) = 0, \quad \nu \in \sigma_p(K) \}.$$

If $\boldsymbol{\gamma}_\nu$ is an eigenvector of K with eigenvalue ν, then

$$\rho_\nu(\boldsymbol{\gamma}_{\nu'}) = \delta_{\nu\nu'}\rho_\nu(\alpha_\nu).$$

Since we may take

$$\boldsymbol{\gamma}_\nu(\mu) = (\nu I - \Sigma^{-1}\mu)^{-1}\boldsymbol{\beta}_\nu,$$

whence, applying the identity

$$(\nu I - s\Sigma^{-1})^{-1}(\nu' I - s\Sigma^{-1})^{-1} = [1/(\nu' - \nu)](\nu - s\Sigma^{-1})^{-1} \; (\nu' - s\Sigma^{-1})^{-1} \qquad (13)$$

for $\nu \ne \nu'$ and Eq. (3), the form on $\boldsymbol{\gamma}_{\nu'}$ becomes

$$\rho_\nu(\boldsymbol{\gamma}_{\nu'}) = \int_{-1}^{1} ds \frac{s}{\nu' - \nu} [\{(\nu I - s\Sigma^{-1})^{-1} - B + \Lambda(\nu')\}\boldsymbol{\beta}_{\nu'}, \boldsymbol{\alpha}]$$

$$= -\frac{1}{\nu' - \nu}[\Lambda(\nu)\boldsymbol{\beta}_{\nu'}, \boldsymbol{\alpha}] = -\frac{1}{\nu' - \nu} \sum_i (\Lambda(\nu)\Lambda_c^t(\nu))_{ii} = 0. \qquad (14)$$

We have used the fact that $\Lambda\Lambda_c^t = 0$.

If the projection P is defined on $X_p(\mathbf{I})$ by

$$P\mathbf{f} = \sum_{\nu \in \sigma_p(K)} (\rho_\nu(\mathbf{f})/\rho_\nu(\boldsymbol{\gamma}_\nu))\boldsymbol{\gamma}_\nu$$

and if P denotes the projection onto the subspace

$$X_p(N)_0 = \{ \mathbf{A} \in X_p(N) | \mathbf{A}(\nu) = 0, \quad \nu \in \mathbf{I} \}$$

of $X_p(N)$ along the subspace

$$X_p(N)_1 = \{ \mathbf{A} \in X_p(N) | \mathbf{A}(\nu) = 0, \quad \nu \in \sigma_p(K) \}$$

then we assert that

$$(I - P)F' = F'(I - \hat{P}).$$

To demonstrate this, we must compute the integrals

$$\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1} \oint_{-1}^{1} dt\, (tI - s\Sigma^{-1})^{-1}\mathbf{A}(t)\, dt +$$

$$+ \int_{-\|\Sigma\|}^{\|\Sigma\|} ds\, s(\nu I - s\Sigma^{-1})^{-1}\Sigma^{-2}V_{\Sigma^{-1}}\mathbf{A}(s)$$

for $\nu \in \sigma_p(K)$ and $\mathbf{A} \in X_p(N)_1 \cap H_p(N)$. With the identity (13) and some rearrangements, these become

$$\int_{-1}^{1} dt (t - \nu)^{-1}(-\Sigma^{-1}\Lambda(\nu) + \Sigma^{-1}\Gamma(t))\mathbf{A}(t)$$

$$- \int_{-1}^{1} dt\, t(t - \nu)^{-1}\Sigma^{-1}\Gamma(t)\mathbf{A}(t).$$

Hence, for $\mathbf{f} \in X_p(N)_1$,

$$\rho_\nu(F'\mathbf{f}) = \int_{-1}^{1} dt (t - \nu)^{-1}[\Sigma^{-1}\Lambda(\nu)\mathbf{A}(t), \boldsymbol{\alpha}],$$

which vanishes by the same reasoning as for Eq. (14).

Now we are prepared to determine Ran$F$. Since $F': \boldsymbol{\beta}_\nu \to PX_p(\mathbf{I})$, $\nu \in \sigma_p(k)$, and $F': X_p(N)_1 \to (I - P)X_p(\mathbf{I})$, it is sufficient to prove F' is one–one on $X_p(N)_1$. Thus, let us suppose $F'(\mathbf{A}) = 0$, $\mathbf{A} \in X_p(N)_1 \cap H_p(N)$, and define

$$(V_{\Sigma^{-1}}N)(z) = \int_{-1}^{1} \nu(\nu I - z\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu.$$

Then the Plemelj formulas give

$$\tfrac{1}{2}V_{\Sigma^{-1}}(\mathbf{N}^+ + \mathbf{N}^-)(z) = \oint_{-1}^{1} \nu(\nu - z\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu$$

and

$$\frac{1}{2\pi i\mu}(\mathbf{N}^+ - \mathbf{N}^-)(\mu) = \mathbf{A}(\mu), \quad \mu \in \mathbf{I}.$$

Substituting these expressions into

$$\oint_{-1}^{1} \nu(\nu - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu)\,d\nu + \Sigma^{-2}(V_{\Sigma^{-1}}\Gamma\mathbf{A})(\mu) = 0,$$

we obtain

$$\pi i \Sigma^2 V_{\Sigma^{-1}}\mu(\mathbf{N}^+ + \mathbf{N}^-) + V_{\Sigma^{-1}}\Gamma(\mathbf{N}^+ - \mathbf{N}^-) = 0,$$

which, with Eq. (9a) and appropriate cancellations, becomes

$$V_{\Sigma^{-1}}(\Lambda^+\mathbf{N}^+ - \Lambda^-\mathbf{N}^-) = 0.$$

Hence, by Liouville's Theorem applied to $\mathbf{J}(z) = \Lambda(z)\mathbf{N}(z)$, we conclude that $\mathbf{A}(\nu) \equiv 0$.

IV. SPECTRAL THEOREM

Lemmas 5, 6, and 8, along with the results of Theorem 1, have as an immediate consequence the following theorem.

Theorem 9: The transformation $F: X_p(\mathbf{I}) \to X_p(N)$ is an invertible bounded linear transformation, and $F^{-1} = F'$. Moreover,

$$FK = \nu F$$

is valid on $X_p(\mathbf{I})$.

We emphasize that Theorem 9, by diagonalizing the bounded operator K, provides effectively a spectral representation of K. This is most transparent in Hilbert space language ($p = 2$), where a new inner product may be introduced on $X_2(\mathbf{I})$,

$$\{f, g\} = (Ff, Fg)_{2,\Gamma},$$

Here, $(\cdot,\cdot)_{2,\Gamma}$ indicates the inner product on $X_2(N)$ derived from the norm $\|\;\|_{2,\Gamma}$. Then if $N \subset \mathbb{R}$,

$$\{Kf, g\} = (FKf, Fg)_{p,\Gamma} = (\nu Ff, Fg)_{p,\Gamma} = (Ff, \nu Fg)_{p,\Gamma} = \{f, Kg\}$$

whence K is self-adjoint, and a similar calculation shows K is normal for $N \subset \mathbb{C}$. Furthermore,

$$FK^n = \nu FK^{n-1} = \cdots = \nu^n F,$$

so, since N is necessarily compact, the map

$$\kappa: K^n \to \nu^n$$

extends to the Gelfand transformation from the C^* algebra generated by K and K^* to the algebra of continuous functions on N with uniform norm. (Actually, by Mergelyan's Theorem, C^* algebra is generated by K alone, even when K is not self-adjoint.[11])

These remarks can equally well be expressed in terms of a spectral resolution for K. Recalling that the Dunford integral was used to obtain

$$\mathbf{f} = \frac{1}{2\pi i}\int_\Gamma dz\,R(z,K)\mathbf{f} = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-1}^{1} d\mu\{R(\mu - i\epsilon, K) - R(\mu + i\epsilon, K)\}\mathbf{f} + \frac{1}{2\pi i}\sum_{\nu\in\sigma_p(K)}\int_{\Gamma_\nu} dz\,R(z,K)\mathbf{f}$$
$$= \int_N d\sigma(\nu)\Phi(\mu,\nu)F(\mathbf{f}),$$

one expects that

$$(E([-1,\omega])\mathbf{f})(\mu) = \int_{-1}^{\omega} d\nu\,\Phi(\mu,\nu)F(\mathbf{f}), \quad \omega \in I, \tag{15a}$$

$$(E(\nu)\mathbf{f})(\mu) = \Phi(\mu,\nu)F(\mathbf{f})(\nu), \quad \nu \in \sigma_p(K), \tag{15b}$$

defines a resolution of the idenity for the normal operator K. This is indeed the case, the essential feature being the fact that in X_2 K is similar to the sum of a self-adjoint operator and a *finite dimensional* normal operator on the span of $\{\gamma_\nu \mid \nu \in \mathbb{C}\backslash\mathbb{R}\}$.

To see this, we recall that the spectral projections can be obtained by the formula[12]

$$E((a,b)) = \lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b+\delta}(R(\mu - \epsilon i, T) - R(\mu + \epsilon i, T))\,d\mu \tag{16}$$

in the strong operator topology, for T any bounded self-adjoint operator on a Hilbert space. It is not difficult to see that this formula extends to operators T which are similar to self-adjoint operators. Further, for any closed operator T, if N_1 is a subset of the spectrum $\sigma(T)$, and Γ is a rectifiable, simple closed curve containing N_1 in its interior and $\sigma(T)\backslash N_1$ in its exterior, then

$$E(N_1) = \frac{1}{2\pi i}\int_\Gamma R(z,T)\,dz \tag{17}$$

is the spectral projection corresponding to N_1. Thus the first of these formulas gives

$$(E([-1,\omega])\mathbf{f})(\mu) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\left(\int_{-1}^{\omega} d\mu\,R(\mu - i\epsilon, K)\mathbf{f} + \int_{\omega}^{-1} d\mu\,R(\mu + i\epsilon, K)\mathbf{f}\right)$$

which reduces to Eq. (15a) by precisely the same steps leading to Eq. (7), and the second formula gives Eq. (15)

This analysis—in particular Eq. (16)—is valid in $X_2(\mathbf{I})$. However, it may be extended to $X_p(\mathbf{I})$ by observing that $\mathbf{M} \equiv X_2(\mathbf{I}) \cap X_p(\mathbf{I})$ is dense in $X_p(\mathbf{I})$ for all $p > 1$. Then the boundedness of the projections E in p-norm follows from the analysis of Lemma 6, as is evident from Eqs. (15), and the algebraic properties of the spectral resolution are a consequence of the boundedness of the projections and the density of $\mathbf{M}$. In more detail, since

$$E([-1,\omega])E([-1,\omega']) = E([-1,\omega'])E([-1,\omega]) = E([-1,\omega])$$

on $\mathbf{M}$ for $\omega' \geq \omega$, and the projections are bounded operators, we immediately obtain this nondecreasing property on $\overline{\mathbf{M}} = X_p(\mathbf{I})$. Likewise, the validity of

$$KE = EK$$

on $\mathbf{M}$ and the boundedness of K on $X_p(\mathbf{I})$ extends the equality to all of $X_p(\mathbf{I})$. The identity

$$E([-1,1]) + \sum_{\nu\in\sigma_p(K)} E(\nu) = I$$

also results from these density arguments, or alternatively, directly from Eqs. (15). Finally, the extension of (strong) right continuity

$$\lim E([-1,\lambda+0]) = E([-1,\lambda])$$

to $X_p(\mathbf{I})$ may be seen easily by using the uniform bound on the projections

$\|E\| \leq 1.$

We state these results in a theorem.

Theorem 10: The spectral decomposition of the transport operator A in $X_p(\mathbf{I})$ is given by

$$K=\int_{-1}^{1}\lambda\, dE(\lambda)+\sum_{\nu\in\sigma_p(K)}\nu E(\nu),$$

where $E(\lambda)$ is obtained from Eq. (15a) and $E(\nu)$ from Eq. (15b).

V. HALF SPACE PROBLEM

The multigroup half space problem consists of the transport equation (1) defined for all $x \geq 0$ along with the boundary conditions

$$\psi(x,\mu)=\mathbf{f}_0(\mu),\quad 0\leq\mu\leq 1,$$

$$\lim_{x\to\infty}\psi(x,\mu)=0,$$

for a given boundary (vector valued) function $\mathbf{f}_0$ on the real interval $\mathbf{J}=[0,1]\subset\mathbf{I}$. Let us define the subspace $X_p(\mathbf{J})\subset X_p(\mathbf{I})$ by $f(\mu)=0$ for $-1\leq\mu\leq 0$ if $\mathbf{f}\in X_p(\mathbf{I})$, so that

$$X_p(\mathbf{I})=X_p(\mathbf{J})\oplus X_p(\mathbf{I}\setminus\mathbf{J}).$$

It is assumed that the given function $\mathbf{f}_0\in X_p(\mathbf{J})$. Then by well-known arguments, the solution of the half space problem is equivalent to the construction of a (non-orthogonal) projection Q satisfying:

(i) $(Q\mathbf{f})(\mu)=\mathbf{f}(\mu),\quad 0\leq\mu\leq 1,$

(ii) $(zI-K)^{-1}Q\mathbf{f}$ analytic in z for $\mathrm{Re}z<0$.

The second condition implies that Q is a projection onto

$$X_p(\mathbf{N})=\left\{\int_0^1 dE(\lambda)+\sum_{\substack{\nu\in\sigma_p(K)\\ R(\nu)>0}}E(\nu)\right\}X_p(\mathbf{I})$$

and the first that Q is a projection along $X_p(\mathbf{I}\setminus\mathbf{J})$. The notation $R(\nu)>0$ signifies that either $\mathrm{Re}\nu>0$ else or $\mathrm{Re}\nu=0$, $\mathrm{Im}\nu>0$. In Caseology language, these conditions ensure that the negative frequency eigenvectors $\Phi(\mu,\nu)$, $\nu<0$ or $\mathrm{Re}\nu<0$ are absent from $Q\mathbf{f}$ for any $\mathbf{f}\in X_p(\mathbf{J})$.

In Ref. 3, some recent results of Mullikin[13] on a certain matrix Riemann problem are utilized to construct the projection Q on $X_2(\mathbf{I})$,

$$(V_\Sigma Q\mathbf{f})(-\mu)=\int_0^1\frac{s}{\mu-s}X^{-1}(\mu)Y^{-1}(-s)\Sigma^2(V_\Sigma\mathbf{f})(s)\,ds,\qquad 0<\mu\leq 1, \tag{18a}$$

$$(Q\mathbf{f})(\mu)=\mathbf{f}(\mu),\quad -1\leq\mu\leq 0, \tag{18b}$$

where the matrices $X(z)$ and $Y(z)$ factor the dispersion matrix,

$$\Lambda(z)=Y(-z)X(z), \tag{19}$$

and satisfy some additional analyticity properties. In particular, X and Y are both continuous and invertible as functions from $[-1,0]$ to matrices on X_p^c, and therefore $X^{-1}(\mu)$ and $Y^{-1}(\mu)$ are bounded as operators on $X_p(\mathbf{J})$.

To extend Eq. (18) to $X_p(\mathbf{I})$ from $\mathbf{M}=X_2(\mathbf{I})\cap X_p(\mathbf{I})$, it is only necessary to observe that Q is a bounded operator on $X_p(\mathbf{I})$. In fact,

$$\|Q\mathbf{f}\|_p\leq\|\Sigma\|^2\|V_\Sigma Q\mathbf{f}\|_p$$
$$\leq M_p\|\Sigma\|^4\;\|X^{-1}\|\;\|\mu Y^{-1}\|\;\|\Sigma\|^4\|\mathbf{f}\|_p,$$

where Lemma 2 has been utilized, and $\|X^{-1}\|$, $\|Y^{-1}\|$ are computed only on $X_p(\mathbf{I}\setminus\mathbf{J})$. Thus the half space theory may be developed for $X_p(\mathbf{I})$. The factorization of Mullikin, Eq. (19), is presently only known for $\|\Sigma^{-1}C\|<\frac{1}{2}$, which, of course, limits these results.

From the viewpoint of expansion theorems, the operator of interest is the product FQ, which is bounded on $X_p(\mathbf{I})$ since each of the factors is. In Ref. 3, this operator is derived for Hölder continuous functions $\mathbf{f}$,

$$(FQ\mathbf{f})(\nu)=\frac{1}{2\pi i\nu}\left(X^{-1}(\nu)^+-X^{-1}(\nu)^-\right)\int_0^1 ds\,\frac{s}{\nu-s}\,Y^{-1}(-s)\Sigma^2$$
$$\times(V_\Sigma\mathbf{f})(s)+\tfrac{1}{2}\left(X^{-1}(\nu)^++X^{-1}(\nu)^-\right)Y^{-1}(-\nu)\Sigma^2(V_\Sigma\mathbf{f})(\nu)$$

for $0\leq\nu\leq 1$, and (20a)

$$(FQ\mathbf{f})(\nu)=\frac{1}{\Omega'(\nu)}\left[\int_0^1 ds\,\frac{s}{\nu-s}\,Y^{-1}(-s)\Sigma^2(V_\Sigma\mathbf{f})(s),\boldsymbol{\alpha}'\right]\boldsymbol{\beta}_\nu \tag{20b}$$

for $\nu\in\sigma_p(K)$, $R(\nu)>0$, where $\boldsymbol{\alpha}'$ is defined by

$$\boldsymbol{\alpha}'=X_c(\nu)\boldsymbol{\beta}_\nu,$$

and X_c is defined analogously to Λ_c.

Theorem 11: If $\|\Sigma^{-1}C\|<\frac{1}{2}$, then Eqs. (18) define a bounded projection on $X_p(N)$ along $X_p(\mathbf{I}\setminus\mathbf{J})$. K^{-1} is semibounded on $QX_p(\mathbf{I})$, and thus is the generator of a holomorphic semigroup. Equations (20) for FQ are valid (almost everywhere) for $\mathbf{f}\in X_p(\mathbf{I})$.

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On the tensor representation for compact groups

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A recent paper of Kasperkovitz and Dirl [J. Math. Phys. **15**, 1203 (1974)] concerning the tensor representation for compact groups is examined critically. The flaw which is found in the main theorem fortunately does not affect the deductions which are made from that theorem.

1. INTRODUCTION

In a recent paper[1] Kasperkovitz and Dirl considered the so-called tensor representation, which can be defined for any compact group by a conjugating action on its group algebra. This generalized earlier work of Gamba and Killingbeck,[2-5] and overlapped with papers of van Zanten and de Vries, and Backhouse.[6,7]

An interesting problem which the authors of Ref. 1 examined is how one correlates group-theoretic properties with the occurrence of a full set of irreducible representations in the tensor representation. Unfortunately, the result which purports to settle this question, theorem 2, is false. It is our first job in this paper to give a counter example to theorem 2 and to explain the flaws in the supposed proof of it given in Ref. 1.

Kasperkovitz and Dirl used theorem 2 to establish certain results about the tensor representation for double point groups (theorem 6) and for the groups $SU(n)$ (theorem 9). Happily, in spite of the downfall of theorem 2, both of these further results are true. This we confirm in Secs. 2 and 3.

Finally, in Sec. 4, we show that the hypotheses of theorems 4 and 8 require strengthening to ensure the validity of their conclusions.

2. TENSOR REPRESENTATION

For a finite group $\mathbf{G}$, the tensor representation T is defined on the group algebra $A(\mathbf{G})$ by $g \to T_g$, where $T_g\, a = gag^{-1}$ for $g \in \mathbf{G}$, $a \in A(\mathbf{G})$. By identifying $A(\mathbf{G})$ as an algebra of group functions, the definition of T can be made to work for compact groups (see Ref. 1). It is evident that T provides a faithful representation of $\mathbf{G}/\mathbf{Z}(\mathbf{G})$, so it can only contain irreducibles (UIR) which are trivial on the center, $\mathbf{Z}(\mathbf{G})$. Theorem 2 of Ref. 1 states that T contains all such UIR if $\mathbf{G}$ possesses a faithful UIR, D^α say. In Ref. 6, there is mention of a centerless group $U_3(3)$ whose tensor representation fails to contain a particular UIR. This group is simple, so all of its nontrivial UIR are faithful; the character table is given in Ref. 8. We therefore have a counter example to theorem 2.

There are two distinct errors in the proof of theorem 2. We discuss the first as if the second does not exist. The authors of Ref. 1 correctly deduced from Burnside's theorem that the n-fold product $(D^\alpha \otimes D^{\alpha *})^n$ contains all UIR of $\mathbf{G}/\mathbf{Z}$ for large enough n, but erroneously concluded that $A(\mathbf{G})$ carries them. The fallacy is most easily appreciated in the finite group case. The dimension of $A(\mathbf{G})$ is finite if $\mathbf{G}$ is finite, so is exceeded by the dimension of $(D^\alpha \otimes D^{\alpha *})^n$ for some n (assume $\dim D^\alpha \neq 1$). It follows, for large enough n, that $A(\mathbf{G})$ does not carry all of $(D^\alpha \otimes D^{\alpha *})^n$, so we lose the power of Burnside's theorem. The other flaw in the argument is that the n-fold products of matrix elements $D^{\alpha *}_{j_1 k_1}(x) D^{\alpha *}_{j_2 k_2}(x) \cdots D^{\alpha *}_{j_n k_n}(x)$, $x \in \mathbf{G}$, transform according to the symmetrized n-fold product of $D^\alpha \otimes D^{\alpha *}$ rather than to the full n-fold product to which Burnside's theorem applies.

The above analysis is regrettably rather negative, in that at best the counterexample indicates that faithful UIR play no useful role in the theory of the tensor representation.

Let us now turn to a reconsideration of theorem 6 in view of its apparent dependence on theorem 2. This is a finite task requiring the checking of UIR of the finite double point groups in their tensor representations. We recall that the frequency of a UIR D^α in the tensor representation of a finite group is equal to the sum of the entries in the αth row of $\mathbf{G}$'s character table. The proof of theorem 6 given in Ref. 1 suggests that the authors only had in mind the proper double point groups, these being subgroups of $\mathbf{SU}(2)$. A character table check confirms theorem 6 in such cases. The improper double point groups are subgroups of $\mathbf{SU}(2)\times \mathbf{Z}_2$. Again we confirm the theorem, but with the exception of those groups where the inversion group $\mathbf{Z}_2$ forms a direct factor. The theorem breaks down for such groups $\mathbf{G}^* = \mathbf{H}^* \times \mathbf{Z}_2$, where $\mathbf{H}^*$ is a proper double point group, because the tensor representation gives a representation of $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*) = \mathbf{H}^*/\mathbf{Z}(\mathbf{H}^*) = \mathbf{H}$, the point group of $\mathbf{H}^*$, not of $\mathbf{G} = \mathbf{H}\times\mathbf{Z}_2$, the point group of $\mathbf{G}^*$.

3. THE TENSOR REPRESENTATION OF SU(n)

In this section, we confirm the validity of theorem 9 of Ref. 1 concerning the occurrence of UIR of $\mathbf{SU}(n)$ in its tensor representation. First, we briefly review the representation theory of $\mathbf{SU}(n)$.

It is well known that a complete set of UIR of $\mathbf{U}(n)$ can be labelled by Young tableaux (YT) $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Diagrammatically λ is a shape con-

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sisting of rows of boxes, λ_1 in the first row, λ_2 in the second row, ..., λ_n in the nth row, where the first boxes in the rows form a vertical column. If we now restrict to to $\mathbf{SU}(n)$ we find the equivalence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$, so to obtain a complete label set for UIR of $\mathbf{SU}(n)$ we need only consider YT with at most $(n-1)$ rows. If D^λ is the UIR of $\mathbf{SU}(n)$ labelled by $\boldsymbol{\lambda}$ then $D^{\lambda*}$ has tableau $\boldsymbol{\lambda}^* = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2)$. Diagrammatically, to obtain $\boldsymbol{\lambda}^*$, we form a rectangle of $n \times \lambda_1$ boxes, remove the YT $\boldsymbol{\lambda}$ and rotate the remaining shape through 180°.

To decompose the Kronecker product $D^\lambda \otimes D^\mu$ we first set the tableaux $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ side by side and write a fixed symbol, a_i, in all of the boxes in the ith row of $\boldsymbol{\mu}$, $i = 1, 2, \ldots, n-1$. Thus, the rows of $\boldsymbol{\mu}$ are distinguishable but the individual boxes within a row are not. Then we adjoin the labelled boxes of $\boldsymbol{\mu}$ to the YT $\boldsymbol{\lambda}$ in all possible ways consistent with the following rules:

(1) At each stage in the process the augmented array must be a YT with at most n rows.

(2) Adjoin all boxes from the ith row before adjoining any boxes from the $(i+1)$th row $(i = 1, \ldots, n-1)$.

(3) No two boxes containing the same symbol can be in the same column.

(4) Each final tableau must be such that if we record the occurrence of the symbols a_1, a_2, etc., reading the rows from right to left starting from the top, then at every step in the count the number of a_i's $\geq$ number a_{i+1}'s $(i = 1, \ldots, n-1)$.

After completion retain only one copy of each tableau with a given distribution of symbols. Finally, reduce all tableaux with n rows to tableaux with $(n-1)$ rows. All possible tableaux resulting correspond to representations in the decomposition of the Kronecker product. A more detailed account of this theory can be found in Refs. 9 and 10.

We also need to know which UIR are trivial on the center $\mathbf{Z}$ of $\mathbf{SU}(n)$, these being the only UIR which can occur in the tensor representation of $\mathbf{SU}(n)$. $\mathbf{Z}$ is the group $\{\omega I_n : \omega^n = 1\}$. Now a basis for D^λ can be formed from linear combinations of $(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1})$ products of basis states for the self-representation of $\mathbf{SU}(n)$, via the $(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1})$ tensor power. Hence D^λ is trivial on $\mathbf{Z}$ if and only if $\lambda_1 + \lambda_2 + \lambda_{n-1} = nq$, where q is a positive integer.

Recalling that the tensor representation of a compact group $\mathbf{G}$ is equivalent to $\oplus_\alpha (D^\alpha \otimes D^{\alpha *})$, summed over all labels α of the inequivalent UIR of $\mathbf{G}$, we see that theorem 9 is a corollary of the following result.

Lemma 1: If D^λ is a UIR of $\mathbf{SU}(n)$, $\lambda_1 + \lambda_2 + \lambda_{n-1} = nq$, q an integer, then D^λ is contained in $D^\mu \otimes D^{\mu *}$, where $\boldsymbol{\mu} = ((n-1)q, (n-2)q, \ldots, 2q, q)$.

Proof: Clearly $\boldsymbol{\mu}^* = \boldsymbol{\mu}$. We show that the YT $\boldsymbol{\lambda}' = (\lambda_1 + (n-2)q, \lambda_2 + (n-2)q, \ldots, \lambda_{n-1} + (n-2)q, (n-2)q) \sim \boldsymbol{\lambda}$ is contained in the product of $\boldsymbol{\mu}$ with itself. Note that the shape defined above has $n(n-1)q$ boxes as required.

Define an integer m by $\lambda_j \geq q$ if $j \leq m$ and $\lambda_j < q$ if $j > m$. Now label all the boxes in the ith row of $\boldsymbol{\mu}^*$ with a fixed symbol a_i, for all i, and add these boxes to the rows of $\boldsymbol{\mu}$ to form the YT $[(n-1)q, (n-1)q, \ldots, (n-1)q] \sim \mathbf{0}$. This is done systematically by adding q blocks of boxes from row i of $\boldsymbol{\mu}^*$ to rows $i+1, i+2, \ldots, n$ of $\boldsymbol{\mu}$, for $i = 1, \ldots, n-1$. To obtain $\boldsymbol{\lambda}'$ we must remove the extra boxes in rows $m+1$ to n and place these in rows 1 to m in such a way that the effect is as if we had carried out the adjunction subject to rules 1–4, above. We remove q boxes from row n, $q - \lambda_{n-1}$ boxes from row $n-1$, etc. Boxes labelled by a_1 are placed in the first row until that row has $\lambda_1 + (n-2)q$ boxes. Boxes labelled by a_2 are placed so as to fill the second row, etc. Finally, we obtain $\boldsymbol{\lambda}'$ by sliding the boxes in rows $m+1$ to n to the left to fill up the gaps. It remains to check that we have not broken any of the rules. The two which are not completely obvious are 3 and 4. In fact, rule 3 is preserved in rows $m+1$ to n because $q - \lambda_j \leq q - \lambda_{j+1}$. Also, promotion of a box to a higher row in the tableau will not break rule 4, and of course we arrange the promoted boxes in accordance with rules 2–4. This concludes the proof.

We can, in fact, go a little further than theorem 9, for as we now show, the UIR which occur in the tensor representation of $\mathbf{SU}(n)$ do so with infinite frequency. First we have

Lemma 2: Let D^λ be a constituent of $D^\mu \otimes D^{\mu *}$. If $\boldsymbol{\mu}'$ denotes the YT $(\mu_1 + 1, \mu_2, \ldots, \mu_{n-1})$, then D^λ is also a constituent of $D^{\mu'} \otimes D^{\mu' *}$.

Proof: The YT $\boldsymbol{\mu}'$ differs from $\boldsymbol{\mu}$ in having an extra box in the first row. The YT $\boldsymbol{\mu}'^*$ differs from $\boldsymbol{\mu}^*$ in having an extra box in each of its $n-1$ rows.

We know that a certain sequence of adjunctions of $\boldsymbol{\mu}$ to $\boldsymbol{\mu}^*$ (note we multiply the other way round) leads to $\boldsymbol{\lambda}$. We perform the same sequence of adjunctions of $\boldsymbol{\mu}'$ to $\boldsymbol{\mu}'^*$, except that for all r, if previously the last box of the rth row of $\boldsymbol{\mu}'$ is placed in the sth row of $\boldsymbol{\mu}'^*$, we now place it in the $(s+1)$th row. This ensures that the final YT has one extra box in every row, including the nth. Finally, by reducing to $n-1$ rows, we obtain $\boldsymbol{\lambda}$.

It follows by induction anchored by the conclusion of Lemma 1 that no irreducible constituent of the tensor representation occurs a finite number of times.

4. SOME REVISED THEOREMS

In this section, we show that both theorem 4 and theorem 8 of Ref. 1 require stronger hypotheses before their proofs become acceptable. In both cases the proofs only work for faithful representations. The important point is that if D is an n-dimensional unitary (resp. permutation) representation of a group $\mathbf{G}$ with kernel $\mathbf{K}$, then D embeds $\mathbf{G}/\mathbf{K}$ in $\mathbf{U}(n)$ (resp. $\mathbf{S}_n$). It follows that D embeds $\mathbf{G}$ in $\mathbf{U}(n)$ (resp. $\mathbf{S}_n$) if and only if $\mathbf{K}$ is trivial, that is to say D is faithful. With the insertion of the word "faithful", theorems 4 and 8 are true. Furthermore, it happens that with the stronger hypotheses, theorem 8 has a stronger conclusion.

Proposition: Let n be an integer and $\mathbf{G}$ a finite group possessing a faithful n-dimensional permutation representation $\Pi^{(n)}$. Denote by $\mathrm{Ext}(\mathbf{G}, \Pi^{(n)})$ the set of groups $\mathbf{G}' \supset \mathbf{G}$ possessing an irreducible representation Π'

such that $\Pi' \downarrow \mathbf{G} = \Pi^{(n)}$. Then Ext. $(\mathbf{G}, \Pi^{(n)})$ contains $\mathbf{S}_{n+1}$. Furthermore, if we take the special case $\mathbf{G} = \mathbf{S}_n$, then $\mathbf{S}_{n+1}$ has minimal order in $\mathrm{Ext}(\mathbf{S}_n, \Pi^{(n)})$.

Proof: We noted above that $\Pi^{(n)}$ embeds $\mathbf{G}$ in $\mathbf{S}_n$, considering the latter as the group of all $n \times n$ permutation matrices. This representation of $\mathbf{S}_n$ is known to be $[n] \oplus [n-1, 1]$ in reduced form. Now the branching laws tell us that $[n] \oplus [n-1, 1] = [n, 1] \downarrow \mathbf{S}_n$, subducing from $\mathbf{S}_{n+1}$. We have $\mathbf{G}$ embedded in $\mathbf{S}_n \subset \mathbf{S}_{n+1}$ and $[n, 1] \downarrow \mathbf{G} = \Pi^{(n)}$, so $\mathbf{S}_{n+1} \in \mathrm{Ext}(\mathbf{G}, \Pi^{(n)})$.

Now take the special case $\mathbf{G} = \mathbf{S}_n$, so that $\Pi^{(n)} = [n-1, 1] \oplus [n]$, and let $\mathbf{G}' \in \mathrm{Ext}(\mathbf{S}_n, \Pi^{(n)})$. $\mathbf{G}'$ possesses an irreducible representation Π' where $\Pi' \downarrow \mathbf{S}_n = \Pi^{(n)}$. By the Frobenius reciprocity theorem $[n] \uparrow \mathbf{G}'$ contains the n-dimensional irreducible representation Π'. Also we always have that $[n] \uparrow \mathbf{G}'$ contains the trivial representation of $\mathbf{G}'$. It follows that $|\mathbf{G}'/\mathbf{S}_n| = \dim([n] \uparrow \mathbf{G}') \geq n+1$. Hence $|\mathbf{G}'| \geq (n+1)n! = |\mathbf{S}_{n+1}|$. Therefore, $\mathbf{S}_{n+1}$ is of least possible order in $\mathrm{Ext}(\mathbf{S}_n, \Pi^{(n)})$. This concludes the proof.

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Quantum field theory on incomplete manifolds*

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A theory of the scalar quantum field on static manifolds is constructed using the language of Feynman Green's functions. By means of examples in which the manifolds are parts of Minkowski space, we show how the "method of images" can be used to solve for the Green's functions. In particular, we consider the Rindler wedge and the space outside a uniformly accelerated conducting sheet. As an example in which the manifold is nonstatic, we consider the region exterior to a conducting sheet which is accelerated impulsively from rest to the speed of light. Finally, we study the steady-state part of de Sitter space where we do not obtain a unique result.

1. INTRODUCTION

In general, a manifold cannot be covered by a single coordinate chart. It is therefore of interest to consider the problem of quantizing a field in a coordinate system covering only part of flat space as a preparation for the more general problem. For example, Fulling[1] has considered the natural quantization of a Klein—Gordon field in flat space using Rindler coordinates which cover only a wedge of the complete manifold. The resulting theory is not equivalent to the usual Minkowski space theory. Quantization on a general manifold is therefore a problem of the same nature as the construction of the usual field theory in Rindler coordinates as if one had no knowledge of Minkowski coordinates. It will appear from our work how one is to recognize the theory appropriate to a given manifold in the case when the manifold is static.

We approach the problem by taking the Feynman Green's function as the basic object. Like any Green's function this is a global object, and must be defined with boundary conditions appropriate to the physical problem under consideration. A trivial example is the Green's function in electrostatics for an infinite plane conductor. This has a singularity in the unphysical region corresponding to an image charge. We expect the same sort of behavior for the Feynman Green's function for a hyperbolic operator L on an incomplete (space-time) manifold M. Suppose M to be an analytic manifold which can be analytically extended to a larger manifold M_0. Given a Green's function G for L on M which satisfies

$$LG(x,x^1)=\delta(x,x^1) \tag{1}$$

on M, one may analytically extend G as a function of x to M_0. There is no reason to believe that the resulting function will be a Green's function on M_0; in general, it will have singularities in M_0-M. We may define a function as the difference between the left- and right-hand sides of (1) when x lies in M_0-M:

$$\rho(x,x^1)=LG(x,x^1)-\delta(x,x^1). \tag{2}$$

We observe that ρ represents a distribution of "image charges" in M_0-M since if L possesses an inverse G_0 on M_0 then a solution of (2) is

$$G(x,x^1)=G_0(x,x^1)+\int dy G_0(x,y)\rho(y,x^1). \tag{3}$$

In the Rindler wedge we find the Feynman Green's function for a scalar field according to the prescription of Sec. 2 by imposing analyticity requirements in the Rindler time coordinate. By exhibiting the Green's function in the form (3) it is shown that it does indeed possess an image charge distribution in the unphysical region. Conversely, by subtracting out the contribution from this image distribution one recovers the Feynman Green's function appropriate to the complete manifold, in this case, of course, the standard Minkowski space Green's function, but expressed in terms of Rindler coordinates.

Seeking a solution for the Feynman function of the form (3) provides, in fact, a powerful method of solving field theories on restricted manifolds. We use this method to solve the problem of a scalar field outside a conductor moving with constant finite acceleration. We are again led to a stable field theory, which moreover goes over into that of the previous case as the acceleration tends to infinity. This provides some insight into the physical significance of the Rindler wedge. In both cases we calculate the vacuum expectation value of the energy-momentum tensor for a massless scalar field in two dimensions and confirm the "conformal anomaly" of Fulling and Davies[2] rather than the result of DeWitt.[3]

As a last example in flat space, we consider the case of a "conducting piston" which remains at rest until time $t=0$, after which it moves out with the velocity of light. The Bogoliubov coefficients are calculated and shown to give a black-body spectrum in the Rindler region. However, by causality, the acceleration of the piston cannot influence the interior region, so we cannot interpret this as a particle flux. The problem is discussed from the point of view of the energy—momentum tensor and the effective Lagrangian.

Finally, we consider an example in curved space—time, namely, the steady-state universe. It is known that the creation of matter here cannot be both uniform and in particle—antiparticle pairs for a realistic model universe (from γ-ray observations), so this example is not chosen for its relevance to cosmology. Rather it is an exactly soluble model in a curved manifold for which, however, the solution is not unique.

In this paper we keep the mass of the field and the dimension of the space arbitrary in order to examine

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the relation between the general case and the massless theory in two dimensions, which is to be distinguished as an exceptional case in view of the properties that it enjoys with respect to conformal transformation. An example of the pathology of the massless two-dimensional case from our point of view is that the Feyman function appropriate to the Rindler wedge diverges at the boundaries of the coordinate patch and so may not be sensibly continued into the unphysical region, whereas in all other cases this may be done.

Two appendices deal with the calculation of the Bogoliubov coefficients for the piston problem and with the evaluation of certain integrals that would otherwise disrupt the narrative.

2. FIELD THEORY ON STATIC MANIFOLDS

In this section we show how to translate the canonical theory for the Klein–Gordon field in static manifolds covariantly into the language of Feyman Green's functions.

Consider a manifold M endowed with a Lorentz metric $\mathbf{g}$ and a global timelike Killing vector field $\partial/\partial t$. Let $\mathbf{V}$ be a vector field parallel to $\partial/\partial t$ but normalized so that $g(\mathbf{V},\mathbf{V})=-1$.

Then associated with $\mathbf{V}$ we may construct a new metric $\mathbf{h}$ on M by

$$h(\mathbf{X},\mathbf{Y})=g(\mathbf{X},\mathbf{Y})+\lambda g(\mathbf{X},\mathbf{V})g(\mathbf{Y},\mathbf{V})$$

with $\mathbf{X}$ and $\mathbf{Y}$ arbitrary vector fields. In coordinate language the components of the new metric are

$$h_{\mu\nu}=g_{\mu\nu}+\lambda V_\mu V_\nu.$$

The essential point is that, for $\lambda>1$, $\mathbf{h}$ is positive definite and therefore in terms of this new metric the operator $\Box-m^2$ is elliptic rather than hyperbolic. Subject to appropriate spacial boundary conditions, it therefore possesses a unique inverse G_λ. We shall see that regarded as a function of λ, G_λ is analytic in the complex plane cut along the real axis from 1 to $-\infty$. G_λ may be analytically continued from $\lambda>1$ to the value corresponding to the physical metric $\lambda=0$ either through the upper half-plane, yielding the Feynman function, or through the lower half-plane yielding the negative of its complex conjugate. It will emerge from the following that this construction yields the unique Green's function analytic in the lower half $(t-t')^2$ plane. We could arrive at this same Green's function by finding a complete set of normal modes that are positive frequency with respect to the Killing field $\partial/\partial t$ and constructing a Fock space in the usual way. In this latter approach the analyticity property with respect to $(t-t')^2$ arises from the evaluation of the vacuum expectation value of a time ordered product of field operators. We prefer, however, to proceed by analytic continuation from a space of positive definite metric since this emphasizes the role played by the global Killing field.

Since M is static, coordinates may be chosen such that the line element corresponding to $\mathbf{g}$ takes the form

$$ds^2=-a^2(\mathbf{x})\,dt^2+\gamma_{jk}(\mathbf{x})\,dx^j\,dx^k,$$

where γ_{jk} is the positive definite metric induced on the hypersurfaces orthogonal to $\partial/\partial t$. In these coordinates the line element corresponding to $\mathbf{h}$ takes the form

$$d\bar{s}^2=(\lambda-1)a^2\,dt^2+\gamma_{jk}\,dx^j\,dx^k$$

and the equation satisfied by the Green's function becomes

$$\left(\frac{1}{\lambda-1}\frac{\partial^2}{\partial t^2}+E\right)G_\lambda(x,x')=-\frac{ia}{\gamma^{1/2}}\frac{\delta(t,t')\delta(\mathbf{x},\mathbf{x}')}{(\lambda-1)^{1/2}}, \tag{4}$$

where E is the elliptic operator

$$(a/\gamma^{1/2})\partial_j(a\gamma^{1/2}\gamma^{jk}\partial_k)-m^2a^2.$$

We shall seek a solution to (4) of the form

$$G_\lambda(x,x')=i(\lambda-1)^{1/4}\int_0^\infty\frac{ds}{(4\pi s)^{1/2}}\times\exp[-(t-t')^2(\lambda-1)^{1/2}/4s]g(s,\mathbf{x},\mathbf{x}')\quad(\lambda>1). \tag{5}$$

A certain loss of generality is involved since (5) exhibits the time dependence of G through a representation that is essentially a Laplace transform. This is equivalent to the imposition of a boundary condition since it is evident from (5) that the representation requires that $G_\lambda\to 0$ as $t\to\pm\infty$. It is also equivalent to the assumption that G should admit an expansion in terms of normal modes that are positive frequency with respect to $\partial/\partial t$. It is therefore essential to choose the Killing field appropriate to the boundary conditions imposed by the problem under consideration. For example, it may be that the points for which $t\to\pm\infty$ are at finite distances, as in the case of Rindler coordinates. The correct Green's function is obtained if these points are the trajectories of perfect conductors (mirrors) in the space–time.

Substituting (5) into (4), we find

$$-\frac{a\delta(t,t')\delta(\mathbf{x},\mathbf{x}')}{\gamma^{1/2}(\lambda-1)^{1/2}}=(\lambda-1)^{-1/4}\int_0^\infty\frac{ds}{(4\pi s)^{1/2}}g(s,\mathbf{x},\mathbf{x}')\left(\frac{\partial}{\partial s}-\frac{1}{2s}\right)\times\exp\left(-\frac{(t-t')^2(\lambda-1)^{1/2}}{4s}\right)+(\lambda-1)^{1/4}\int_0^\infty\frac{ds}{(4\pi s)^{1/2}}\times\exp\left(-\frac{(t-t')^2(\lambda-1)^{1/2}}{4s}\right)Eg(s,\mathbf{x},\mathbf{x}').$$

Integrating by parts and using the relation

$$\lim_{s\to 0}\frac{\exp[-(t-t')^2(\lambda-1)^{1/2}/4s]}{(4\pi s)^{1/2}}=(\lambda-1)^{-1/4}\delta(t,t'),$$

we see that $g(s,\mathbf{x},\mathbf{x}')$ satisfies the diffusion equation

$$Eg=\frac{1}{(\lambda-1)^{1/2}}\frac{\partial g}{\partial s} \tag{6}$$

with the initial conditions

$$g(0,\mathbf{x},\mathbf{x}')=\frac{a(\mathbf{x})}{\gamma^{1/2}(\mathbf{x})}\,\delta(\mathbf{x},\mathbf{x}'). \tag{7}$$

We assume that, subject to suitable restrictions on the spatial metric γ_{jk} and spatial boundary conditions, the solution to (6) is unique and that the large s behavior of the integrand is such as to converge the integral (5). We observe that G_λ is, as anticipated, analytic in the complex λ plane cut along the real axis from $-\infty$ to 1. Using (6), we approach $\lambda = 0$ through the upper half-plane to obtain

$$G(x,x') = \exp\left(\frac{3i\pi}{4}\right)\int_0^\infty \frac{ds}{(4\pi s)^{1/2}} \exp\left[-\frac{i(t-t')^2}{4s}\right] g(s,\mathbf{x},\mathbf{x}'). \tag{8}$$

It is evident from (8) that G is analytic in the lower half $(t-t')^2$ plane.

Alternatively, if G is analytic in the lower half $(t-t')^2$ plane and tends to zero as $(t-t')^2 \to \infty$ through the lower half-plane, then the $(t-t')^2$ dependence of G may be represented by a half-range Fourier integral. That is, there will exist some function $g(s,\mathbf{x},\mathbf{x}')$ such that (8) holds. The uniqueness of this representation then follows by substitution of (8) into the equation

$$(\Box - m^2)G(x,x') = -\delta(x,x')/g^{1/2}$$

and by the uniqueness of the solution to the associated parabolic problem (6).

3. THE RINDLER WEDGE

Take n-dimensional Minkowski space with standard coordinates $(t, x, x_2, \ldots, x_{n-1})$ and consider the coordinate transformation in the x, t plane

$$t = \xi \sinh\tau, \quad x = \xi\cosh\tau.$$

In the new coordinates the line element is

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + d\mathbf{x}^2$$

and the transformation is regular in the wedge $x > |t|$, which we shall call the Rindler wedge (region I in Fig. 1). The whole of Minkowski space, with the exception of the lines $x = \pm t$, can be covered by four coordinate patches of this type in an obvious way. We shall use this later to coordinatise points in region II (Fig. 1) and in regions F and P (Fig. 6). Returning to the Rindler wedge, it is clear from the expression for the metric that $\partial/\partial\tau$ is a global timelike Killing field and the corresponding normalized timelike vector field is $\xi^{-1}\partial/\partial\tau$.

We could at this stage proceed as in the previous section in order to obtain the Feynman Green's function appropriate to this manifold. The calculation involved is straightforward, and the solution to the associated parabolic problem (6) turns out to be

$$g(s,\xi,\xi',\mathbf{x},\mathbf{x}') = 4\int \frac{d^{n-2}\mathbf{k}}{(2\pi)^n} \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] \int_0^\infty d\nu\, \nu \sinh\nu\pi \times \exp[-\nu^2 s(\lambda-1)^{1/2}] K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi'),$$

where we have set $\mu = (\mathbf{k}^2 + m^2)^{1/2}$.

However, we shall require a knowledge of the normal modes for the problem in Sec. 5 so we shall employ them here in order to calculate the Feynman function. The Klein-Gordon equation in these coordinates is

$$\left\{-\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} + \sum_2^n \frac{\partial^2}{\partial x_k^2} - m^2\right\}\phi = 0$$

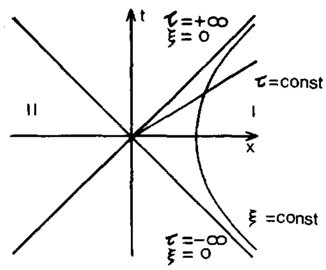

FIG. 1. The (x,t) plate of Minkowski space showing the Rindler wedge (I) bounded by the lines $x = \pm t$. The lines ξ = constant are the trajectories of the Killing vector.

and a complete set of solutions that are of positive frequency with respect to $\partial/\partial\tau$ is

$$V_{\mathbf{k}\nu}(x) = [2/(2\pi)^{n/2}](\sinh\pi\nu)^{1/2}\exp(-i\nu\tau)K_{i\nu}(\mu\xi) \times \exp(i\mathbf{k}\cdot\mathbf{x}),$$

where $K_{i\nu}(\mu\xi) = K_{-i\nu}(\mu\xi)$ is a Macdonald function (Bessel function of the third kind). With respect to the usual inner product on τ equal to constant hypersurfaces

$$(V_{\mathbf{k}\nu}, V_{\mathbf{k}'\nu'}) = i\int d\mathbf{x}\int \frac{d\xi}{\xi} V^*_{\mathbf{k}\nu} \overleftrightarrow{\frac{\partial}{\partial\tau}} V_{\mathbf{k}'\nu'}$$

these solutions are orthonormal: we have

$$\begin{aligned}(V_{\mathbf{k}\nu}, V_{\mathbf{k}'\nu'}) &= \frac{4}{(2\pi)^2}\int\frac{d\xi}{\xi}K_{i\nu}(\mu\xi)K_{i\nu'}(\mu\xi)(\nu+\nu') \times(\sinh\nu\pi\sinh\nu'\pi)^{1/2}\delta(\mathbf{k},\mathbf{k}')\\ &= \delta(\nu,\nu')\delta(\mathbf{k},\mathbf{k}').\end{aligned}$$

We can decompose the field operator $\phi(x)$ with respect to the $V_{\mathbf{k}\nu}$ as

$$\phi(x) = \int_0^\infty \frac{d\nu}{(2\pi)^{n/2}} 2(\sinh\nu\pi)^{1/2}\int d\mathbf{k}K_{i\nu}(\mu\xi) \times\{a_{\nu\mathbf{k}}\exp(-i\nu\tau + i\mathbf{k}\cdot\mathbf{x}) + a^+_{\nu\mathbf{k}}\exp(i\nu\tau - i\mathbf{k}\cdot\mathbf{x})\} \tag{9}$$

with $a^+_{\nu\mathbf{k}}$, $a_{\nu\mathbf{k}}$ creation and annihilation operators for "Rindler particles." The Feynman function is defined by

$$G(x,x') = i\langle 0|T\phi(x)\phi(x')|0\rangle, \tag{10}$$

where $|0\rangle$ is the state annihilated by all the $a_{\nu\mathbf{k}}$. Since $a^+_{\nu\mathbf{k}}$ and $a_{\nu\mathbf{k}}$ satisfy the usual commutation relations, we have

$$\langle 0|a_{\nu\mathbf{k}}a^+_{\nu'\mathbf{k}'}|0\rangle = \langle 0|[a_{\nu\mathbf{k}}, a^+_{\nu'\mathbf{k}'}]|0\rangle = \delta(\nu,\nu')\delta(\mathbf{k},\mathbf{k}'). \tag{11}$$

Using (9) and (11) in (10) we obtain

$$G(x,x') = i\int_0^\infty \frac{4}{(2\pi)^n}\sinh\nu\pi\, d\nu\int d\mathbf{k}\,K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi') \times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - i\nu|\tau-\tau'|], \tag{12}$$

This is analytic in the region $\mathrm{Re}|\tau-\tau'| > 0$, $\mathrm{Im}|\tau-\tau'| < 0$ [i.e., in the lower half $(\tau-\tau')^2$ plane] as required.

To evaluate this integral, we need the relation[4]

$$K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi') = \tfrac{1}{2}\int_{-\infty}^\infty d\lambda\exp(i\nu\lambda)K_0(\mu\gamma_1), \tag{13}$$

where $\gamma_1^2 = \xi^2 + \xi'^2 + 2\xi\xi'\cosh\lambda$.

Let $u = |\tau-\tau'|$. It will be convenient to suppose initially that $\mathrm{Im}u < -\pi$; then we may interchange orders of integration and with the aid of (13) write (12) as

$$G(x,x') = i\int_{-\infty}^{\infty} d\lambda \int_0^{\infty} \frac{d\nu}{2\pi^2}\sinh\pi\nu\,\exp[-i\nu(u-\lambda)] \times \int \frac{d^{n-2}\mathbf{k}}{(2\pi)^{n-2}}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]K_0(\mu\gamma_1).$$

The ν integration is straightforward and yields

$$\int_0^{\infty}\frac{d\nu}{2\pi^2}\sinh\pi\nu\,\exp[-i\nu(u-\lambda)] = \frac{-1}{2\pi[(\lambda-u)^2+\pi^2]}\,.$$

To perform the $\mathbf{k}$ integration, we use the integral representation

$$K_0(\mu\gamma_1) = \int_0^{\infty}\frac{dz}{2z}\exp\left[-\frac{1}{2}\left(z+\frac{\mu^2\gamma_1^2}{z}\right)\right]$$

and interchange the orders of the $\mathbf{k}$ and z integration. Setting $z = m\gamma_1^2[\gamma_1^2+(\mathbf{x}-\mathbf{x}')^2]^{-1/2}v$ and $\gamma_2^2 = \gamma_1^2+(\mathbf{x}-\mathbf{x}')^2$ we have

$$\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]K_0(\mu\gamma_1) = \int_0^{\infty}\frac{dv}{2v}\left(\frac{mv}{2\pi\gamma_2}\right)^{(n-2)/2}\exp\left[-\frac{m}{2}\gamma_2\left(v+\frac{1}{v}\right)\right] = \left(\frac{m}{2\pi\gamma_2}\right)^{(n-2)/2}K_{(n-2)/2}(m\gamma_2). \qquad (14)$$

Thus for $\mathrm{Im}\,u < -\pi$

$$G(x,x') = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{(\lambda-u)^2+\pi^2}\left(\frac{m}{2\pi\gamma_2}\right)^{(n-2)/2}K_{(n-2)/2}(m\gamma_2).$$

The integrand contains poles at $u\pm i\pi$ lying below the real axis, i.e., below the contour of integration. To analytically continue to $\mathrm{Im}\,u=0$, we must ensure that the contour of integration is deformed so as to remain above the poles. Therefore, for $\mathrm{Im}\,u=0$, we integrate along the contour shown in Fig. 2.

The contribution from the pole at $\lambda = u+i\pi$ is

$$G_0(x,x') = \frac{i}{2\pi}\left(\frac{m}{2\pi(2\sigma)^{1/2}}\right)^{(n-2)/2}K_{(n-2)/2}((2m^2\sigma)^{1/2}),$$

where $(2\sigma)^{1/2}$ is γ_2 evaluated at $\lambda = u+i\pi$. Since $(2\sigma)^{1/2}$ is just the geodesic separation of x and x' expressed in Rindler coordinates, $G_0(x,x')$ is seen to be the Feynman function for Minkowski space. So far we have

$$G(x,x') = G_0(x,x') - \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{(\lambda-\tau')^2+\pi^2}\left(\frac{m}{2\pi\gamma}\right)^{(n-2)/2} \times K_{(n-2)/2}(m\gamma),$$

where

$$\gamma = \gamma_2(\lambda-\tau) = [\xi^2+\xi'^2+2\xi\xi'\cosh(\lambda-\tau)+(\mathbf{x}-\mathbf{x}')^2]^{1/2}$$

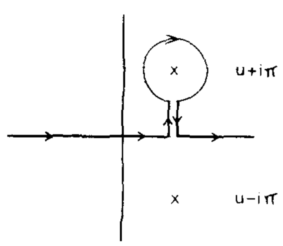

FIG. 2. The contour for the λ integration.

is the geodesic distance between the point $(\tau,\xi,\mathbf{x})$ in region I and the point $(\lambda,\xi',\mathbf{x}')$ in region II. Thus we can write

$$G(x,x') = G_0(\sigma) - \int_{-\infty}^{\infty}\frac{d\lambda}{(\lambda-\tau')^2+\pi^2}\,G_0(\bar{\sigma}),$$

the second term representing a contribution from an image charge density $-[(\lambda-\tau')^2+\pi^2]^{-1}$ distributed on a line $\mathbf{x}=\mathbf{x}'$, $\xi=\xi'$ in region II (Fig. 3).

The effective Lagrangian[5] is calculated from

$$\mathcal{L}_1 = \tfrac{1}{2}ig^{1/2}\int G(x,x)\,dm^2.$$

The ultraviolet and infrared (in the massless case) divergences of $G(x,x)$ may be isolated in the contribution from $G_0(x,x)$ (except when $m=0$ and $n=2$):

$$\mathcal{L}_1 = \mathcal{L}_\infty - g^{1/2}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2+\pi^2}\left(\frac{m}{2\pi\gamma}\right)^{n/2}K_{n/2}(m\gamma)$$
$$(\gamma = 2\xi\cosh(\lambda/2)).$$

It is of interest to compute the effective energy momentum tensor. In two dimensions, where the canonical and "new improved" versions agree, we have

$$\langle T_{\mu\nu}\rangle = -i\lim_{x\to x'}\{\partial_{\mu\nu'} - \tfrac{1}{2}g_{\mu\nu}(\partial_\alpha^{\alpha'} - m^2)\}\,G(x,x').$$

Dropping the infinite term from G_0 and using the integral representation of $K_0(m\gamma)$, we obtain

$$\langle T_\xi{}^\xi\rangle = -\frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2+\pi^2}\int_0^{\infty}\frac{ds}{s}\left(\frac{\gamma^2}{4}-m^2\right)\times\exp\left(-m^2s-\frac{\gamma^2}{4s}\right).$$

In the limit $m\to 0$, this gives

$$\langle T_\xi{}^\xi\rangle = -\langle T_\tau{}^\tau\rangle = -\frac{1}{8\pi\xi^2}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2+\pi^2}\,\frac{1}{\cosh^2\lambda/2}\,.$$

The integral may be evaluated by residues since

$$\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda^2+\pi^2}\,\frac{1}{\cosh^2\lambda/2} = \frac{1}{2\pi i}\int_C\frac{d\lambda}{\lambda-i\pi}\,\frac{1}{\cosh^2\lambda/2} = \frac{1}{3}$$

with the contour C consisting of the real axis and the line $(-\infty+2\pi i, \infty+2\pi i)$. Thus,

$$\langle T_\xi{}^\xi\rangle = -\langle T_\tau{}^\tau\rangle = -1/24\pi\xi^2,$$
$$\langle T_\xi{}^\tau\rangle = \langle T_\tau{}^\xi\rangle = 0.$$

This confirms the existence of a "conformal anomaly" as found by Davies and Fulling. The origin of the anomaly is clear enough in our treatment. We have regularized the divergences of the theory by using a representation of the form (3). In two dimensions this is permitted only if we make the theory massive, thereby explicitly breaking the conformal symmetry. What we have shown is that the symmetry is not restored in the limit $m\to 0$.

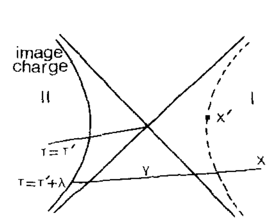

FIG. 3. The image charge distribution for the Rindler wedge.

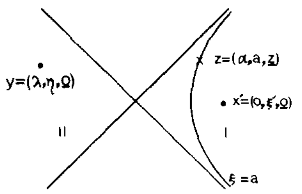

FIG. 4. Coordinates for the moving mirror problem. The mirror is the surface $\xi=a$.

4. A MOVING MIRROR PROBLEM

The $\xi=$const surfaces in the Rindler wedge are surfaces of constant acceleration. Choose $\xi=a$ to be replaced by a perfect conductor for the ϕ field such that ϕ is required to vanish on $\xi=a$ which then becomes a moving mirror (Fig. 4).

Let $x=(\tau,\xi,\mathbf{x})\in$ I, $x'=(\tau',\xi',\mathbf{x}')\in$ II then the Minowski space Green's function for a massive scalar field is

$$G_0(x,x')=\frac{i}{2\pi}\left(\frac{m}{2\pi\gamma}\right)^{(n-2)/2}K_{(n-2)/2}(m\gamma)$$

with

$$\gamma^2=\xi^2+\xi'^2+2\xi\xi'\cosh(\tau-\tau')+(\mathbf{x}-\mathbf{x}')^2.$$

Using (14) with γ_2 replaced by γ, and the inverse Fourier transform of (13), we can write

$$G_0(x,x')=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[-i\nu(\tau-\tau')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\times K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi'). \tag{15}$$

If we require both $x,x'\in I$ then some care is necessary to satisfy the requirement that G_0 be analytic in the upper half $\gamma^2=(x-x')^2$ plane, with now $\gamma^2=\xi^2+\xi'^2-2\xi\xi'\cosh(\tau-\tau')+(\mathbf{x}-\mathbf{x}')^2$. To this end, we replace the smaller of ξ and ξ' in (15) by $\xi_<\exp(i\theta)$ and analytically continue G_0 to $\theta=\pi$. For $\theta=\pi-\epsilon$ we have

$$\gamma^2=\xi_<^2+\xi_>^2-2\xi_>\xi_<\cosh(\tau-\tau')+2i\xi_<\epsilon(\xi_>\cosh(\tau-\tau')-\xi_<)$$

which manifests the correct analyticity property. This yields

$$G_0(x,x')=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[-i\nu(\tau-\tau')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\times K_{i\nu}(\mu\xi_>)K_{i\nu}(\exp(i\pi)\mu\xi_<) \tag{16}$$

as our expression in normal modes for the Minkowski space Green's function in the Rindler wedge.

We are seeking a Green's function which vanishes on $\xi=a$. In view of the result of Sec. 3 let us assume that this can be obtained by adding a contribution from an image distribution $\rho(y;x')$ located in region II. Thus, we take

$$G(x,x')=G_0(x,x')+\int d^ny\,G_0(x,y)\rho(y;x') \tag{17}$$

with the boundary condition

$$0=G_0(z,x')+\int d^ny\,G_0(z,y)\rho(y;x') \tag{18}$$

for all z in the boundary $\xi=a$. This is an integral equation for ρ. Substituting for G_0 from (16) and noting that the relation must be valid for all z in the boundary gives

$$0=K_{i\nu}(\mu\xi')K_{i\nu}(\exp(i\pi)\mu a)+\int_{-\infty}^{\infty}d\lambda\exp(i\nu\lambda)\int_0^{\infty}d\eta'\eta'\times K_{i\nu}(\mu a)K_{i\nu}(\mu\eta')\rho(\lambda,\eta';\xi'). \tag{19}$$

Using the relation (valid for $C>0$)

$$\frac{1}{i\pi}\int_{C-i\infty}^{C+i\infty}d\mu\,\mu I_{i\nu}(\mu\eta)K_{i\nu}(\mu\eta')=\frac{1}{\eta}\delta(\eta,\eta'),$$

we obtain the solution for ρ:

$$\rho(\lambda,\eta;\xi')=-\frac{1}{2\pi^2 i}\int d\nu\exp(-i\nu\lambda)\int_{C-i\infty}^{C+i\infty}d\mu\,\mu\times\frac{I_{i\nu}(\mu\eta)K_{i\nu}(\mu\xi')K_{i\nu}(\exp(i\pi)\mu a)}{K_{i\nu}(\mu a)}.$$

Substituting for G_0 from (16) in (17) and using (19) to eliminate ρ gives us $G(x,x')$ in the form

$$G(x,x')=\frac{i}{\pi}\int\frac{d\nu}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[-i\nu(\tau-\tau')+i\mathbf{k}(\mathbf{x}-\mathbf{x}')]\times f(\nu,\mathbf{k}\,|\,\xi,\xi'), \tag{20}$$

where

$$f(\nu,\mathbf{k}\,|\,\xi,\xi')=K_{i\nu}(\mu\xi_>)K_{i\nu}(\exp(i\pi)\mu\xi_<)-\frac{K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi')K_{i\nu}(\exp(i\pi)\mu a)}{K_{i\nu}(\mu a)}. \tag{21}$$

Now the analytic continuation of $K_{i\nu}(\zeta\exp(i\theta))$ to $\theta=\pi$ is given by[4]

$$K_{i\nu}(\zeta\exp(i\pi))=\exp(\pi\nu)K_{i\nu}(\zeta)-i\pi I_{i\nu}(\zeta). \tag{22}$$

This enables us to write

$$f(\nu,\mathbf{k}\,|\,\xi,\xi')=i\pi\frac{K_{i\nu}(\mu\xi_>)}{K_{i\nu}(\mu a)}\,[K_{i\nu}(\mu\xi_<)I_{i\nu}(\mu a)-K_{i\nu}(\mu a)I_{i\nu}(\mu\xi_<)]. \tag{23}$$

Using this in (20) it is straightforward to check that $G(x,x')$ is a Green's function for the Klein-Gordon equation vanishing on $\xi=a$.

The expression (23) for $f(\nu,\mathbf{k}\,|\,\xi,\xi')$ has singularities in the plane at the zeros of $K_{i\nu}(\mu a)$. These singularities lie on the real axis. To obtain the Feyman Green's function we must choose the contour for the ν integration to avoid these singularities in such a way that the resulting function be analytic in the lower half $(\tau-\tau')^2$ plane. The appropriate contour lies below the poles on the negative real axis and above those on the positive real axis. (This will be verified presently.) A possible choice C: $(-\infty\exp(i\epsilon),\infty\exp(i\epsilon))$ is shown in Fig. 5.

To perform the integration, we now show that the contour may be rotated in the lower half-plane so as to envelop the poles on the positive real axis. Using the relation

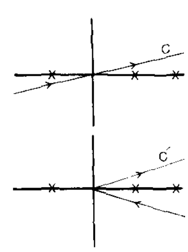

FIG. 5. Rotation of the contour for the ν integration.

$$K_{i\nu}(\zeta)=\frac{i\pi}{2\sinh\pi\nu}\left[I_{-i\nu}(\zeta)-I_{i\nu}(\zeta)\right], \tag{24}$$

we see that the expression $K_{i\nu}(\mu\xi_<)I_{i\nu}(\mu a)-K_{i\nu}(\mu a)\times I_{i\nu}(\mu\xi_<)$ is invariant under $\nu\to-\nu$ as is the contour of integration. This means that we may take $(\tau-\tau')>0$ in (20) without loss of generality.

For $|\nu|$ large and $i\nu\neq\pm N$ $(N=0,1,\cdots)$, we have

$$\begin{aligned} &I_{i\nu}(\zeta)\sim[\Gamma(1+i\nu)]^{-1}(\tfrac{1}{2}\zeta)^{i\nu},\\ &K_{i\nu}(\zeta)\sim\tfrac{1}{2}[\Gamma(i\nu)(\tfrac{1}{2}\zeta)^{-i\nu}+\Gamma(-i\nu)(\tfrac{1}{2}\zeta)^{i\nu}]. \end{aligned}\tag{25}$$

For $-\pi+\epsilon<\arg\nu<-\epsilon$ write $i\nu=R\exp(i\beta)$, $-\frac{1}{2}\pi+\epsilon<\beta<\frac{1}{2}\pi-\epsilon$. Then

$$|\Gamma(i\nu)|\sim(2\pi/R)^{1/2}\exp[-R(\cos\beta+\sin\beta)]R^{R\cos\beta},$$

$$\Gamma(-i\nu)=\frac{\pi}{\nu\sinh\pi\nu}\,\frac{1}{\Gamma(i\nu)}\ .$$

Using these asymptotic forms, we find

$$f(\nu,\mathbf{k}|\,\xi,\xi')\sim-(\pi i/2R)\exp(-i\beta)(\xi_</\xi_>)^{R\exp(i\beta)}$$
$$\to 0\ \text{ as } R\to\infty.$$

By Jordan's lemma, and the absence of zeros of $K_{i\nu}(\mu a)$ in $\operatorname{Im}\nu<0$, we may close the contour in the lower half ν plane to obtain the contour C' (Fig. 5).

For a given $k=|\mathbf{k}|$, let ν_{rk} be the rth zero of $K_{i\nu}(\mu a)$. Then the ν integral in (20) along C' is evaluated by residues to give

$$G(x,x')=i\sum_{r=1}^{\infty}\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\,N_{rk}\exp[-i\nu_{rk}|\tau-\tau'|+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]$$
$$\times K_{i\nu_{rk}}(\mu\xi)K_{i\nu_{rk}}(\mu\xi'),$$

where

$$N_{rk}=\lim_{\nu\to\nu_{rk}}\{(\nu-\nu_{rk})[I_{i\nu}(\mu a)/K_{i\nu}(\mu a)]\}.$$

Here we see explicitly the analyticity in the lower half $|\tau-\tau'|$ plane. This expression could have been obtained by starting from an expansion of the field operator in terms of normalized basis functions

$$U_{rk}(x)=\frac{N_{rk}^{1/2}}{(2\pi)^{(n-2)/2}}\,K_{i\nu_{rk}}(\mu\xi)\exp(-i\nu_{rk}\tau+i\mathbf{k}\cdot\mathbf{x}).$$

There are now two interesting points to be made in relation to this result. First, when $x\to x'$, we find $G(x,x)$ is purely imaginary, thus the effective Lagrangian is purely real and we again have a stable field theory (no particle production).

Second, we show that we regain the result of the previous case of the Rindler wedge by letting $a\to 0$ (i.e., the acceleration of the mirror, a^{-1}, $\to\infty$).

Consider again the ν integration in (20) round the contour C'. On the part of the contour in the lower half-plane we proceed as before with the expression (23) for $f(\nu,\mathbf{k}|\,\xi,\xi')$. In the upper half-plane we use instead of (22) the equivalent formula

$$K_{i\nu}(\zeta\exp(i\pi))=\exp(-\pi\nu)K_{i\nu}(\zeta)-i\pi I_{-i\nu}(\zeta)$$

to obtain an expression for f. Thus

$$\begin{aligned} G(x,x')=\frac{i}{\pi}\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}&\exp[-i\nu|\tau-\tau'|+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\\ \times\Bigg\{\int_0^{\infty\exp(i\epsilon)}\frac{d\nu}{2\pi}&\Bigg[-i\pi K_{i\nu}(\mu\xi_>)I_{-i\nu}(\mu\xi_<)\\ &+i\pi K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi')\frac{I_{-i\nu}(\mu a)}{K_{i\nu}(\mu a)}\Bigg]\\ -\int_0^{\infty\exp(-i\epsilon)}\frac{d\nu}{2\pi}&\Bigg[-i\pi K_{i\nu}(\mu\xi_>)I_{i\nu}(\mu\xi_<)\\ &+i\pi K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi')\frac{I_{i\nu}(\mu a)}{K_{i\nu}(\mu a)}\Bigg]\Bigg\}. \end{aligned}$$

The $a\to 0$ asymptotic forms for the Bessel functions $I_{i\nu}(\mu a)$, $K_{i\nu}(\mu a)$ are the same as the $\nu\to\infty$ forms given by (25). The integrand on each part of the contour has been chosen so that for $a\to 0$ the terms involving a ratio of Bessel functions tend to zero. Taking this limit and using (24) to combine the remaining terms leaves

$$\begin{aligned}\lim_{a\to 0}G(x,x')=[i/(2\pi)^n]\int_0^{\infty}d\nu\,4\sinh\pi\nu\\ \times\int d\mathbf{k}\,K_{i\nu}(\mu\xi)K_{i\nu}(\mu\xi')\exp[-i\nu|\tau-\tau'|\\ +i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\end{aligned}$$

which is precisely the expression derived in Sec. 3 for the Green's function in the Rindler wedge. In an exactly analogous way it may be shown that the image distribution for this problem goes over to that of the previous problem in the limit $a\to 0$.

Consider now the two-dimensional massless case for which we wish to find the vacuum expectation value of the energy momentum tensor.

The renormalized vacuum expectation value of the energy momentum tensor $T^{\mu}{}_{\nu}$ will be diagonal in Rindler coordinates and satisfy $T^{\mu}{}_{\mu}=0$. Thus we may write

$$T^{\mu}{}_{\nu}=p(a,\xi)\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

On dimensional grounds the function p must be of the form

$$p(a,\xi)=(1/\xi^2)q(a/\xi).$$

The divergence condition $T^{\mu\nu}{}_{;\nu}=0$ implies that q is in fact constant. We may determine the value of q by taking the limit $a\to 0$. Thus by comparison with the results of the previous section we find

$$T^{\mu}{}_{\nu}=-\frac{1}{24\pi\xi^2}\begin{pmatrix}-1&0\\0&1\end{pmatrix}\ ,$$

confirming again the anomaly of Fulling and Davies.

5. THE CASE OF THE IMPULSIVE PISTON

We consider a plane conducting piston perpendicular to the x axis at rest at the origin for $t<0$, which for $t>0$ moves out along the positive x axis with the veloc-

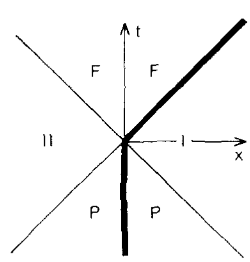

FIG. 6. Showing the regions I, II, F, P in the Minkowski x-t plane of the impulsive piston; the motion of the piston is shown by the thick line.

ity of light (Fig. 6). In contrast to the previous examples, there is no longer a global timelike Killing vector field. From this, and from the similarity of the morphology of the conformal diagram to that of the region exterior to a collapsing black hole,[7] one might expect a flux of particles. However, a causality argument shows that the piston cannot influence the exterior region and from this point of view one expects no flux and a stable vacuum.

The mathematical situation is that at early times ($t<0$) the Green's function must vanish at infinity and on the conductor at $x=0$. This is achieved if we decompose the field operator ϕ with respect to basis functions appropriate to Minkowski time:

$$\phi(x)=\sum_i \{u_i(x)a_i+u_i^*(x)a_i^*\}, \tag{26}$$

where

$$u_{K\mathbf{k}}(x)=[2\sqrt{\pi}/(2\pi)^{n/2}]\exp(-i\mu t\cosh K)\times\sin(x\mu\sinh K)\exp(i\mathbf{k}\cdot\mathbf{x}).$$

(We denote adjoint operators by an asterisk rather than a dagger in anticipation of a matrix notation.) At late times ($t>0$), the Green's function is subject to Rindler-type boundary conditions, we decompose ϕ in terms of the basis functions of Sec. 3:

$$\phi(x)=\sum_i \{v_i(x)b_i+v_i^*(x)b_i^*\}. \tag{27}$$

The "late time" basis functions v_i are related to the "early time" basis functions u_i by a linear transformation

$$v_i=\sum_j \{\alpha_{ij}u_j+\beta_{ij}u_j^*\}, \tag{28}$$

where α_{ij} and β_{ij} are the Bogoliubov coefficients for the problem. The orthonormality of the respective sets of basis functions requires the Bogoliubov coefficients to satisfy the identities

$$\begin{pmatrix}\alpha^+ & -\beta^\sim\\ -\beta^+ & \alpha^\sim\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}\begin{pmatrix}\alpha^+ & -\beta^\sim\\ -\beta^+ & \alpha^\sim\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{29}$$

where we have employed a matrix notation.[6]

The Feynman function for the problem is

$$G(x,x')=i\frac{\langle\text{out}\,|\,T\phi(x)\phi(x')\,|\,\text{in}\rangle}{\langle\text{out}\,|\,\text{in}\rangle}$$

with $|\text{in}\rangle$ and $|\text{out}\rangle$ defined as the "asymptotic" vacua annihilated by the a_i and b_i, respectively. Using (26) and (27) in (30), we obtain

$$G(x,x')=i\sum_{jk} v_j(x_>)u_k^*(x_<)\frac{\langle\text{out}\,|\,b_ja_k^*\,|\,\text{in}\rangle}{\langle\text{out}\,|\,\text{in}\rangle}, \tag{30}$$

where $x_>$ and $x_<$ denote the later and earlier of the space-time points x and x'.

In view of (28) and the identities (29), we have

$$a_i=\sum_j (b_j\alpha_{ji}+b_j^*\beta_{ji}^*),$$

therefore,

$$\delta_{ij}\langle\text{out}\,|\,\text{in}\rangle=\langle\text{out}\,|\,[a_i,a_j^*]\,|\,\text{in}\rangle$$

$$=\sum_k \langle\text{out}\,|\,(b_k\alpha_{ki}+b_k^*\beta_{ki}^*)a_j^*\,|\,\text{in}\rangle$$

$$=\sum_k \alpha_{ki}\langle\text{out}\,|\,b_ka_j^*\,|\,\text{in}\rangle$$

and hence

$$\frac{\langle\text{out}\,|\,b_ja_k^*\,|\,\text{in}\rangle}{\langle\text{out}\,|\,\text{in}\rangle}=(\alpha^\sim)^{-1}_{jk}.$$

Substituting this result into (30) yields (suppressing indices)

$$G(x,x')=iv(x_>)\alpha^{\sim-1}u^*(x_<)=iu^*(x_<)\alpha^{-1}v(x_>).$$

Using (28), we obtain the equivalent relations

$$G(x,x')=G_{\text{in}}(x,x')+iu^*(x)\alpha^{-1}\beta u^*(x') \tag{31a}$$

$$=G_{\text{out}}(x,x')-iv(x)\beta\alpha^{-1}v(x'), \tag{31b}$$

where

$$G_{\text{in}}(x,x')=iu^*(x_>)u(x_<)$$

and

$$G_{\text{out}}(x,x')=iv^*(x_>)v(x_<)$$

are the Feynman functions appropriate to a plane conductor at rest and one with infinite acceleration, respectively.

The Bogoliubov coefficients α and β are computed in Appendix A and are found to be

$$\alpha_{\nu\mathbf{k},K\mathbf{k}'}=(1-\exp(-2\nu\pi))^{-1/2}(2/\pi)^{1/2}\sin\nu K\,\delta(\mathbf{k}-\mathbf{k}'),$$

$$\beta_{\nu\mathbf{k},K\mathbf{k}'}=(\exp(2\nu\pi)-1)^{-1/2}(2/\pi)^{1/2}\sin\nu K\,\delta(\mathbf{k}+\mathbf{k}'). \tag{32}$$

It is trivial to verify that (32) satisfy the identities (29) which, since α and β are real reduce to

$$\alpha^\sim\alpha-\beta^\sim\beta=\alpha\alpha^\sim-\beta\beta^\sim=1,$$

$$\alpha^\sim\beta-\beta^\sim\alpha=\alpha\beta^\sim-\beta\alpha^\sim=0.$$

In order to proceed with our calculation of the Feynman function we shall employ (31a) since the functions u have a simple form throughout the manifold whereas the functions v lose their simple form outside region I.

We see from (32) that

$$(\alpha^{-1})_{K\mathbf{k}',\nu\mathbf{k}}=(1-\exp(-2\nu\pi))^{1/2}(2/\pi)^{1/2}\sin\nu K\,\delta(\mathbf{k}-\mathbf{k}')$$

and hence that

$$(\alpha^{-1}\beta)_{K\mathbf{k},K'\mathbf{k}'}=\delta(\mathbf{k}+\mathbf{k}')\frac{2}{\pi}\int_0^\infty d\nu\,\exp(-\nu\pi)\sin\nu K\sin\nu K'$$

$$=\delta(\mathbf{k}+\mathbf{k}')\left\{\frac{1}{\pi^2+(K-K')^2}-\frac{1}{\pi^2+(K+K')^2}\right\}. \tag{33}$$

The term to be added to G_{in} is

$$\Delta G_{\text{in}}(x,x')=iu^*(x)\alpha^{-1}\beta u^*(x')$$

$$=\int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\Delta g_{\text{in}}(\mathbf{k}\,|\,x,x',t,t'),$$

where

$$\Delta g_{\text{in}}(\mathbf{k}\,|\,x,x',t,t')$$
$$=(i/\pi)\int dK\int dK'([\pi^2+(K-K')^2]^{-1}-[\pi^2+(K+K')^2]^{-1})$$
$$\times\exp[i\mu(t\cosh K+t'\cosh K')]\sin(\mu x\sinh K)\sin(\mu x'\sinh K')$$

$$= -(i/4\pi)\int dK \int dK'([\pi^2+(K-K')^2]^{-1}-[\pi^2+(K+K')^2]^{-1})$$
$$\times \exp[i\mu(x\sinh K + x'\sinh K' + t\cosh K + t'\cosh K')]. \quad (34)$$

This integral is evaluated in Appendix B; we find that for $x' \in I$

$$G(x,x') = G_{\rm in}(x,x') + \int_{-\infty}^{\infty} d\lambda\{(\lambda+\tau'+i\epsilon)^{-1}(\lambda+\tau'-2\pi i)^{-1} - [(\lambda-\tau')^2+\pi^2]^{-1}\} G_0(\bar\gamma),$$

i.e.,

$$G(x,x') = G_0(\gamma) + \int_{-\infty}^{\infty} d\lambda\{(\lambda+\tau'-i\epsilon)^{-1}(\lambda+\tau'-2\pi i)^{-1} - [(\lambda-\tau')^2+\pi^2]^{-1}\} G_0(\bar\gamma)$$

with γ the geodetic interval between x and x' and $\bar\gamma$ the geodetic interval between x and an image point in region II labelled by ξ' and λ.

For $x' \in P$ we find

$$G(x,x') = G_{\rm in}(x,x') + \int_{-\infty}^{\infty} d\lambda\{[(\lambda+\tau')^2+\pi^2]^{-1} - [(\lambda-\tau')^2+\pi^2]^{-1}\} G_0(\bar\gamma),$$

where in the case $\bar\gamma$ represents the geodetic interval between x and an image point in region F labelled by ξ' and λ (Figs. 7 and 8).

We appear now to have several methods by which to calculate a putative particle flux. Directly from the Bogoliubov coefficients (32), and the relation between the a_i and the b_i, we may compute the number of "particles" produced with quantum numbers $(\nu, \mathbf{k})$:

$$\langle \mathrm{in} | b^*_{\nu\mathbf{k}} b_{\nu\mathbf{k}} | \mathrm{in}\rangle = (\beta\beta^\sim)_{\nu\mathbf{k},\nu\mathbf{k}} = \delta^{(n-1)}(0)/[\exp(2\pi\nu)-1].$$

The presence of the term $\delta^{(n-1)}(0)$ reflects the infinite volume of spacetime occupied by these quanta. Even extracting this factor, the spectrum gives rise to a divergent integral over $\mathbf{k}$ because of the infinite phase space arising from the infinite plane geometry. Let us therefore restrict our attention to two dimensions where this problem does not arise. In that case, the spectrum is Planckian [a uniformly accelerating observer whose worldline is $\xi = $ const would observe a frequency ν as corresponding to an energy $\xi^{-1}\nu$ and would conclude that the spectrum corresponded to a temperature $(2\pi\xi)^{-1}$], this is reminiscent of Hawking's calculation.[8]

Alternatively, we can calculate $\langle \mathrm{in}| T_{\mu\nu} |\mathrm{in}\rangle$ directly from $G_{\rm in}$. This is simplified by the following argument. Since $\langle \mathrm{in}| T_{\mu\nu} |\mathrm{in}\rangle$ is to be evaluated relative to the "in" vacuum, in Minkowski coordinates it must be a function of the x coordinate alone. Since also $T_{\mu\nu}$ must be invariant under arbitrary boosts parallel to the plane $x=0$, it is of the form

$$T_{\mu\nu} = \mathrm{diag}(-a, b, a, \ldots, a)$$

with a and b functions of x. The divergence condition $T^{\mu\nu}{}_{;\nu} = 0$ requires b to be constant. Now the vacuum expectation value of $T_{\mu\nu}$ could be evaluated by letting a certain differential operator act on $G_{\rm in}$, viz.,

$$T_{\mu\nu} = \lim_{x\to x'} T_{\mu\nu'} G_{\rm in}(x,x')$$

before renormalization; after renormalization we have

$$T_{\mu\nu} = \lim_{x\to x'} T_{\mu\nu'} G_{\rm in}(\bar\gamma),$$

where $\bar\gamma(x,x')$ is the distance from x to the image point of x' in the plane $x=0$. We see from this that $T_{\mu\nu} \to 0$ as $x \to \infty$ so that the constant b is zero. Taking the trace of the energy momentum tensor, we find

$$(n-1)a = -im^2 G_0(\bar\gamma)$$

and therefore

$$T_{\mu\nu} = -\frac{im^2 G_0(\bar\gamma)}{n-1}\,\mathrm{diag}(-1, 0, 1, \ldots, 1).$$

Thus $\langle T_{\mu\nu}\rangle$ corresponds to a pure vacuum polarization for an inertial observer, as indeed we expect from the causality argument. Transforming to Rindler coordinates we find a (τ, ξ) component, $\langle T^\tau{}_\xi\rangle$ which is nonzero; it follows that a uniformly accelerated observer would see a flux. This flux has no relation to the Bogoliubov coefficients, depending only on the vacuum polarization of the conductor at rest, and vanishing for $m=0$.

One can throw some light on the origin of the discrepancy in the observations of inertial and accelerated observers (without entirely resolving all the problems raised) by noting that $\langle T_{\mu\nu}\rangle$ is a tensor under coordinate transformations only if we normal order $T_{\mu\nu}$ relative to a given vacuum. The black-body spectrum above arises from a normal ordering with respect to the Rindler vacuum, which is perhaps natural for observers accelerated for all time. Our calculation of the Green's function involves subtracting a vacuum energy corresponding to Minkowski space in the discarding of $G_0(x,x)$ and hence a normal ordering with respect to Minkowski time. Whilst the Green function $G(x,x')$ is given uniquely according to our theory by the boundary conditions, the extraction of physical information from G must be carried out with some care!

Finally we know from the Bogoliubov coefficients that the in and out vacua do not differ by simply a phase and we would expect this to manifest itself in the effective Lagrangian. Indeed it does, for on the null plane $x+t=0$ we observe from the expression for $\Delta g_{\rm in}$ that $G(x,x)$ is infinite there (even after renormalization),

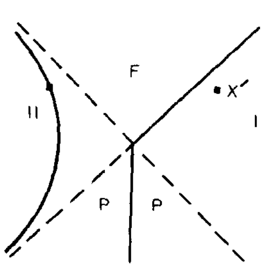

FIG. 7. The charge distribution for the impulsive piston when x' is in region I.

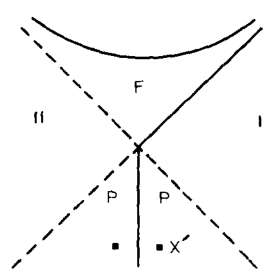

FIG. 8. The charge distribution for the impulsive piston when x' is in region P.

and this results in an infinite contribution to $\mathrm{Im}\int \mathcal{L}_1 dx$ (in the massless case this is in fact the only contribution to $\mathrm{Im}\int \mathcal{L}_1$ since $\mathrm{Im}\,\mathcal{L}_1$ vanishes when integrated over the rest of the manifold).

6. THE STEADY-STATE UNIVERSE

We use the steady-state model to illustrate a soluble problem on an incomplete curved manifold for which the solution is not unique. In the case of the static metric considered in Sec. 2 it was possible to represent the time dependence of the Green's function by an integral of the form (5) in terms of the essentially unique solution of an elliptic equation. However, the steady-state universe is not static, so the time dependence of the Green's function is more complicated. If we attempt to find a representation of the form (5) for the $(t-t')^2$ dependence of G, we find that g depends on time. Thus g satisfies a differential equation in which there appear derivatives with respect to time, and this will not have a unique solution without the imposition of further boundary (or initial) conditions. This is just the statement that the incompleteness of the manifold allows a certain amount of arbitrariness in the dependence of G on $t+t'$, and this was not present in the static case. We would expect this to be the generic situation, so that in general a further physical principle is required to determine a unique theory.

The steady-state metric in conformally flat form is

$$ds^2 = (1/K\eta^2)(-d\eta^2 + d\mathbf{x}^2)$$

with $\mathbf{x} = (x_1, \ldots, x_{n-1})$. K is a measure of the radius of curvature related to the Ricci scalar by $R = -n(n-1)K$.

The Klein-Gordon equation with the "conformal term" included,

$$L\phi \equiv \left(\Box - m^2 - \frac{1}{4}\left(\frac{n-2}{n-1}\right)R\right)\phi = 0$$

becomes

$$\left(\frac{\partial^2}{\partial\eta^2} - \nabla^2 + \frac{m^2}{K\eta^2}\right)\phi = 0$$

with the replacement $\phi \to (K^{1/2}\eta)^{-1+n/2}\phi$.

Choose new variables $u = \eta - \eta'$, $v = \eta + \eta'$. We know that the Feynman function G is symmetric under $\eta \leftrightarrow \eta'$. Therefore, we decompose L into a symmetric and antisymmetric part under $u \to -u$, $L = L_+ + L_-$ and require

$$L_- G = 0.$$

Explicitly this is

$$\left[\frac{\partial^2}{\partial u\,\partial v} - \frac{4uvm^2}{K(u^2-v^2)}\right] G = 0$$

by seeking a solution of the form $P(u^2-v^2)Q(v)$ we find

$$\exp[i\lambda(u^2-v^2) + 2i\lambda v^2][\lambda(u^2-v^2)]^{1/2} Z_{i\alpha}[\lambda(u^2-v^2)]$$

is a solution for arbitrary λ, with Z a solution of Bessel's equation and

$$\alpha^2 = m^2/K - \tfrac{1}{4}.$$

Let $\xi = v^2 - u^2$, $\chi = v^2 + u^2$ and seek a solution of $L_+ G = -\delta$ of the form

$$G(\xi,\chi,r) = \int f(r,s)\exp(-is\chi)(s\xi)^{1/2} Z_{i\alpha}(s\xi)\,ds \tag{33}$$

with $r^2 = (\mathbf{x}-\mathbf{x}')^2$. Explicitly L_+ is

$$L_+ = 4\chi\frac{\partial^2}{\partial\chi^2} + 4\chi\frac{\partial^2}{\partial\xi^2} + 8\xi\frac{\partial^2}{\partial\chi\,\partial\xi} + 4\frac{\partial}{\partial\chi} + \frac{4m^2\chi}{K\xi^2} - \nabla^2. \tag{34}$$

Apply this to G, using $L_- G = 0$, replacing derivatives with respect to ξ by derivatives with respect to s, and integrating by parts yields, as in Sec. 2,

$$\begin{aligned} L_+ G = 4\int_C &\left\{-2is^{3/2}f + 2i\frac{\partial}{\partial s}(s^{5/2}f) - \frac{s^{1/2}}{4}\nabla^2 f\right\}\xi^{1/2} \\ &\times\exp(-is\chi)Z_{i\alpha}(s\xi)\,ds + 4[-2is^{5/2} \\ &\times\exp(-is\chi)Z_{i\alpha}f\xi^{1/2}]_C. \end{aligned} \tag{35}$$

For the integrand to vanish in n dimensions we require

$$-2is^{3/2}f + 2i\frac{\partial}{\partial s}(s^{5/2}f) = \frac{s^{1/2}}{4r^{n-2}}\frac{\partial}{\partial r}\left(r^{n-2}\frac{\partial f}{\partial r}\right).$$

The appropriate solution is found to be

$$f(r,s) = s^{n/2-2}\exp(2isr^2)$$

in order to satisfy the analylicity properties below.

We now have to choose Z and C, such that the integrated term in (35) gives rise to a δ function. It is convenient to use a representation of the δ function in terms of the geodetic distance $(2\sigma)^{1/2}$ in the $(n+1)$ dimensional Minkowski space, in which the manifold can be embedded,[9] rather than the geodetic distance as measured in the manifold. Calculation of σ in our coordinates gives $1 - \frac{1}{2}K\sigma = (\chi - 2r^2)\xi^{-1}$, which shows that we require a representation of the δ function as the boundary value of a function analytic in the lower half $(\chi - 2r^2)\xi^{-1}$ plane of the form

$$\delta^{(n)}(\eta,\eta',r) = \lim_{\epsilon\to 0}\frac{i\exp(-in\pi/4)}{(4\pi\epsilon)^{n/2}}\exp[i(r^2 - (\eta-\eta')^2)/4\epsilon].$$

We write $Z_{i\alpha} = aH^{(1)}_{i\alpha} + abH^{(2)}_{i\alpha}$, since any Bessel function can be written in this form, and choose $C:[0,\infty)$. The contribution from $s=0$ to the final term in (35) vanishes; for the contribution from $s\to\infty$ we use the asymptotic form $(\xi > 0)$

$$H^{(1)}_{i\alpha} \sim (2/\pi s\xi)^{1/2}\exp[i(s\xi - \pi/4) + \pi\alpha/2],$$
$$H^{(2)}_{i\alpha} \sim (2/\pi s\xi)^{1/2}\exp[-i(s\xi - \pi/4) - \pi\alpha/2].$$

Putting $s = 1/2\epsilon$ we find

$$a = -\tfrac{1}{8}(2/\pi)^{(n-1)/2}\exp[-i(n-1)\pi/4 - \pi\alpha/2].$$

The coefficient b is completely undetermined since it multiplies a term which is singular outside the physical region $\eta, \eta' > 0$. In fact, this term represents a δ-function singularity at the point $(-\eta', \mathbf{x}')$ antipodal to $(\eta', \mathbf{x}')$, with arbitrary complex charge b. Taking $b=0$, we obtain

$$\begin{aligned} G(x,x') = -\tfrac{1}{8}(2/\pi)^{(n-1)/2}&\exp[-i(n-1)\pi/4 - \pi\alpha/2] \\ &\times\int_0^\infty s^{n/2-2}\exp(i2sr^2 - is\chi)H^{(1)}_{i\alpha}(s\xi)\,ds. \end{aligned}$$

Using the relation

$$H_{i\alpha}^{(1)}(s\xi)=\left(\frac{2}{\pi s\xi}\right)^{1/2}\exp[i(s\xi-\pi/4)+\pi\alpha/2]$$

$$\times{}_2F_0\left(\tfrac{1}{2}+i\alpha\,,\,\tfrac{1}{2}-i\alpha;\,\frac{1}{2i\xi s}\right)$$

and Goldstein's integral,[10] we obtain

$$G(\eta,\eta',r)=\frac{iK^{n/2-1}}{(4\pi)^{n/2}}\left(\frac{\xi K}{4}\right)^{-n/2+1}\Gamma\left(\frac{n-1}{2}+i\alpha\right)\Gamma\left(\frac{n-1}{2}-i\alpha\right)$$

$$\times{}_2F_1\left(\frac{n-1}{2}+i\alpha,\,\frac{n-1}{2}-i\alpha;\,\frac{n}{2};\,\frac{\chi-2r^2}{\xi}\right).$$

Transforming back to the original field variables $(K^{1/2}\eta)^{n/2-1}\phi$ removes the term $(\xi K/4)^{-n/2+1}$ and we obtain the Feynman function appropriate to the complete de-Sitter manifold.[11]

Taking $b=1$ we obtain again the result for the half-space calculation of our previous paper.[11]

For general b, the addition to the de-Sitter Green's function in four dimensions, and with coincident points, is

$$\Delta G=\frac{biK}{(4\pi)^2}\Gamma(\tfrac{3}{2}+i\alpha)\Gamma(\tfrac{3}{2}-i\alpha),\quad \alpha=\left(\frac{m^2}{K}-\frac{1}{4}\right)^{1/2}.$$

We have passed directly to four dimensions here since this term is finite.

Using

$$\frac{\partial\mathcal{L}_1}{\partial m^2}=\frac{i}{2}g^{1/2}G(x,x),$$

we obtain the addition $\Delta\mathcal{L}_1$ to the de Sitter effective Lagrangian:

$$\Delta\mathcal{L}_1=-\frac{b}{2}\frac{1}{(4\pi)^2}\int\Gamma(\tfrac{3}{2}+i\alpha)\Gamma(\tfrac{3}{2}-i\alpha)d\left(\frac{m^2}{K}\right).$$

Asymptotically, for $m^2/K\to\infty$, we use

$$\Gamma(\tfrac{3}{2}+i\alpha)\Gamma(\tfrac{3}{2}-i\alpha)\sim 2\pi\alpha^2\exp(-\pi\alpha)\quad\text{as}\quad\alpha\to\infty$$

to find

$$\mathrm{Im}\,\Delta\mathcal{L}_1\sim\mathrm{Im}\left\{\frac{b}{4\pi^2}\left(\frac{m^2}{K}\right)^{3/2}\exp\left(-\frac{\pi m}{\sqrt{K}}\right)\right\}. \tag{36}$$

Alternatively, for $m^2/K\to 0$, we have $i\alpha\sim\frac{1}{2}-m^2/K$ and expanding in powers of m^2/K gives

$$\mathrm{Im}\,\Delta\mathcal{L}_1\sim\mathrm{Im}\{-(b/2)[m^2/(4\pi)^2K]\}. \tag{37}$$

We require $\mathrm{Im}\,\Delta\mathcal{L}_1=\mathrm{Im}\,\mathcal{L}_1>0$ in order that the vacuum to vacuum transition probability, $\exp[-2\,\mathrm{Im}\int\mathcal{L}_1 dx]$, be less than unity. If (i) m is real and (ii) b is independent of m^2/K, (36) and (37) cannot both be positive. It then follows that b must be real and there exists no particle production. However, both of these conditions can be broken: (i) the "conformal" addition to the action $-\frac{1}{12}R\phi^2=K\phi^2$ is of the same form as the mass term, so should be regarded as defining an effective squared mass $m^2=m^2+2K$ which can be positive for negative m^2;

(ii) the m^2/K dependence of b is entirely arbitrary. These conditions may be exploited to construct a $\Delta\mathcal{L}_1$ such that

$$\mathrm{Im}\int\Delta\mathcal{L}_1 dx>0$$

in which models this may be interpreted as particle production. These models are of course highly contrived, but illustrate the need for further physical input, for example, the asymptotic state of the system. Such considerations would be of importance in realistic cosmological models.[12]

APPENDIX A

In this appendix, we calculate the Bogoliubov coefficients α and β of Sec. 5.

We have

$$u_{K\mathbf{k}}(x)=[2\sqrt{\pi}/(2\pi)^{n/2}]\exp(-i\mu t\cosh K)$$

$$\times\sin(x\mu\sinh K)\exp(i\mathbf{k}\cdot\mathbf{x})$$

and

$$v_{\nu\mathbf{k}}(x)=2[(\sinh\nu\pi)^{1/2}/(2\pi)^{n/2}]\exp(-i\nu t)K_{i\nu}(\mu\xi)$$

$$\times\exp(i\mathbf{k}\cdot\mathbf{x})$$

with

$$\mu=(\mathbf{k}^2+m^2)^{1/2}.$$

In matrix notation α and β are determined by

$$v(x)=\alpha u(x)+\beta u^*(x). \tag{A1}$$

Setting $\alpha_{\nu\mathbf{k},K\mathbf{k}'}=\alpha_{\nu K}\delta(\mathbf{k}-\mathbf{k}')$

and

$$\beta_{\nu\mathbf{k},K\mathbf{k}'}=\beta_{\nu K}\delta(\mathbf{k}+\mathbf{k}'),$$

where $\alpha_{\nu K}$ and $\beta_{\nu K}$ are the Bogoliubov coefficients that obtain in two dimension, (A1) reduces to

$$(\sinh\nu\pi)^{1/2}\exp(-i\nu\tau)K_{i\nu}(\mu\xi)$$

$$=\sqrt{\pi}\int_0^\infty dK\sin(x\mu\sinh K)$$

$$\times[\alpha_{\nu K}\exp(-i\mu t\cosh K)+\beta_{\nu K}\exp(i\mu t\cosh K)]. \tag{A2}$$

Since the functions u and v are solutions of the Klein-Gordon equation, (A2) will hold for all time if (A2) and its derivative with respect to τ hold for $\tau=0$. Recalling that $x=\xi\cosh\tau$ and $t=\xi\sinh\tau$ in (A1) we obtain the following equations:

$$(\sinh\nu\pi)^{1/2}K_{i\nu}(\mu\xi)=\pi^{1/2}\int_0^\infty dK\sin(\xi\mu\sinh K)(\alpha+\beta)_{\nu K},$$

$$\nu(\sinh\nu\pi)^{1/2}K_{i\nu}(\mu\xi)=\pi^{1/2}\int_0^\infty dK\sin(\xi\mu\sinh K)\xi\mu$$

$$\times\cosh K(\alpha-\beta)_{\nu K}.$$

These integrals may be inverted and the resulting expressions evaluated using the formula

$$\int_0^\infty d\xi\,\xi^\lambda K_{i\nu}(\mu\xi)\sin(\xi\mu\sinh K)$$

$$=\frac{2^\lambda}{\mu^{\lambda+1}}\sinh K\,\Gamma\left(\frac{2+\lambda+i\nu}{2}\right)\Gamma\left(\frac{2+\lambda-i\nu}{2}\right)$$

$$\times{}_2F_1\left(\frac{2+\lambda+i\nu}{2},\frac{2+\lambda-i\nu}{2};\,\frac{3}{2};\,-\sinh^2K\right).$$

When $\lambda=-1$ the hypergeometric function simplifies to $\sin\nu K/\nu\sinh K$ and when $\lambda=0$ it simplifies to $\sin\nu K/\nu\sinh K\cosh K$.

Thus, we obtain finally

$$\alpha_{\nu K}=[1-\exp(-2\nu\pi)]^{-1/2}(2/\pi)^{1/2}\sin\nu K,$$

$$\beta_{\nu K} = [\exp(2\nu\pi) - 1]^{-1/2}(2/\pi)^{1/2}\sin\nu K$$

which establishes (32).

APPENDIX B

This appendix deals with the evaluation of integral (34). We have

$$\begin{aligned}\Delta g_{\text{in}} = &-\frac{i}{4\pi}\int_{-\infty}^{\infty} dK \int_{-\infty}^{\infty} dK' \\ &\times[(\pi^2 + (K-K')^2)^{-1} - (\pi^2 + (K+K')^2)^{-1}] \\ &\times \exp[i\mu(x\sinh K + x'\sinh K' + t\cosh K + t'\cosh K')] \\ = &\frac{i}{8\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [(\lambda^2+\pi^2)^{-1} - (\lambda'^2+\pi^2)^{-1}] \\ &\times \exp\left[i\mu\left(x\sinh\left(\frac{\lambda+\lambda'}{2}\right) + x'\sinh\left(\frac{\lambda-\lambda'}{2}\right)\right.\right. \\ &\left.\left. + t\cosh\left(\frac{\lambda+\lambda'}{2}\right) + t'\cosh\left(\frac{\lambda-\lambda'}{2}\right)\right)\right],\end{aligned}$$

where we have set

$$\lambda = K + K', \quad \lambda' = K - K'.$$

There are four cases to be considered according as x and x' are located in I or P.

In the first instance, let us take both x and x' in I; then we may set

$$x = \xi\cosh\tau, \quad x' = \xi'\cosh\tau',$$

$$t = \xi\sinh\tau, \quad t' = \xi'\sinh\tau',$$

and

$$\begin{aligned}\Delta g_{\text{in}} = &\frac{i}{8\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [(\lambda^2+\pi^2)^{-1} - (\lambda'^2+\pi^2)^{-1}] \\ &\times \exp\left\{i\mu\left[\xi\sinh\left(\frac{\lambda+\lambda'}{2}+\tau\right) + \xi'\sinh\left(\frac{\lambda-\lambda'}{2}+\tau'\right)\right]\right\}.\end{aligned} \tag{B1}$$

In the λ integration displace the contour as shown in Fig. 9. This yields

$$\begin{aligned}\Delta g_{\text{in}} = &\frac{i}{8\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' [(\lambda - i\epsilon)(\lambda+2\pi i)^{-1} - (\lambda'^2+\pi^2)^{-1}] \\ &\times \exp\left\{-\mu\left[\xi\cosh\left(\frac{\lambda+\lambda'}{2}+\tau\right) + \xi'\cosh\left(\frac{\lambda-\lambda'}{2}+\tau'\right)\right]\right\} \\ = &\frac{i}{4\pi}\int_{-\infty}^{\infty} d\lambda(\lambda - i\epsilon)^{-1}(\lambda+2\pi i)^{-1}\int_0^{\infty}\frac{d\Lambda'}{\Lambda'} \\ &\times \exp\left\{-\frac{\mu}{2}\left[\Lambda' + \frac{1}{\Lambda'}(\xi^2+\xi'^2+2\xi\xi'\cosh(\lambda+\tau+\tau'))\right]\right\} \\ &-\frac{i}{4\pi}\int_{-\infty}^{\infty} d\lambda'(\lambda'^2+\pi^2)^{-1}\int_0^{\infty}\frac{d\Lambda}{\Lambda} \\ &\times \exp\left\{-\frac{\mu}{2}\left[\Lambda + \frac{1}{\Lambda}(\xi^2+\xi'^2+2\xi\xi'\cosh(\lambda+\tau-\tau'))\right]\right\},\end{aligned} \tag{B2}$$

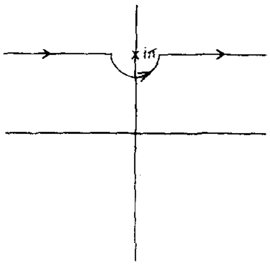

FIG. 9. Deformation of the contour of integration appropriate to (B1).

where we have employed the substitutions

$$\Lambda = \exp(\lambda/2)[\xi\exp(\lambda'/2+\tau) + \xi'\exp(-\lambda'/2+\tau')],$$

$$\Lambda' = \exp(\lambda'/2)[\xi\exp(\lambda/2+\tau) + \xi'\exp(-\lambda/2-\tau')].$$

Therefore,

$$\begin{aligned}\Delta g_{\text{in}} = (i/2\pi)\int_{-\infty}^{\infty} d\lambda[&(\lambda+\tau'+i\epsilon)^{-1}(\lambda+\tau'-2\pi i)^{-1} \\ &-((\lambda-\tau')^2+\pi^2)^{-1}]K_0(\mu\gamma_1)\end{aligned}$$

with

$$\gamma_1 = [\xi^2 + \xi'^2 + 2\xi\xi'\cosh(\lambda-\tau)]^{1/2}.$$

Now

$$\Delta G_{\text{in}} = \int\frac{d\mathbf{k}}{(2\pi)^{n-2}}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\Delta g_{\text{in}}$$

and the integral may be performed as in Sec. 3 to yield

$$\begin{aligned}\Delta G_{\text{in}} = \int_{-\infty}^{\infty} d\lambda[&(\lambda+\tau'+i\epsilon)^{-1}(\lambda+\tau'-2\pi i)^{-1} \\ &-((\lambda-\tau')^2+\pi^2)^{-1}]G_0(\gamma)\end{aligned}$$

with γ as in Sec. 3.

If $x' \in I$ but $x \in P$ then we may set

$$x = -\xi\sinh\tau, \quad x' = \xi'\cosh\tau',$$

$$t = -\xi\cosh\tau, \quad t' = \xi'\sinh\tau'.$$

This parametrization may be obtained from the previous case by making the replacements

$$\xi \to i\xi$$

and

$$\tau \to \tau + i\pi/2.$$

In (B1) we may make the replacements

$$\xi \to \xi\exp(i\theta),$$

$$\tau \to \tau + i\theta$$

and continue analytically from $\theta = 0$ to $\theta = \pi/2$.

The result has the same form as (B2) except that now

$$\begin{aligned}\gamma^2 &= -\xi^2\exp(-2i\epsilon) + \xi'^2 - 2\xi\xi'\exp(-i\epsilon)\sinh(\tau-\lambda-i\epsilon) \\ &= -\xi^2 + \xi'^2 - 2\xi\xi'\sinh(\tau-\lambda) + i\epsilon[2\xi^2 + 2\xi\xi'\sinh(\tau-\lambda)].\end{aligned} \tag{B3}$$

x and x' are null separated when

$$-\xi^2 + \xi'^2 - 2\xi\xi'\sinh(\tau-\lambda) = 0$$

so that (B3) is equivalent to

$$\gamma^2 = -\xi^2 + \xi'^2 - 2\xi\xi'\sinh(\tau-\lambda) + i\epsilon. \tag{B4}$$

We observe that our analytic continutation has resulted in the correct prescription for the Feynman function in integral (B2). γ as evaluated from (B4) is just the geodetic length between x and an image point located in II and labelled by ξ' and λ.

For the case $x' \in P$ and $x \in I$ we make the replacements

$$\xi' \to \xi' \exp(i\theta'),$$

$$\tau' \to \tau' + i\theta',$$

and

$$\lambda \to \lambda + i\pi/2$$

in (B1). Proceeding as before we obtain from (B2)

$$\Delta G_{\text{in}}(x, x') = \int_{-\infty}^{\infty} d\lambda [((\lambda + \tau')^2 + \pi^2)^{-1} - ((\lambda - \tau')^2 + \pi^2)^{-1}] G_0(\gamma),$$

where we find in a manner analogous to the previous case that

$$\gamma^2 = \xi^2 - \xi'^2 + 2\xi\xi' \sinh(\tau - \lambda) + i\epsilon$$

is seen to be the geodetic length between x and an image point located in F and labelled by ξ' and λ.

The final case when both x and x' lie in P is dealt with by making the simultaneous replacements

$$\xi \to \xi \exp(i\theta), \quad \xi' \to \xi' \exp(i\theta'),$$

$$\tau \to \tau + i\theta, \qquad \tau' \to \tau' + i\theta'$$

in (B1) and continuing both θ and θ' to $\pi/2$. This gives an expression of the same form as (B5) with

$$\gamma^2 = -\xi^2 - \xi'^2 - 2\,\xi\xi' \cosh(\tau - \lambda),$$

i.e., γ is the geodetic length between x and an image point located in F labelled by ξ' and λ.

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Systems of differential inequalities and stochastic differential equations. III

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Consider the system of stochastic differential equations $x'(t,\omega)=f(t,x(t,\omega),\omega)$, $x(t_0,\omega)=x_0(\omega)$, where $f(t,x(t,\omega),\omega)$ is a product measurable n-dimensional random vector function whenever $x(t,\omega)$ is a product measurable random function, and it satisfies the desired regularity conditions to ensure the existence of solution process. By developing systems of random differential inequalities, a very general comparison theorem in the framework of a vector Lyapunov function is developed, and furthermore sufficient conditions are given for the stability of solutions in probability, in the mean and with probability one.

1. INTRODUCTION

The stability analysis of stochastic differential systems of Itô type and differential systems with Markov coefficients has been investigated by several workers, and it has been documented in Refs. 1–3. However, the stability analysis of a system of differential equations under nonwhite excitations is very far from the saturation state. In fact, most of the stability study is devoted to linear systems.[3–9] Recently, the stability analysis has been extended to nonlinear random systems by Khas'minskii[3,10] in the framework of Lyapunov's second method. A good deal of stability results have been surveyed in a recent monograph by Morozan.[3] The stability results for random differential systems[3] are centered around either the use of the single or scalar Lyapunov function, or the use of the variation of constants formula.

Very recently, by developing very general comparison theorems[11–15] for Itô type stochastic differential equations and differential systems with Markov coefficients in the context of a single as well as a vector Lyapunov functions and the theory of differential inequalities, sufficient conditions are given for the stability and boundedness of solutions of these stochastic differential systems in a systematic and unified way. Furthermore, very recently, the comparison theorems in Refs. 13 and 14 have been utilized by Ladde and Šiljak[16,17] to study the connective stability of the large-scale stochastic systems in engineering and ecology.

In this paper, we develop the theory of systems of random differential inequalities, and obtain a very general comparison theorem in the framework of a vector Lyapunov function and the systems of random differential inequalities. As indicated above, these extensions have several advantages over a single Lyapunov function. In particular, a system may be unstable according to a single Lyapunov function approach, but it may be stable in the context of a vector Lyapunov function approach.[13,18] In addition, very recently,[16] it has been demonstrated that the concept of vector Lyapunov function and the theory of differential inequalities seems to be a promising tool for undertaking the study of "complexity vs stability" problem in the model ecosystems.

This paper is organized as follows:

In Sec. 2, depending on the mode of convergence, namely in probability, in the mean and with probability one, we define various notions of stability. In Sec. 3, we formulate the basic theory of systems of random differential inequalities. These results include the deterministic results[18] as special cases. In Sec. 4, we develop a very general comparison theorem for random differential systems based on the vector Lyapunov function and the systems of random differential inequalities. In Sec. 5, we give sufficient conditions for stability of solutions in probability, in the mean and in the almost surely sample sense or with probability one. These results include the earlier results in Ref. 3 as special cases. Finally, examples are given to illustrate the usefulness of our results.

2. NOTATIONS AND DEFINITIONS

Let R^n denote the n-dimensional Euclidean space with a convenient norm $\|\cdot\|$. We also denote by the same symbol $\|\cdot\|$ the corresponding norm of a matrix. Let R_+ and R denote the nonnegative real and real lines, respectively. For $\infty \geq \rho > 0$, $D=D(0,\rho,R^n)=\{x\in R^n: \|x\| < \rho\}$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Let $S(R^n)$ denote the set of random n vectors defined on $(\Omega,\mathcal{F},P)$ into R^n. For $x\in S(R^n)$, the qth moment of x is defined by $E(\|x\|^q)=\int_\Omega \|x(\omega)\|^q P(d\omega)$, $0<q<\infty$, and for $1\leq q<\infty$ let $L^q(\Omega)$ be the space of n-dimensional random vectors with the norm $\|x\|_q=[E(\|x\|^q)]^{1/q}$. Let $AC[R_+,S(R^n)]$ denote the set of all almost surely absolutely sample continuous random functions defined on R_+ into R^n. For $\rho>0$, $D(S(R^n))=D(0,\rho,S(R^n))=\{x\in S(R^n) = \|x(\omega)\|<\rho$ with probability $1\}$. In this paper we shall consider p, q such that $1\leq \rho \leq q\leq\infty$. We shall mean by $M[R_+\times D,S(R^n)]$ the class of random functions $f(t,x,\omega)$ defined on $R_+\times D\times\Omega$ into R^n such that $f(t,x(t,\omega),\omega)$ is product measurable whenever $x(t,\omega)$ is product measurable.

Consider the system of stochastic differential equations of the type

$$x'(t,\omega)=f(t,x(t,\omega),\omega),\quad x(t_0,\omega)=x_0(\omega), \tag{2.1}$$

where $x\in R^n$, $f\in M[R_+ x D, S(R^n)]$, and f is smooth enough to guarantee the existence of a sample solution $x(t,\omega) = x(t,t_0,x_0,\omega)$ of (2.1) for $t\geq t_0$. For existence and uniqueness theorems, see Refs. 3 and 19.

We shall assume that $f(t,0,\omega)\equiv 0$, with probability 1 (abbreviated w.p. 1), so that the system (2.1) possesses the trivial solution $x(t)\equiv 0$, w.p. 1.

Now, depending on the mode of convergence in the probabilistic analysis, we shall formulate some definitions of stability.

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Definition 2.1: The trivial solution of (2.1) is said to be

(SP_1) *stable in probability,* if for each $\epsilon > 0$, $\eta > 0$, $t_0 \in R_+$, there exists $\delta = \delta(t_0, \epsilon, \eta) > 0$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$$

implies

$$P\{\omega : \|x(t, \omega)\| \geq \epsilon\} < \eta, \quad t \geq t_0;$$

(SP_2) *asymptotically stable in probability,* if it is stable in probability and, if for any $\epsilon > 0$, $\eta > 0$, $t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon, \eta)$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$$

implies

$$P\{\omega : \|x(t, \omega)\| \geq \epsilon\} < \eta, \quad t \geq t_0 + T;$$

(SM_1) *stable in the mean,* if for each $\epsilon > 0$, $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \epsilon)$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta$$

implies

$$E[\|x(t, \omega)\|] < \epsilon, \quad t \geq t_0;$$

(SM_2) *asymptotically stable in the mean,* if it is stable in the mean and, if for any $\epsilon > 0$, $t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta_0$$

implies

$$E[\|x(t, \omega)\|] < \epsilon, \quad t \geq t_0 + T;$$

(SS_1) *stable with probability* 1 (or almost surely sample stable), if for $\epsilon > 0$, $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \epsilon)$ such that the inequality

$$\|x_0(\omega)\| \leq \delta \text{ w.p. 1}$$

implies

$$\|x(t, \omega)\| < \epsilon, \quad t \geq t_0 \text{ w.p. 1};$$

(SS_2) *asymptotically stable with probability* 1 (or almost surely sample asymptotically stable), if it is stable with probability 1 and, if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $0 < \delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality

$$\|x_0(\omega)\| \leq \delta \text{ w.p. 1}$$

implies

$$\|x(t, \omega)\| < \epsilon, \quad t \geq t_0 + T \text{ w.p. 1}.$$

Definition 2.2: The trivial solution of (2.1) is said to be:

(USP_1) uniformly stable in probability, (USM_1) uniformly stable in the mean, and (USS_1) uniformly stable with probability 1, if the δ's in Definition 2.1 (SP_1), (SM_1), and (SS_1) are independent of t_0, respectively;

(USP_2) uniformly asymptotically stable in probability, (USM_2) uniformly asymptotically stable in the mean, and (USS_2) uniformly asymptotically stable w.p. 1, if (SP_1), (SM_1), and (SS_1) hold, and the corresponding δ's and T's in Definition 2.1 (SP_2), (SM_2), and (SS_2) are independent of t_0, respectively.

Based on Definitions 2.1 and 2.2, one can formulate other definitions of stability and boundedness,[13,14] analogously.

Consider now the auxiliary stochastic differential system

$$u'(t, \omega) = g(t, u(t, \omega), \omega), \quad u(t_0, \omega) = u_0(\omega), \qquad (2.2)$$

where $g \in LC[R_+ \times R^m, S(R^m)]$, $LC[R_+ \times R^m, S(R^m)]$ stands for the class of random functions $g(t, u, \omega)$ defined on $R_+ \times R^m \times \Omega$ into R^m such that $g(t, u, \omega)$ satisfies the Caratheodory condition in (t, u) for almost all $\omega \in \Omega$, i.e., $g(t, u, \omega)$ is continuous in u for each $t \in R_+$ and Lebesgue measurable in t for each fixed u with probability 1, and there exists a product measurable random function K: $R_+ \times \Omega \to R_+$ which is summable on R_+ with probability 1, such that $\|g(t, u, \omega)\| \leq K(t, \omega)$ for $\|u\| \leq \rho$, $0 < \rho < \infty$ w.p. 1; $g(t, u, \omega)$ is quasimonotone nondecreasing in u for fixed $t \in R_+$ w.p. 1. Under these conditions, existence of maximal and minimal solutions with probability one can be shown analogous to the deterministic case[18] with simple modifications. Let $u(t, \omega) = u(t, t_0, u_0, \omega)$ be any solution of (2.2).

Relative to auxiliary differential system (2.2), we need the corresponding definitions in our discussion that may be defined analogously. For example, the definition of stable in probability (SP_1^*) runs as follows:

Definition 2.3: The trivial solution of (2.2) is said to be stable in probability, if given $\epsilon > 0$, $\eta > 0$, $t_0 \in R_+$, there exists $\delta = \delta(t_0, \epsilon, \eta)$ such that

$$P\left\{\omega : \sum_{i=1}^{m} u_{i0}(\omega) > \delta\right\} < \eta$$

implies

$$P\left\{\omega : \sum_{i=1}^{m} u_i(t, \omega) \geq \epsilon\right\} < \eta, \quad t \geq t_0.$$

Definition 2.4: A function $b(r)$ is said to belong to the class $\mathcal{K}$ if $b \in C[R_+, R_+]$, $b(0) = 0$, $b(r)$ is strictly increasing in r.

Definition 2.5: A function $b(r)$ is said to belong to the class $V\mathcal{K}$ if $b \in C[R_+, R_+]$, $b(0) = 0$, $b(r)$ is a convex and strictly increasing in r.

Definition 2.6: A function $a(t, r)$ is said to belong to the $C\mathcal{K}$ if $a \in C[R_+ \times R_+, R_+]$, $a(t, 0) \equiv 0$, and $a(t, r)$ is concave and increasing in r for each fixed $t \in R_+$.

Definition 2.7: Let G be a function defined on R^n into R^m. The function G is said to be convex if each component G_i of G is convex for $1 \leq i \leq m$, and G is said to be concave if $-G_i$ is convex.

In order to avoid monotonicity, hereafter, it will be understood, unless otherwise specified, that all equalities, inequalities, and relations that involve the random processes will hold with probability 1.

3. RANDOM DIFFERENTIAL INEQUALITIES

In this section, we shall establish the result that will be widely useful in the qualitative analysis of random differential systems of the type (2.1).

Theorem 3.1: Assume that

(i) $g \in LC[R_+ \times R^m, S(R^m)]$, $g(t, u, \omega)$ is quasimonotone nondecreasing in u for each fixed $t \in R_+$, with probability 1, and $r(t, \omega) = r(t, t_0, u_0, \omega)$ be the maximal solution process of the system of random differential equations (2.2) existing for $t \geq t_0$;

(ii) $m \in AC[R_+, S(R^m)]$, and $m(t, \omega)$ satisfies

$$m'(t, \omega) \leq g(t, m(t, \omega), \omega) \tag{3.1}$$

almost everywhere on $(t, \omega) \in R_+ \times \Omega$.

Then

$$m(t_0, \omega) \leq u_0(\omega) \quad \text{w.p. } 1 \tag{3.2}$$

implies

$$m(t, \omega) \leq r(t, t_0, u_0, \omega), \quad t \geq t_0 \quad \text{w.p. } 1. \tag{3.3}$$

Proof: For any $i \in I = \{1, 2, \ldots, m\}$, we define the function

$$\bar{g}_i(t, u, \omega) = g_i(t, \bar{u}, \omega), \tag{3.4}$$

where $u \in R^m$, $m(t, \omega) \leq \bar{u}$, and for $j \in I$

$$\bar{u}_j = \begin{cases} u_j, & \text{if } m_j(t, \omega) \leq u_j, \\ m_j(t), & \text{if } m_j(t, \omega) > u_j. \end{cases} \tag{3.5}$$

It is easy to observe that $\bar{g}(t, u, \omega) \in LC[R_+ \times R^m, S(R^m)]$ and satisfies the quasimonotone nondecreasing property in u, for fixed $t \in R_+$, w.p. 1. Let $\bar{r}(t, \omega) = \bar{r}(t, t_0, u_0, \omega)$ be the maximal solution of

$$u'(t, \omega) = \bar{g}(t, u(t, \omega), \omega), \quad u(t_0, \omega) = u_0(\omega). \tag{3.6}$$

From (3.4) and (3.5), one can easily see that $\bar{r}(t, \omega) = r(t, \omega)$, whenever $m(t, \omega) \leq \bar{r}(t, \omega)$. In view of this, the validity of the inequality (3.3) is immediate, if we can show that

$$m(t, \omega) \leq \bar{r}(t, \omega), \quad t \geq t_0. \tag{3.7}$$

If (3.7) is false, there exists an index $i \in I$, t_1 and t_2 with $t_0 < t_1 < t_2$, and $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that

(a) $m_i(t_1, \omega) = \bar{r}_i(t_1, \omega), \quad \omega \in \Omega_1$,

(b) $m_i(t, \omega) > \bar{r}_i(t, \omega), \quad t \in (t_1, t_2)$ and $\omega \in \Omega_1$,

(c) $m_j(t, \omega) \leq \bar{r}_j(t, \omega)$, w.p. 1, $t \in [t_1, t_2)$ for all $j \in I$.

For almost every $t \in (t_1, t_2)$, and $\omega \in \Omega_1$, we obtain from (3.1) and (3.6) the inequality

$$m_i'(t, \omega) - \bar{r}_i'(t, \omega) \leq g_i(t, m(t, \omega), \omega) - \bar{g}_i(t, \bar{r}(t, \omega), \omega). \tag{3.8}$$

From (3.4), (3.5), (a)—(c) and quasimonotone nondecreasing property of g give

$$m_i'(t, \omega) - \bar{r}_i'(t, \omega) \leq 0,$$

which implies

$$\int_{t_1}^{t} m_i'(s, \omega) \leq \bar{r}_i(t, \omega) - \bar{r}_i(t_1, \omega), \quad t \in (t_1, t_2) \text{ and } \omega \in \Omega_1. \tag{3.9}$$

Since $m(t) \in LC[R_+, S(R^m)]$, we have

$$\int_{t_1}^{t} m_i'(s, \omega) = m_i(t, \omega) - m_i(t_1, \omega). \tag{3.10}$$

From (3.9), (3.10), (a), and (c), we have the contradiction

$$m_i(t, \omega) \leq \bar{r}_i(t, \omega) < m_i(t, \omega)$$

which establishes the relation (3.7). Thus the proof of the theorem is complete.

Remark 3.1: Theorem 3.1 is analogous to deterministic Corollary 1.7.1 in Ref. 18. Furthermore, it is direct extension of the deterministic Theorem 1.10.1[18] that is obtained in the context of differential inequalities of Caratheodory type.

Remark 3.2: If, in Theorem 3.1, the inequalities (3.1) and (3.2) are reversed, then the conclusion (3.3) is to be replaced by

$$m(t, \omega) \geq \rho(t, \omega), \quad t \geq t_0,$$

where $\rho(t, \omega) = \rho(t, t_0, u_0, \omega)$ is the minimal solution process of (2.2).

4. COMPARISON THEOREMS

In this section, we shall develop some results which furnish a very general comparison theorem for random differential systems. This is achieved by employing the system of random differential inequalities that are developed in Sec. 3, and by introducing the concept of random vector Lyapunov function analogous to the earlier work.[13] These results play an important role not only in studying the qualitative behavior of (2.1), but also in studying the qualitative behavior of competitive processes in biological, physical and social sciences.

Let the function $V \in L[R_+ \times D, S(R^m)]$, where $L[R_+ \times D, S(R^m)]$ stands for collection of random functions $V(t, x, \omega)$ defined on $R_+ \times D \times \Omega$ into R^m such that $V(t, x, \omega)$ is locally Lipschitzian in $(t, x) \in R_+ \times D$ w.p. 1. We define a vector

$$D^+_{(2.1)} V(t, x, \omega) = \limsup_{h \to 0} (1/h)[V(t+h, x + hf(t, x, \omega), \omega) - V(t, x, \omega)] \tag{4.1}$$

for $(t, x) \in R_+ \times D$. Note that $D^+_{(2.1)} V(t, \partial, \omega)$ is a product measurable random vector in view of the assumptions on V.

From here on, we shall assume that the function g in (2.2) and the function V satisfy the following hypothesis:

(H_1) $g \in LC[R_+ \times R^m, S(R^m)]$, $g(t, u, \omega)$ is quasimonotone nondecreasing in u, for fixed $t \in R_+$, w.p. 1.

(H_2) Let $r(t, t_0, u_0, \omega) = r(t, \omega)$ be the maximal solution process of the auxiliary random system (2.2) existing for $t \geq t_0$.

(H_3) Assume that $g(t, 0, \omega) \equiv 0$ almost everywhere (a.e.) on $(t, \omega) \in R_+ \times \Omega$.

(H_4) $V \in L[R_+ \times D, S(R^m)]$, $V(t, x, \omega)$ is Lipschitzian in $(t, x) \in R_+ \times D$ w.p. 1. Furthermore, for $(t, x) \in R_+ \times D$,

$$D^+_{(2.1)} V(t, x, \omega) \leq g(t, V(t, x, \omega), \omega), \quad \text{w.p. } 1. \tag{4.2}$$

(H_5) Assume that the hypotheses (H_4) holds except that the inequality (4.2) is strengthened to

$$A(t) D^+_{(2.1)} V(t, x, \omega) + A'(t) V(t, x, \omega) \leq g(t, A(t) V(t, x, \omega), \omega), \tag{4.3}$$

where $D^+_{(2.1)} V(t, x, \omega)$ is defined in (4.1).

Here $A(t) = (a_{ij}(t))$, $a_{ij} \in L[R_+, S(R_+)]$; $A^{-1}(t)$ exists w.p. 1; $A^{-1}(t)A'(t)$ is (t, ω) measurable, and its off-diagonal elements are nonpositive w.p. 1, for $t \geq 0$.

(H_6) For $(t, x) \in R_+ \times D$,

$$b(\|x\|) \leq \sum_{i=1} V_i(t, x, \omega) \leq a(t, \|x\|) \tag{4.4}$$

where $b, a(t, \cdot) \in \mathcal{K}$.

(H_7) For $(t, x) \in R_+ \times D$, (4.4) holds with $b \in V\mathcal{K}$, $a \in C\mathcal{K}$.

(H_8) In addition to the hypothesis (H_6), we assume that $a(t, r) = a(r)$.

(H_9) Assume that (H_7) holds with $a(t, r) = a(r)$.

We shall state and prove the following comparison theorem.

Theorem 4.1: Let the hypotheses (H_1), (H_2), and (H_4) be satisfied. Assume that for any sample solution process $x(t, \omega) = x(t, t_0, x_0, \omega)$ of (2.1) with $x_0 \in D(S(R^n))$ and

$$V(t_0, x_0(\omega), \omega) \leq u_0(\omega). \tag{4.5}$$

Then

$$V(t, x(t, \omega), \omega) \leq r(t, t_0, u_0, \omega), \tag{4.6}$$

as long as $x(t, \omega) \in D(S(R^n))$ for $t \geq t_0$.

Proof: Set

$$m(t, \omega) = V(t, x(t, \omega), \omega), \quad m(t_0, \omega) = V(t_0, x_0(\omega), \omega). \tag{4.7}$$

Since $x(t, \omega)$ is a sample solution of (2.1) and $V \in L[R_+ \times D, S(R^m)]$, we conclude from the Rademacher's theorem[20] that $m(t, \omega)$ is sample absolutely continuous w.p. 1 for $t \geq t_0$. For small $h > 0$, we have

$$\begin{aligned} m(t+h, \omega) &- m(t, \omega) \\ &= V(t+h, x(t+h, \omega), \omega) - V(t, x(t, \omega), \omega) \\ &= V(t+h, x(t+h, \omega), \omega) - V(t+h, x(t, \omega) \\ &\quad + hf(t, x(t, \omega), \omega), \omega) + V(t+h, x(t, \omega) \\ &\quad + hf(t, x(t, \omega), \omega), \omega) - V(t, x(t, \omega), \omega) \\ &\leq K\|x(t+h, \omega) - x(t, \omega) - hf(t, x(t, \omega), \omega)\| \\ &\quad + V(t+h, x(t, \omega) + hf(t, x(t, \omega), \omega), \omega) \\ &\quad - V(t, x(t, \omega), \omega), \end{aligned}$$

where K is the local Lipschitz constant. This, together with (2.1), (4.2), and sample absolute continuity of $m(t, \omega)$ yields the inequality

$$m'(t, \omega) \leq g(t, m(t, \omega), \omega) \tag{4.8}$$

almost everywhere on $(t, \omega) \in R_+ \times \Omega$. From (4.5) and (4.7), $m(t_0, \omega) \leq u_0(\omega)$. Hence, by Theorem 3.1, we have

$$m(t, \omega) \leq r(t, t_0, u_0, \omega)$$

as long as $x(t, \omega) \in D(S(R^n))$ to the right of t_0. The proof is complete.

The following variant of Theorem 4.1 is often made useful in applications.

Theorem 4.2: Let the hypotheses of Theorem 4.1 hold except (H_4) is replaced by (H_5). Then, $V(t_0, x_0(\omega), \omega) \leq u_0(\omega)$ implies

$$V(t, x(t, \omega), \omega) \leq R(t, t_0, v_0, \omega) \tag{4.9}$$

as long as $x(t, \omega) \in D(S(R^n))$ where $R(t, t_0, v_0, \omega)$ is the maximal solution process of the auxiliary random differential system

$$v'(t, \omega) = A^{-1}(t)[-A'(t)v(t, \omega) + g(t, A(t)v(t, \omega), \omega)] \tag{4.10}$$

existing for $t \geq t_0$.

Proof: Set $W(t, x, \omega) = A(t)V(t, x, \omega)$. Because of (4.3), we have

$$\begin{aligned} D^+_{(2.1)} W(t, x, \omega) &= A(t)D^+_{(2.1)} V(t, x, \omega) + A'(t)V(t, x, \omega) \\ &\leq g(t, W(t, x, \omega), \omega). \end{aligned}$$

Now, by following the arguments used in Theorem 2.3 in Ref. 13 and Theorem 4.1, the proof of the theorem can be constructed, analogously.

Remark 4.1: Note that the comparison theorems in Refs. 12—14 are developed for Itô type stochastic differential systems. However, our present comparison theorems are for random differential systems in the context of random vector Lyapunov functions and systems of sample random differential inequalities.

5. STABILITY RESULTS

In this section, we employ the comparison theorems developed in the preceding section to study stability properties of the trivial solution of the random differential system (2.1).

The following result establishes the stability properties of (2.1) in the sense of probability.

Theorem 5.1: Let the hypotheses (H_1), (H_2), (H_3), (H_4), and (H_6) be satisfied. Assume that $f(t, o, \omega) \equiv 0$. Then,

(i) (SP_1^*) implies (SP_1),

(ii) (SP_2^*) implies (SP_2).

Proof: Let us prove the statement (i). Let $\eta > 0$, $0 < \epsilon < \rho$, and $t_0 \in R_+$ be given. Assume that (SP_1^*) holds. Then, $b(\epsilon)$, $\eta > 0$ and $t_0 \in R_+$, there exists a positive function $\delta_1 = \delta_1(t_0, \epsilon, \eta)$ such that

$$P\left\{\omega : \sum_{i=1}^{m} u_i(t, t_0, u_0, \omega) \geq b(\epsilon)\right\} < \eta, \quad t \geq t_0 \tag{5.1}$$

provided

$$P\left\{\omega : \sum_{i=1}^{m} u_{i0}(\omega) > \delta_1\right\} < \eta. \tag{5.2}$$

Let us choose $u_0 = (u_{10}, u_{20}, \ldots, u_{m0})$ so that $V(t_0, x_0(\omega), \omega) \leq u_0(\omega)$ and

$$\sum_{i=1}^{m} u_{i0}(\omega) = a(t_0, \|x_0(\omega)\|) \text{ for } x_0 \in D(S(R^n)). \tag{5.3}$$

Since $a(t_0, \cdot) \in \mathcal{K}$, we can find a $\delta = \delta(t_0, \epsilon, \eta)$ such that

$$P\{\omega : a(t_0, \|x_0(\omega)\|) > \delta_1\} = P\{\omega : \|x_0(\omega)\| > \delta\}. \tag{5.4}$$

Now, we claim that (SP_1) holds. Suppose that this claim is false. There would exist a solution process $x(t, \omega)$ of (2.1) with $P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$ and a $t_1 > t_0$ such that

$$P\{\omega : \|x(t_1, \omega)\| \geq \epsilon\} = \eta. \tag{5.5}$$

On the other hand, by Theorem 4.1, the inequality

$$V(t, x(t, \omega), \omega) \leq r(t, t_0, u_0, \omega) \tag{5.6}$$

is valid as long as $x(t, \omega) \in D(S(R^n))$. From (4.4) and (5.6), we have

$$b(\|x(t, \omega)\|) \leq \sum_{i=1}^{m} V_i(t, x(t, \omega), \omega) \leq \sum_{i=1}^{m} r_i(t, t_0, u_0, \omega). \tag{5.7}$$

The relations (5.1), (5.5), and (5.7) lead us to the contradiction

$$\eta = P\{\omega : \|x(t_1, \omega)\| \geq \epsilon\} = P\{\omega : b(\|x(t_1, \omega)\|) \geq b(\epsilon)\} = P\left\{\omega : \sum_{i=1}^{m} r_i(t, t_0, u_0, \omega) \geq b(\epsilon)\right\} < \eta,$$

thus proving statement (i).

To prove statement (ii), it is enough to prove that for any $\epsilon > 0$, $\eta > 0$, and $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon, \eta)$ such that $P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$ implies

$$P\{\omega : \|x(t, \omega)\| \geq \epsilon\} < \eta, \quad t \geq t_0 + T. \tag{5.8}$$

Assume that (SP_2^*) holds. Then given $b(\epsilon) > 0$, $\eta > 0$, and $t_0 \in R_+$, there exist numbers $\delta^0(t_0) = \delta^0$ and $T(t_0, \epsilon, \eta) = T > 0$ such that

$$P\left\{\omega : \sum_{i=1}^{m} u_i(t, t_0, u_0, \omega) \geq b(\epsilon)\right\} < \eta, \quad t \geq t_0 + T, \tag{5.9}$$

whenever

$$P\left\{\omega : \sum_{i=1}^{m} u_{i0}(\omega) > \delta^0\right\} < \eta.$$

As before, we choose u_0 so that (5.3) holds, and choose $\delta_0(t_0) = \delta_0 > 0$ such that

$$P\{\omega : a(t_0, \|x_0(\omega)\|) > \delta^0\} = P\{\omega : \|x_0(\omega)\| > \delta_0\}.$$

We claim that (5.8) holds. Otherwise, there exists a sequence $\{t_n\}$, $t_n \geq t_0 + T$, $t_n \to \infty$ as $n \to \infty$ such that for some solution process of (2.1) satisfying $P\{\omega : \|x_0(\omega\| > \delta_0\} < \eta$, it will satisfy the relation

$$P\{\omega : \|x(t_n, \omega)\| \geq \epsilon\} = \eta, \quad t_n \geq t_0 + T.$$

This together with (5.7) and (5.9) will establish the validity of (5.8). This completes the proof of the theorem.

Remark 5.1: If we replace the hypothesis (H_6) in Theorem 5.1 by the weaker hypothesis, namely,

$$(H_6^*) \quad \inf_{\substack{t>0 \\ \|x\|>\epsilon}} \left(\sum_{i=1}^{m} V_i(t, x, \omega)\right) = b(\epsilon), \quad \text{whenever } \epsilon > 0,$$

then the conclusions of Theorem 5.1 remain true. Under (H_6^*), the proof of Theorem 5.1 can be formulated analogously, except for a few modifications.

Remark 5.2: In the context of Remark 5.1, we note that the scalar version of Theorem 5.1 contains the results in Ref. 10 as special cases, whenever $f(t, x, \omega)$ in (2.1) is defined by $f(t, x, \omega) = F(t, x) + \sigma(t, x)y(t, \omega)$, where $F \in R^n$, σ is a $k \times n$ matrix, and $y \in M[R_+, S(R^k)]$ and it is sample continuous, and $\|y(t)\|$ satisfies the law of large numbers. In fact, under the hypotheses of Theorem 2.1 in Ref. 10, we have $g(t, u, \omega) = (-c_1 + Lc_2\|y(t)\|)u$ where c_1, L, and c_2 are defined in Ref. 10. Note that under the hypothesis on $\|y(t)\|$, c_1, L, and c_2, $u' = g(t, u, \omega)$ is stable in probability.

The following result establishes the stability properties of (2.1) in the sense of first moment.

Theorem 5.2: Assume that the hypotheses of the Theorem 5.1 holds except that (H_6) is replaced by (H_7). Then,

(i) (SM_1^*) implies (SM_1),

(ii) (SM_2^*) implies (SM_2).

Proof: First, we prove (i). Let $\rho > \epsilon > 0$, $t_0 \in R_+$ be given. Assume that (SM_1^*) holds. Then $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon)$ such that

$$\sum_{i=1}^{m} E[u_{i0}(\omega)] \leq \delta_1$$

implies

$$\sum_{i=1}^{m} E[u_i(t, t_0, u_0, \omega)] < b(\epsilon), \quad t \geq t_0. \tag{5.10}$$

We choose u_0 such that $V(t_0, x_0(\omega), \omega) \leq u_0(\omega)$ and

$$\sum_{i=1}^{m} E[u_{i0}(\omega)] = a(t_0, E[\|x_0(\omega)\|]) \quad \text{for } x_0(\omega) \in D. \tag{5.11}$$

Since $a(t_0, \cdot) \in K$, we can find a $\delta = \delta(t_0, \epsilon)$ such that

$$E[\|x_0(\omega)\|] \leq \delta \quad \text{implies} \quad a(t_0, E[\|x_0(\omega)\|]) < \delta_1. \tag{5.12}$$

Now, we claim that if $E[\|x_0(\omega)\|] \leq \delta$, then $E[\|x(t, \omega)\|] < \epsilon$, $t \geq t_0$. Suppose that this is false. Then, there would exist a solution process $x(t, t_0, x_0)$ with $E[\|x_0(\omega)\|] \leq \delta$ and a $t_1 > t_0$ such that

$$E[\|x(t_1, \omega)\|] = \epsilon \quad \text{and} \quad E[\|x(t, \omega)\|] \leq \epsilon, \quad t \in [t_0, t_1]. \tag{5.13}$$

By following the proof of the Theorem 5.1, we have the inequality (5.7), and hence by the convexity of b, we have

$$b(E[\|x(t, \omega)\|]) \leq \sum_{i=1}^{m} E[V_i(t, x(t, \omega), \omega)] \leq \sum_{i=1}^{m} E[r_i(t, t_0, u_0, \omega)]. \tag{5.14}$$

The relations (5.10), (5.13), and (5.14) lead to the contradiction

$$b(\epsilon) \leq \sum_{i=1}^{m} E[V_i(t_1, x(t_1, \omega), \omega)] \leq \sum_{i=1}^{m} E[r_i(t_1, t_0, u_0, \omega)] < b(\epsilon),$$

proving (i).

Based on the proof of (i) and the proof of (i) and the proof of Theorem 5.1, the proof of (ii) can be similarly formulated.

Remark 5.3: In the light of Remark 5.2 and $p = 1$, our Theorem 5.2 includes Theorems 2.2 and 3.2 in Ref. 10 as special cases.

The following result establishes the stability property of (2.1) in the sense of a probability one or almost sure sample.

Theorem 5.3: Assume that the hypotheses of Theorem 5.1 hold. Then,

(i) (SS_1^*) implies (SS_1),

(ii) (SS_2^*) implies (SS_2).

Proof: The proof of the theorem can be formulated by following the arguments used in proofs of Theorems 5.1 and 5.2 and the deterministic version[18] of the theorem. We omit the details.

Remark 5.4: Again in the light of Remark 5.2, our Theorem 5.3 includes Theorems 2.1 and 3.1 in Ref. 10, whenever the random processes $\|y(t)\|$ in Remark 5.2 satisfies the sharper law of large numbers, i.e., $\lim_{t\to\infty}(1/t)\int_0^t \|y(s)\| ds = \lim_{t\to\infty}(1/t)\int_0^t E(\|y(s)\|)\, ds$ w.p. 1.

Under this condition, the trivial solution of the comparison random differential equation $u' = g(t, u, \omega)$ is asymptotically stable with probability one, where $g(t, u, \omega)$ is as defined in Remark 5.2.

Remark 5.5: In general, we may not be able to find the auxiliary random differential system (2.2) whose trivial solution has (SP*), (SM*), and (SS*) properties. In such cases, the comparison Theorem 4.2 is useful in discussing the stability properties of (2.1). Further detail discussion about the usefulness of Theorem 4.2 can be formulated by following the discussion about the usefulness of the comparison Theorem 3.2 in Ref. 13 relative to Itô type stochastic differential equations.

Remark 5.6: Note that one could formulate the results corresponding to uniform notions under the hypotheses of the previous theorems except that (H_6) and (H_7) are replaced by (H_8) and (H_9), respectively, and the corresponding notions relative to auxiliary system (2.2) are uniform.

Remark 5.7: We also note that our stability results are local in nature. If, on the other hand $\rho = \infty$, then $D = R^n$, and the previous stability results would be of global character.

6. EXAMPLES

In this section, we give some examples in order to demonstrate the scope of our results.

The following example shows that the use of the comparison theorems and theory of differential inequalities is an alternative approach for the use of the variation of constants formula[3,8] for studying the stability properties of systems of random differential equations.

Example 6.1: Consider the system of random differential equations

$$x'(t, \omega) = F(t, \omega)x(t, \omega), \quad x(t_0, \omega) = x_0(\omega), \tag{6.1}$$

where $x \in R^n$, $F(t, \omega)$ is an $n\times n$ random matrix function whose n column vectors belong to $LC[R_+, S(R^n)]$. We further assume that

(a) $P\ \{\omega : \limsup_{t\to\infty}[\int_{t_0}^t \mu(F(s, \omega))\, ds] < \infty\} = 1$;

(b) for some positive real number a,

$$P\left\{\omega : \limsup_{t\to\infty}[(1/t)\int_{t_0}^t \mu(F(s, \omega))\, ds] \le -a\right\} = 1,$$

where $\mu(F(t, \omega))$ is the logarithmic norm for the random matrix function $F(t, \omega)$ defined by

$$\mu(F(t, \omega)) = \lim_{h\to 0^+}(1/h)[\|I + hF(t, \omega)\| - 1], \quad \text{w.p. 1.}, \tag{6.2}$$

which is a direct analog of the logarithmic norm for deterministic matrices.[18] Further details about the logarithmic norm of random matrices and its properties and scope, appears in Ref. 21. We further note that the value of $\mu(F(t, \omega))$ depends on the particular norm used for vector and matrices. However, in the following discussion, we use $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|F(t, \omega)\| = \sup_j[\sum_{i=1}^n |f_{ik}(t, \omega)|]$. In this case, $\mu(F(t, \omega))$ is given by

$$\mu(F(t, \omega)) = \sup_k[f_{kk}(t, \omega) + \sum_{\substack{i=1\\ i\ne k}}^n |f_{ik}(t, \omega)|]. \tag{6.3}$$

This is analogous to the deterministic case.[18]

Take $m = 1$, and $V(t, x, \omega) = \|x\|_1$. For $h > 0$, from (6.2), we obtain

$$\begin{aligned}\|x + hF(t, \omega)x\| &\le \|I + hF(t, \omega)\|\ \|x\| \\ &\le [h\mu(F(t, \omega)) + 1 + o(h)]\|x\|\end{aligned} \tag{6.4}$$

where $o(h)/h \to 0$ as $h \to 0^+$. From (6.4) and the definitions of $V(t, x, \omega)$, $D^+_{(6.1)}V(t, x, \omega)$, we have

$$D^+_{(6.1)}V(t, x, \omega) \le \mu(F(t, \omega))V(t, x, \omega). \tag{6.5}$$

The auxiliary or comparison random differential equation is $u'(t, \omega) = \mu(F(t, \omega))u(t, \omega)$. If (a) holds, then $u = 0$ is stable with probability 1. On the other hand if (b) holds, then $u = 0$ is asymptotically stable with probability 1. Thus, all the hypotheses of Theorem 5.3 are satisfied. Hence, relative to (6.1), the conclusion of Theorem 5.3 remains true.

Remark 6.1: Note that one can state suitable conditions on $\mu(F(t, \omega))$ so that the other types of stability properties of (6.1) can be similarly derived.

In the following, we discuss a simple example that shows that our approach is not only an alternative approach over the variation of constants formula approach, but also shows certain gain over the earlier stability analysis.[3,10]

Example 6.2: Consider the special type of random differential system

$$x'(t, \omega) = A(\omega)x + B(t, \omega)x, \tag{6.6}$$

where $x \in R^n$, $\|A(\omega)\| \le M$ w.p. 1, $E[\|B(t, \omega)\|] < \infty$ for $t \in R_+$, and the elements $b_{ij}(t, \omega)$ of the random matrix function $B(t, \omega) = (b_{ij}(t, \omega))$ are product measurable.

We further assume that

$$P\left\{\omega : \limsup_{t\to\infty}[(1/t)\int_{t_0}^t \mu(A(\omega) + B(s, \omega))] \le -a\right\} = 1,$$
$$\text{for } a > 0. \tag{6.7}$$

As before, take $V(t, x, \omega) = \|x\|_1$ and by following the argument in Example 6.1, we conclude that the trivial solution of (6.6), asymptotically stable with probability one.

In order to compare, the stability condition (6.7) with the earlier stability condition,[3,8] we use the property of logarithmic norm, $\mu(A(\omega)+B(t,\omega)) \leqslant \mu(A(\omega)) + \mu(B(t,\omega))$.[21] In the light of this, the inequality (6.5) relative to (6.6) reduces to

$$D^+_{(6.6)}V(t,x,\omega) \leqslant [\mu(A(\omega)) + \mu(B(t,\omega))]V(t,x,\omega). \tag{6.8}$$

Similarly, the stability condition (6.7) reduces to

$$P\Big\{\omega : \mu(A(\omega)) + \limsup_{t\to\infty}[(1/t)\textstyle\int_{t_0}^t \mu(B(s,\omega))\,ds] \leqslant -a\Big\} = 1$$

for $a > 0$. (6.9)

Again by using the property $\mu(B(t,\omega)\|B(t,\omega)\|$,[21] if we further majorize (6.8), we get

$$D^+_{(6.6)}V(t,x,\omega) \leqslant [\mu(A(\omega)) + \|B(t,\omega)\|]V(t,x,\omega). \tag{6.10}$$

In this case, the stability condition (6.9) becomes

$$P\Big\{\omega : \mu(A(\omega)) + \limsup_{t\to\infty}[(1/t)\textstyle\int_{t_0}^t \|B(s,\omega)\|ds] \leqslant -a\Big\} = 1. \tag{6.11}$$

From (6.3), the stability conditions (6.7), (6.9), and (6.11) are equivalent to

$$P\left\{\omega : \lim_{t\to\infty}\inf\left[\frac{1}{t}\int_{t_0}^t \inf_j[-(a_{jj}(\omega)+b_{jj}(s,\omega)) - \sum_{\substack{i=1\\i\neq j}}^n |a_{ij}(\omega)+b_{ij}(s,\omega)|]ds\right] \geqslant a\right\} = 1, \tag{6.12}$$

$$P\left\{\omega : \inf\left[-(a_{jj}(\omega)) - \sum_{\substack{i=1\\i=j}}^n |a_{ij}(\omega)|\right] + \liminf_{t\to\infty}\left[\frac{1}{t}\int_{t_0}^t \inf_j\left(-(b_{jj}(s,\omega)) - \sum_{\substack{i=1\\i\neq j}}^n |b_{ij}(s,\omega)|\right)ds\right] \geqslant a\right\} = 1, \tag{6.13}$$

and

$$P\left\{\omega : \inf\left[-(a_{jj}(\omega)) - \sum_{\substack{i=1\\i\neq j}}^n |a_{ij}(\omega)|\right] - \limsup_{t\to\infty}\left[\frac{1}{t}\int_{t_0}^t \|B(s,\omega)\|\,ds\right] \geqslant a\right\} = 1, \tag{6.14}$$

respectively.

With respect to (6.6), if we further assume that the elements $b_{ij}(t,\omega)$ of the random matrix function $B(t,\omega)$ are strictly stationary metrically transitive stochastic processes,[22] then the stability conditions (6.12), (6.13), and (6.14) reduce to

$$P\left\{\omega : -(a_{jj}(\omega) + E[b_{jj}(0,\omega)]) - \sum_{\substack{i=1\\i\neq j}}^n E[|a_{ij}(\omega)+b_{ij}(0,\omega)]| \geqslant a\right\} = 1, \tag{6.15}$$

$$P\left\{\omega : -(a_{jj}(\omega) + E[b_{jj}(0,\omega)]) - \sum_{\substack{i=1\\i\neq j}}^n |a_{ij}(\omega)| + \sum_{\substack{i=1\\i\neq j}}^n E[|b_{ij}(0,\omega)|] \geqslant a\right\} = 1, \tag{6.16}$$

and

$$P\left\{\omega : -(a_{jj}(\omega) + E[b_{jj}(0,\omega)]) - \left[\sum_{\substack{i=1\\i\neq 1}}^n |a_{ij}(\omega)| + E[\|B(0,\omega)\|]\right] \geqslant a\right\} = 1, \tag{6.17}$$

respectively. We note that the stability condition (6.17) implies

$$P\{\omega : a_{jj}(\omega) < 0\} = 1 \text{ and } P\ \omega : |a_{jj}(\omega)| - \sum_{\substack{i=1\\i\neq j}}^n |a_{ij}(\omega)| \geqslant a\ = 1,$$

and hence $P\{\omega : \mu(A(\omega)) \leqslant -a\}$. This together with the property that $\mathrm{Re}\lambda(\omega) \leqslant \mu(A(\omega))$ for any eigenvalue $\lambda(\omega)$ of $A(\omega)$ yields

$$P\ \omega : \max_{\lambda(\omega)\in\sigma(A(\omega))}\mathrm{Re}\lambda(\omega) < -a\ = 1. \tag{6.18}$$

From (6.18), the random matrix $A(\omega)$ is P-stable.[3,6] This shows that the stability condition (6.17) is stronger than the P-stability of $A(\omega)$. Recently,[3,8] by assuming the P-stability of $A(\omega)$ and measurability, strictly stationary metrically transitive property of the coefficients of random matrix $B(t,\omega)$, the stability of the trivial solution is established, whenever $E[\|B(0,\omega)\|]$ is sufficiently small. Now, by comparing the different set of the stability conditions (6.15), (6.16), and (6.17), one can immediately conclude that our approach is certainly more advantageous over the earlier approach.[3,6,8] Furthermore, the particular stability condition (6.16) that is more restrictive than (6.15), shows that the matrix $A(\omega)$ may not be P-stable. From (6.16), we can conclude that the trivial solution of (6.6) is asymptotically stable with probability one, if at least one of the matrices $A(\omega)$ and $E[B(0,\omega)]$ is P-stable, and the matrices $A(\omega)$, $E[B(0,\omega)]$ satisfy the relation (6.16). This shows an important gain over the earlier approach.

In the following, we shall make further remarks which are important by-products of our above discussion.

Remark 6.2: We note that the special type of random differential system (6.6) is not a very restrictive assumption. In fact, the differential system (6.1) can be rewritten as (6.6) whenever $E[F(t,\omega)]$ exists. For instance, we set

$$A(t) = E[F(t,\omega)] \text{ and } B(t,\omega) = F(t,\omega) - E[F(t,\omega)].$$

Then, the system (6.1) can be rewritten as

$$x'(t,\omega) = A(t)x + B(t,\omega)x. \tag{6.19}$$

Now, one can formulate the stability conditions for (6.19) analogous to the stability conditions of (6.6). For details, see Ref. 21.

Remark 6.3: We also note that in the case of white noise coefficients,[3,13,16] the randomness is a destabilizing agent, however, in the case of nonwhite coefficients such as strictly stationary metrically transitive random coefficients, the randomness may be a stabilizing agent. This remark can be justified from (6.15) and (6.16). Further note that this observation confirms the Note 3.2 made by Khas'minskii.[10]

In the following, we give an example to illustrate the comparison principle relative to the system (2.1).

Example 6.3: Suppose that (2.1) satisfies

$$\|x_i + hf_i(t, x, \omega)\| \leq \|x_i\| + h\left(\sum_{j=1}^{m} a_{ij}(t, \omega)\|x_j\|\right), \quad i = 1, 2, \ldots, m, \tag{6.20}$$

for $(t, x) \in R_+ \times D$, sufficiently small $h > 0$, where

$$x_i \in R^{n_i}, \ n = \sum_{i=1}^{m} n_i, \ a_{ij} \in LC[R_+, S(R)], \ a_{ij} \geq 0 \text{ for } i \neq j.$$

Take

$$V(t, x, \omega) = (V_1(t, x), V_2(t, x), \ldots, V_m(t, x))^T \tag{6.21}$$

where $V_i(t, x) = \|x_i\|$, $i = 1, 2, \ldots, m$. Note that

$$b(\|x\|) \leq \sum_{i=1}^{m} V_i(t, x) \leq a(\|x\|), \tag{6.22}$$

where $b(\|x\|) = \|x\|$, $a(\|x\|) = \sqrt{m}\,\|x\|$.

From (6.20) we have the vectorial inequality

$$D^+ V(t, x, \omega)_{(2.1)} \leq g(t, V(t, x, \omega)),$$

where

$$g(t, u, \omega) = A(t, \omega)u.$$

It is obvious that $g(t, u, \omega) \in LC[R_+ \times R^m, S(R^m)]$ and $g(t, u, \omega)$ satisfies the quasimonotone nondecreasing property in u for fixed $t \in R_+$. Assume that the trivial solution of (2.2) is uniformly stable in probability. Thus all the hypotheses of Theorem 5.1 and Remark 5.6 are satisfied. Hence, we conclude that the trivial solution of the system (2.1) is uniformly stable in probability. Note that one can draw similar conclusions with respect to the stability in the mean and the stability with probability one in the context of Theorems 5.2 and 5.3 together with Remark 5.6, analogously.

Remark 6.5: In order to show an advantage of a vector Lyapunov function over a single Lyapunov function, an example similar to the Example 5.3 in Ref. 13 can be analogously constructed. To avoid monotonicity, we do not want to discuss further details.

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ERRATA

Erratum: Clebsch-Gordan coefficients for crystal space groups [J. Math. Phys. 16, 227 (1975)]

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It has been pointed out to us that due to our use of condensed notation, certain of the equations could be misread. In order to avoid this possibility we give below some equations, including expanded notation.

(1) Equation (3.8) can be clarified by writing

$$P_{\phi_\sigma}\psi^m_{\tau\nu}=\frac{1}{\omega_{\sigma,\sigma^{-1}}}\sum_{\mu=1}^{l_m}H^m(\phi_{\sigma^{-1}}\phi_g\phi_\tau)_{\mu\nu}\,\psi^m_{\sigma\mu}$$

$$=\frac{\omega_{\sigma^{-1},g\tau}\,\omega_{g,\tau}}{\omega_{\sigma,\sigma^{-1}}}\sum_{\mu=1}^{l_m}H^m(\phi_{\sigma^{-1}g\tau})_{\mu\nu}\,\psi^m_{\sigma\mu}.$$

(2) Equation (5.15) states

$$\omega^k_{\alpha,\alpha^{-1}}=1$$

This is to be read (in expanded notation) as

$$\omega^k_{\{\phi_\alpha|\tau_\alpha\},\{\phi_\alpha|\tau_\alpha\}^{-1}}=1.$$

It is *not* to be read as

$$\omega^k_{\{\phi_\alpha|\tau_\alpha\},\{\phi_{\alpha^{-1}}|\tau_{\alpha^{-1}}\}}=1.$$

Thus what appears in Eq. (5.16) is

$$\dot{D}(\{\phi_\sigma|\tau_\sigma\}^{-1}\{\phi_g|\tau_g\}\{\phi_\tau|\tau_\tau\})$$

and *not*

$$\dot{D}(\{\phi_{\sigma^{-1}}|\tau_{\sigma^{-1}}\}\{\phi_g|\tau_g\}\{\phi_\tau|\tau_\tau\}).$$

We thank Professor R. Dirl for suggesting that we clarify our notation.

Erratum: Optimal factor group for nonsymmorphic space groups [J. Math. Phys. 17, 1051 (1976)]

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In Table I, p. 1054, some group operations are not in the right place. The correct Table I should read as follows on the right. The relation $R_3^{-1}R_2R_3=R_4$ in the first column of p. 1054 must be replaced by $R_3^{-1}R_2R_3=tR_2$. In the first row of Table III the right values of $D_1(R_3)_{12}$ and $D_1(R_4)_{12}$ are respectively $D_1(R_3)_{12}=1$ and $D_1(R_4)_{12}=-1$.

TABLE I. Multiplication table of Q_D.

	R_1	R_2	R_3	R_4
R_1	R_1	R_2	R_3	R_4
R_2	R_2	R_1	tR_4	tR_3
R_3	R_3	R_4	R_1	R_2
R_4	R_4	R_3	tR_2	t

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